Algebra, Math 250B
Homework six, Due March 10

1. Prove that if the (complex) variety of the polynomials $f_1, \ldots, f_r$ in $\mathbb{Q}[x_1, \ldots, x_n]$ is empty, then there is a Nullstellensatz certificate whose coefficients are in fact all polynomials with rational coefficients.

2. Find and example over F_5 where the statement of Hilbert's Nullstellensatz is false.

3. Hungerford page 382-383: 12,18 (on primary decomposition)

4. Describe how would you find the intersection of two monomial ideals. Determine when a monomial ideal is (a) prime, (b) primary (c) irreducible.

5. Rotman 379-380: 5.75,5.77

6. Show that the result of applying the Euclidean Algorithm in one variable to any pair of polynomials f, g is a reduced Gröbner basis for the ideal they generated.

7. Rotman page 389-390: 5.80,5.81,5.82 (Maple ok! in last 2 problems).

8. Using Gröbner bases (in Maple or similar system),

   a) find the critical points of the function

   $$f(x, y) = (x^2 + y^2 - 4)(x^2 + y^2 - 1) + (x - 3/2)^2 + (y - 3/2)^2.$$

   b) Find the maximum of $f = x^2 + y^2 + xy$ subject to $x^2 + 2y^2 = 1$.

9. For each of the following systems of polynomial equations, use MAPLE to answer the following questions: 1) Is the system solvable? 2) Does it have a finite number of solutions or is it infinite?

   a) $x^2 - 2x + 5, xy^2 + yz^3, 3y^2 - 8z^3$.

   b) $x^2z^2 + x^3, xz^4 + 2x^2z^2 + x^3, y^2z - 2yz^2 + z^3$.

10. Is $f = xy^3 - z^2 + y^5 - z^3$ in the ideal generated by $-x^3 + y, x^2y - z$? Same question, is $g = x^3z - 2y^2$ in the ideal generated by $xz - y, xy + 2z^2, y - z$?

11. True or false? The ideals $I = (x + xy, y + xy, x^2, y^2)$ and $J = (x, y)$ in $K[x, y]$, for any field K are identical.