Algebra, Math 250B
Homework seven, Due March 24

1. let $R = \mathbb{Q}[x]$, $M = \mathbb{Q}[x]/\langle (x-2)^3 \rangle$.

   (a) List all R-submodules of M.

   (b) With respect to the $\mathbb{Q}$-basis $B = \{(x-2)^2, (x-2), 1\}$, write out the matrix A for the linear transformation on M given by left multiplication by x.

   (c) For each of your submodules from part (a) describe what they mean in terms of the matrix A

2. Let k be a field. Verify the following data are equivalent

   (a) a $k[x]$-module V with $dim_k(V) = n$

   (b) an n-dimensional k-vector-space V and a linear map from V to V.

   (c) A fixed matrix $A \in M_n(k)$

   Now let $R = \mathbb{C}[x] \supseteq I =< x^3 - 1 >$. Let M be the $\mathbb{C}[x]$-module $V = R/I$.

   (d) Describe M as in (c)

   (e) Is A diagonalizable? If so, determine its eigen-basis, i.e. the basis with respect to which it is diagonal

   (f) relate this basis/diagonalizability to the Chinese Remainder Theorem.

3. What is the Abelian group with generators a, b, c, d, e whose relations are given by

$$a - 7b + 21c + 14d = 0, \quad 5a - 7b - 2c + 10d - 15e = 0,$$
$$3a - 3b - 2c + 6d - 9e = 0, \quad a - b + 2d - 3 = 0.$$

4. Use your knowledge of Smith normal forms to find all solutions (if any!) for the linear Diophantine system of equations

$$12x_1 + 6x_2 + 7x_3 = 8, \quad 2x_1 + 9x_2 + 4x_3 = 7$$

   where x_i is an integer-valued variable.

5. Rotman pages 663-664: 8.31, 8.32, 8.34

6. Rotman pages 670-671: 8.35, 8.41

7. Rotman page 682: 8.46.