Algebra, Math 250B
Midterm 1 (Take-home) Due Friday February 17

You can discuss the problems with me or with MAPLE, but not with other students without me being present too. Justify your answers! In particular, attach any MAPLE code you used to solve the problems. You may use your notes or any book you wish, but cite references used, and include complete proofs.

1. Problems on Field Theory

   (a) We have seen already it is true over $\mathbb{Q}$, but prove that the polynomial ring $K[x]$ over any field has infinitely many irreducible polynomials.

   (b) If K is a field of characteristic different from 2, prove that $\sqrt{1-x^2}$ is not a rational function over K.

   (c) Suppose u is an algebraic number over $\mathbb{Q}$ with minimal polynomial f of degree n. Recall that $\mathbb{Q}[u] = \mathbb{Q}(u)$ (see Theorem V 1.6 in Hungerford). Given a polynomial $g(x) \in \mathbb{Q}[x]$, with $g(u)$ nonzero, how you would write $\frac{1}{g(u)}$ in terms of powers of u?

   (d) Let α be a root of $x^3 - x - 1 \in \mathbb{Q}[x]$. Consider the extension $K = \mathbb{Q}[\alpha]$. Now pick β a root of the polynomial $x^2 + (\alpha^2+1)x + (7\alpha^2+5\alpha)$ with coefficients in K. Is there a minimum degree irreducible polynomial over $\mathbb{Q}$ for β? If yes, what is it? If not, why not?

   (e) Let p, q primes. Consider the field extension $K = \mathbb{Q}[\sqrt{p}, \sqrt{q}]$ of $\mathbb{Q}$. What is the degree of this extension? What is its Galois group? Is this extension actually a simple extension? I.e., does there exist γ such that $K = \mathbb{Q}[\gamma]$?

2. Problems on Galois Theory

   (a) A finite Galois extension E of K has degree 8100. Is there a field F such that $K \subset F \subset E$ and $[F : K] = 100$?

   (b) Determine (by hand) the Galois group of the polynomial $x^4 + 4x^3 - 4x^2 - 16x + 72$.

   (c) Consider the family of all quartic polynomials of the form $x^4 + rx + s \in \mathbb{Q}[x]$. For which values of r, s are these polynomials separable? Suppose now $s = -1$, what Galois groups are possible for varying integer values of r? Can you find explicit values of r that achieve those groups you think are possible?

   (d) We saw in class the quintic $x^5 - 4x + 2$ is not solvable by radicals. For this we used the fact two of its roots are complex. Can you find a non-solvable quintic whose 5 roots are real?

   (e) Exhibit a rational polynomial f of degree 6, whose Galois group over the rationals is the dihedral group D_6.