Algebra, Math 250B
Midterm 2 (Take-home) Due Friday February 17

INSTRUCTIONS:

1. You can discuss the problems with me or with MAPLE, but not with other students without me being present too.
2. Write each problem in a separately, dont use same sheet for two problems.
3. Justify your answers very well! In particular, attach any MAPLE code you used to solve the problems. You may use your notes or any book you wish, but cite references used, and include *complete* proofs (the only exception is stuff I proved in class and it is in your notes).

Material in this test is roughly that of Rotman chapter 5 (basics commutative algebra) and thus Homeworks 4,5,6.

1. If $P_1, P_2, \ldots, P_n$ are prime ideals in a commutative ring R and I is an ideal such that $I \not\subset P_i$ for all i, then there is $r \in I$, such that $r \notin P_i$ for all i.
2. If R is a commutative ring with identity, and S a multiplicative subset. The operation of localization was briefly described in class on how to construct new commutative rings. Show that every ideal J of $S^{-1}R$ is of the form $S^{-1}P$ for some ideal P of R. Is P uniquely determined by J? Why or why not?
3. Consider the ring of formal power series in x with real coefficients (i.e., infinite series with no conditions of convergence). What are the ideals in this ring? Is it Noetherian? Similarly, is the following ring Noetherian? the ring of rational functions on z having no pole on the circle $|z| = 1$.
4. On what follows let $S = K[x_1, x_2, \ldots, x_n]$. Prove or disprove the following statements:
   (a) Let $>$ be a monomial order and $f, g \in S$, $in(fg) = in(f)in(g)$.
   (b) the monomial 1 is a necessarily the smallest monomial under any such ordering.
   (c) If I is a monomial ideal any Gröbner bases is made up of monomials.
5. Prove that the intersection, sum, product, and quotient of any two monomial ideals is again a monomial ideal.
6. Find an implicit equation(s) in x, y, z for the surface parametrized by $x = t + u,\ y = t^2 + 2tu,\ z = t^3 + 3t^2u$.

7. Construct a computer algorithm to check whether a polynomial belongs to the radical of an ideal I. Use it to decide whether $x+y$ is in the radical of $< x^3, y^3, xy(x+y) >$. HINT: Your algorithm must actually END in all situations and not go in an infinite loop. Prove it stops. If you study the Rabinowitz trick you will come up with ideas.

8. Given an ideal I of the polynomial ring $K[x_1, \ldots, x_n]$, let $G = \{g_1, \ldots, g_s\}$ be Gröbner basis for I with respect to some monomial order. Prove or disprove the following statements:

   (a) The remainder of the division of f by G is unique.

   (b) If G' is another Gröbner basis of I for a different monomial order the remainders for f are the same.

   (c) A basis for the K-vector space $K[x_1, \ldots, x_n]/I$ is given by the cosets $\{x^a + I | LM(g_i) \text{ does not divide } x^a\}$.

   (d) The set $\{hg_1, hg_2, \ldots, hg_s\}$ is also a Gröbner basis for the ideal it generates.

   (e) No finite set of polynomials can be a Gröbner basis for the ideal I for ALL monomial orders

9. Recall the elementary symmetric functions $e_1, e_2, \ldots, e_n$ in $R_n = \mathbb{Q}[x_1, \ldots, x_n]$ are simply the coefficients of the generic univariate polynomial of degree n $g(z) = (z - x_1)(z - x_2)\cdots(z - x_n)$. E.g., $e_1 = x_1 + x_2 \ldots + x_n$, and $e_n = x_1 x_2 \cdots x_n$.

   a) Give a "Gröbnerian" style proof of the fact the ring of symmetric polynomials in $x_1, \ldots, x_n$ equals the $\mathbb{Q}$-algebra generated by $e_1, \ldots, e_n$.

   b) Use your algorithm to write the Discriminant of a cubic equation in terms of in terms of e_1, e_2, e_3. Prove that these 3 elementary symmetric functions are in fact algebraically independent.

   c) Let I_n be the ideal generated by the the elementary symmetric functions $e_1, e_2, \ldots, e_n$. What is the $\mathbb{C}$-vector space dimension of the quotient R_2/I_2, R_3/I_3, R_4/I_4? Show vector space bases for each of them. What are the varieties of I_2, I_3? Describe them the best you can.

10. if I is an ideal in S generated by *binomials* (i.e., all generators are of the form $x^a - x^b$ for two different monomials in compact monomial notation), then any reduced Gröbner basis is generated by binomials too.