Combinatorial Geometry & Topology arising in Game Theory and Optimization

Jesús A. De Loera

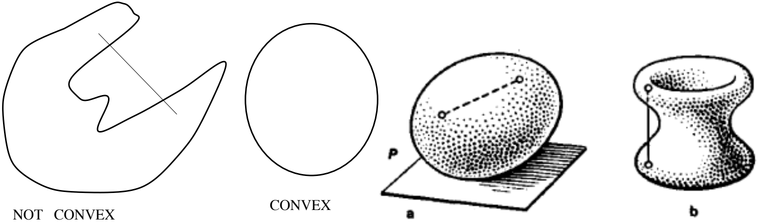
University of California, Davis

LAST EPISODE...

We discuss the content of the course...

Convex Sets

A set is CONVEX if it contains any line segment joining two of its points:

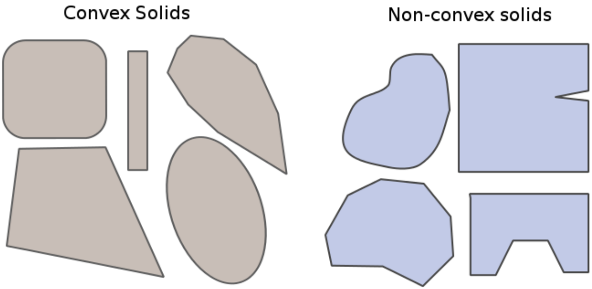


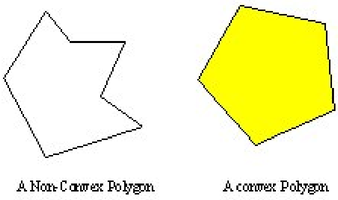
The line segment between x and y is given by

$$[x, y] := \{\alpha x + (1 - \alpha)y : 0 \leq \alpha \leq 1\}$$

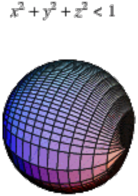
EXERCISE Prove or disprove: the image of a convex set under a linear transformation is again a convex set.

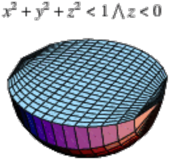
Examples

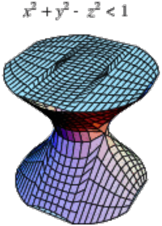

Convex Solids
Non-convex solids

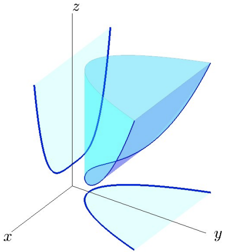

A Non-Convex Polygon
A convex Polygon

TEST: Which of the following are convex sets?









$$x^4 - (z-1) \leq 0 \quad \textbf{and} \quad x^2 - (y-1) \leq 0 \quad \textbf{and} \quad z \geq 0$$

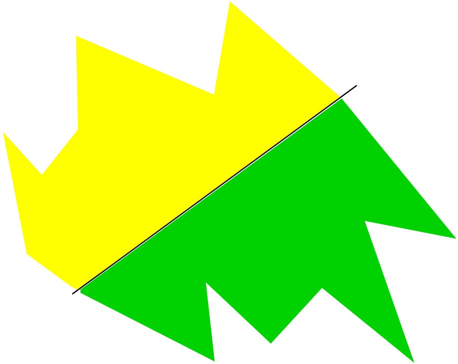
Proposition: The intersection of convex sets is always convex.

HYPERPLANES

- A linear functional $f : \mathbf{R}^d \rightarrow \mathbf{R}$ is given by a vector $c \in \mathbf{R}^d, c \neq 0$.
- For a number $\alpha \in \mathbf{R}$ we say that $H_\alpha = \{x \in \mathbf{R}^d : f(x) = \alpha\}$ is an *affine hyperplane* or hyperplane for short.
- The intersection of finitely many hyperplanes is an *affine space*.
- The *affine hull* of a set A is the smallest affine space containing A.
- Affine spaces are important examples of convex sets in particular because they allow us to speak about dimension:
- The dimension of an affine set is the largest number of *affinely independent points* in the set minus one.
- The dimension of a convex set in $\mathbf{R}^d$ is the dimension of its affine hull.

HALF-SPACES

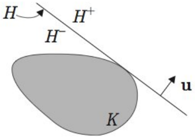
A hyperplane divides $\mathbf{R}^d$ into two halfspaces $H_\alpha^+ = \{x \in \mathbf{R}^d : f(x) \geq \alpha\}$ and $H_\alpha^- = \{x \in \mathbf{R}^d : f(x) \leq \alpha\}$.



Half-spaces are convex sets each denoted formally by a linear inequality:

$$a_1 x_1 + a_2 x_2 + \cdots + a_d x_d \leq b$$

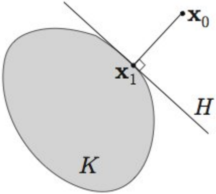
- For a convex set S in $\mathbf{R}^d$. A linear inequality $f(x) \leq \alpha$ is said to be valid on S if every point in S satisfies it.
- A set $F \subset S$ is a face of S if there exists a linear inequality $f(x) \leq \alpha$ which is valid on P and such that $F = \{x \in P : f(x) = \alpha\}$.
- The hyperplane defined by f is a supporting hyperplane of F. It defines a supporting half-space



K has a supporting hyperplane orthogonal to $\mathbf{u}$.

- A face of dimension 0 is called a vertex. A face of dimension 1 is called an edge, and a face of dimension $dim(P) - 1$ is called a facet.

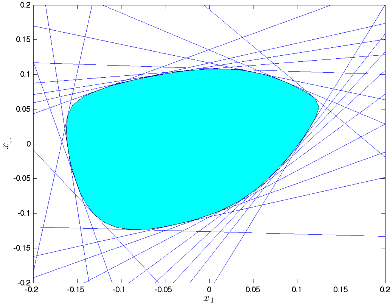
- Let K be a closed and bounded convex set in $\mathbf{R}^d$. Let $x_0 \notin K$. Then,
  - There is a unique nearest point x_1 of K to x_0.
  - The hyperplane H through x_1 orthogonal to $x_1 - x_0$ is a supporting hyperplane of K.



- A hyperplane H red separates sets X and Y if and only if X and Y lie in different closed halfspaces of H. If X and Y lie in different open halfspaces, we say that H *strictly separates* X and Y.

CONVEX BODIES ARE INTERSECTION OF HALF-SPACES!!!

Theorem A convex body K is the intersection of its closed supporting half-spaces.



Theorem convex bodies are the sets of solutions of **systems of LINEAR inequalities**.

WARNING: It may require **infinitely many hyperplanes**

POLYHEDRA: THE INTERSECTION OF FINITELY MANY HALF-SPACES

Examples

- **d-dimensional unit cube**

$$C_d = \{x \in \mathbf{R}^d : 0 \leq x_i \leq 1,\ i = 1..n\}$$

- **the $(d-1)$-dimensional standard simplex**

$$\Delta_{n-1} = \{x \in \mathbf{R}^d : \sum_i^d x_i = 1,\ x_i \geq 0\}.$$

- **the d-dimensional cross-polytope**

$$O_n = \{x \in \mathbf{R}^d : \sum_i^d |x_i| \leq 1\}.$$

- **a simplotope** is the Cartesian product of several simplices

$$\Delta_{m_1} \times \Delta_{m_2} \times \cdots \times \Delta_{m_r}.$$

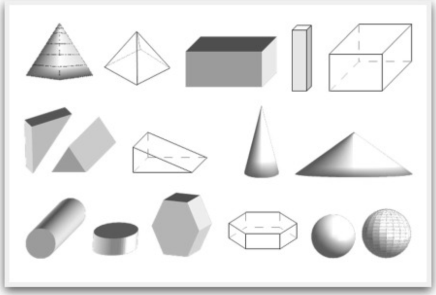
SOLVABILITY OF SYSTEMS OF LINEAR INEQUALITIES

Find a vector $(x_1, x_2, \ldots, x_d)$, satisfying:

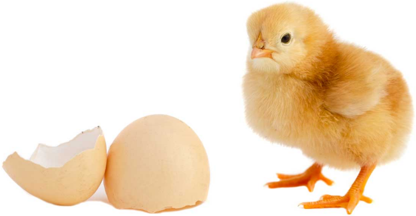
$$\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,d}x_d &\leq b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,d}x_d &\leq b_2 \\ &\vdots \\ a_{k,1}x_1 + a_{k,2}x_2 + \cdots + a_{k,d}x_d &\leq b_k \end{aligned}$$

This is the Linear feasibility problem

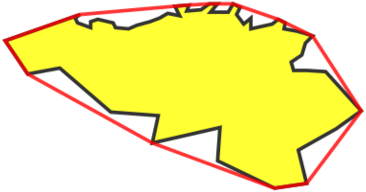
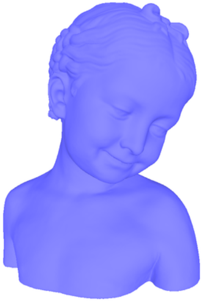
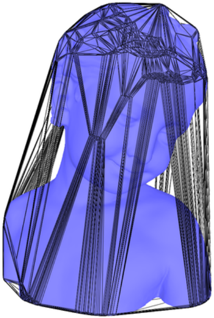
Convex Sets are EVERYWHERE!



and ALTHOUGH not all sets in nature are convex!



Convex Sets APPROXIMATE ALL SHAPES!

Let $A \subset \mathbf{R}^d$. The convex hull of A, denoted by $conv(A)$, is the intersection of all the convex sets containing A. The smallest convex set that contains A.

A polytope is the convex hull of a finite set of points in $\mathbf{R}^d$. It is the smallest convex set containing the points.

linear convex and conic combinations

- **Definition:** Given finitely many points $A := \{x_1, x_2, \ldots, x_n\}$ we say the linear combination $\sum \gamma_i x_i$ is
  - a conic combination is one with all γ_i non-negative.
  - an affine combination if $\sum \gamma_i = 1$.
  - a convex combination if it is affine and $\gamma_i \geq 0$ for all i.
- **Lemma:** (EXERCISE) For a set of points A in $\mathbf{R}^d$ we have that $conv(A)$ equals all finite convex combinations of A:

$$conv(A) = \{\sum_{x_i \in A} \gamma_i x_i : \gamma_i \geq 0 \; and \; \gamma_1 + \ldots \gamma_k = 1\}$$

- **Definition** A set of points $x_1, \ldots, x_n$ is affinely dependent if there is a linear combination $\sum a_i x_i = 0$ with $\sum a_i = 0$. Otherwise we say they are affinely independent.
- **Lemma:** A set of $d+2$ or more points in $\mathbf{R}^d$ is affinely dependent.
- **Lemma:** A set $B \in \mathbf{R}^d$ is affinely independent $\iff$ every point has a unique representation as an affine combination of points in B.
- A k-dimensional **simplex** is the convex hull of $k+1$ affinely independent points.

Weyl-Minkowski: How to represent the points of a polyhedron?

- There are **TWO ways** to represent a convex set: As the intersection of half-spaces OR as the convex/conic hull of extreme points.
- For polyhedra, even better!! Either as a finite system of inequalities or with finitely many generators.

Weyl-Minkowski Theorem

- **Theorem**: (Weyl-Minkowski's Theorem): For a polyhedral subset P of $\mathbf{R}^d$ the following statements are equivalent:
  - P is an **H-polyhedron**, i.e., P is given by a system of linear inequalities $P = \{x : Ax \geq b\}$.
  - P is a **V-polyhedron**, i.e., For finitely many vectors $v_1, \ldots, v_n$ and $r_1, \ldots, r_s$ we can write

$$P = conv(v_1, v_2, \ldots, v_n) + cone(r_1, r_2, \ldots, r_s)$$

- $R + S$ denotes the Minkowski sum of two sets, $R + S = \{r + s : r \in R, s \in S\}$.
- There are algorithms for the conversion between the H-polyhedron and V-polyhedron.
- **NOTE:** Any cone can be decomposed into a pointed cone plus a linear space.