Triangulations of Convex Polytopes and Point Configurations.

Winter 2016

Jesús A. De Loera

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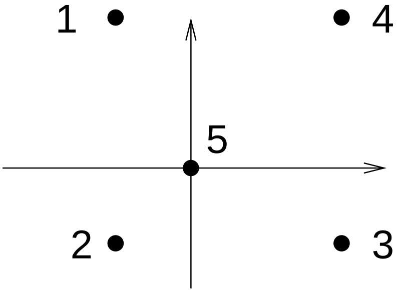

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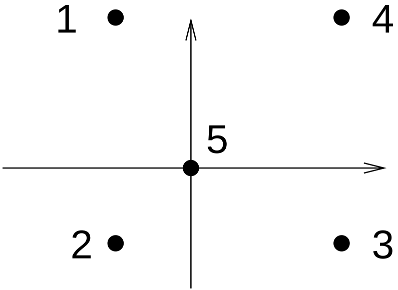

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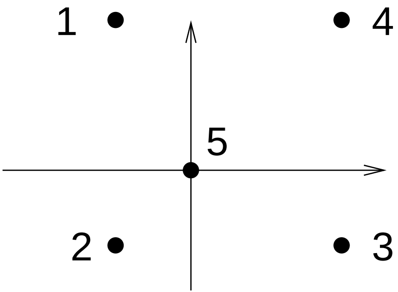

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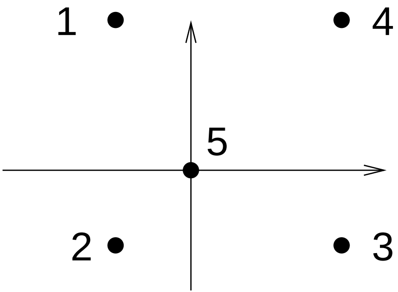

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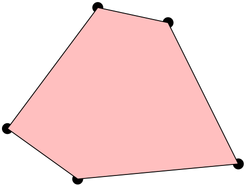
A convex **polytope** is the convex hull of finitely many points

$$\mathrm{conv}(p_1, \ldots, p_n) := \{\sum \alpha_i p_i : \alpha_i \geq 0 \ \forall i = 1, \ldots, n, \sum \alpha_i = 1\}$$



A **simplex** is the convex hull of any set of **affinely independent points**.

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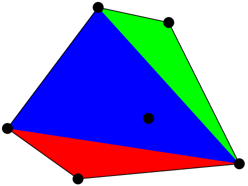
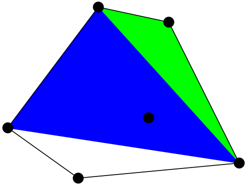
Remark: We do **not** need to use all (interior) points!, but cannot add points



Exercises

- Let F be a k-dimensional face of a d-dimensional polytope P. And let T be a triangulation of P. Denote by T^k the k-dimensional faces of all the simplices of T. Prove that the set $\{\tau : \tau \subset F,\ \tau \in T^k\}$ is a triangulation of F.
- Let D be a compact subset of a polytope $P \subset \mathbb{R}^d$. Prove that D meets any triangulation T of P only in a finite number of simplices.
- Let T be a triangulation of a d-polytope. Prove that if σ is a $d-1$ simplex inside T it is either a facet of either 1 d-simplex or 2 d-simplices of T

The following are **not** triangulations:



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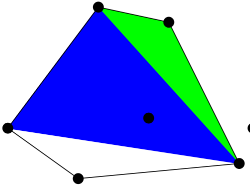


Remark: We have a simplicial complex, a ball, with an explicit coordinate realization!

diameter, mesh size

Given a triangulation T of a polytope, the **diameter** of a simplex $\sigma \in T$ is given by

$$diam(\sigma) = \max\{||x - y|| : x, y \in \sigma\}$$

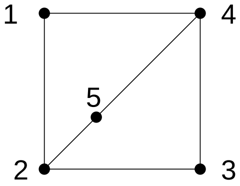
The **mesh size** of the triangulation T is given by the

$$mesh(T) = sup\{diam(\sigma) : \sigma \in T\}$$

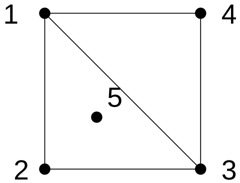
By adding more and more points and refining the triangulation we reduce the mesh size.

All triangulations of a point configuration

When the number of vertices allowed is finite, we have finitely many possible triangulations.



124, 234



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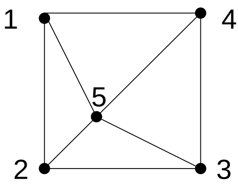

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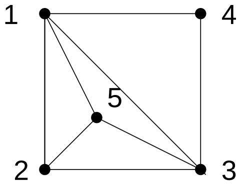

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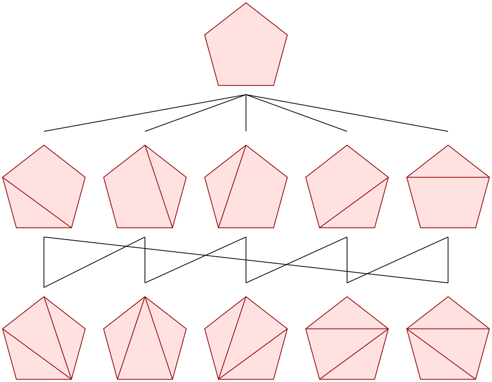
BIG WISH to give structure to the set of all triangulations!!

Important: There is a natural generalization to **subdivisions**: Pieces are not simplices!

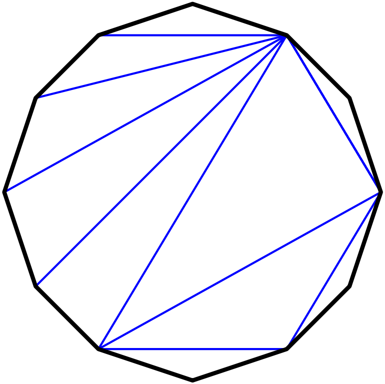


Subdivisions form a partially ordered set:

$$\Gamma_1 < \Gamma_2 \text{ if } \Gamma_1 \text{is finer than } \Gamma_2.$$

A key example: Triangulations of a convex n-gon

To triangulate the n-gon, you just need to insert $n-3$ non-crossing diagonals:



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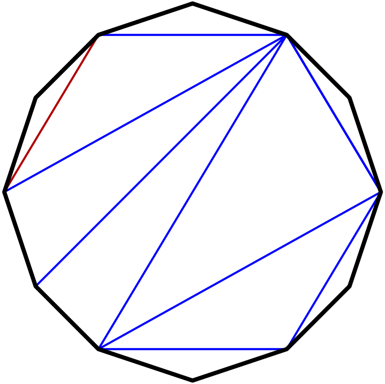
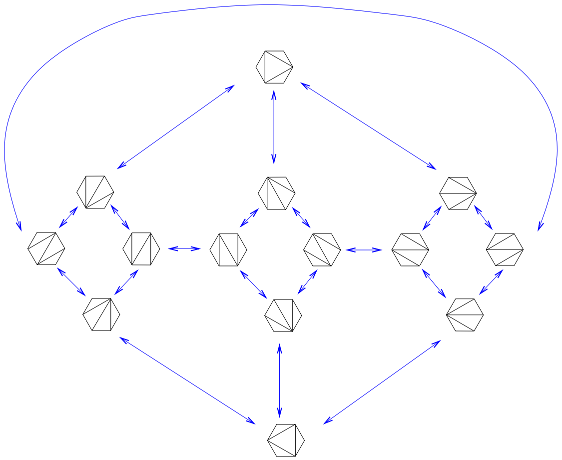


The flips of a Hexagon



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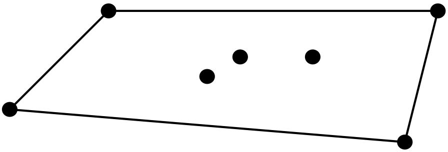
THIS HAS FAR REACHING GENERALIZATIONS!!

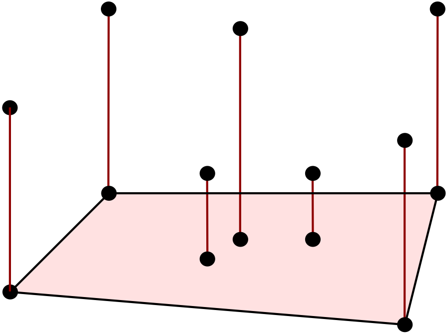
Regular triangulations

Let $A = \{a_, \ldots, a_n\} \subset \mathbb{R}^d$ be a vector configuration. Let $h = (h_1, \ldots, h_n) \in \mathbb{R}^n$ be a vector of **heights**.

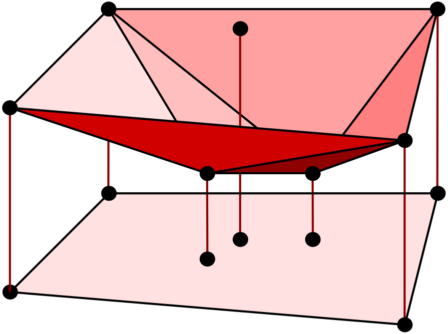


Regular triangulations

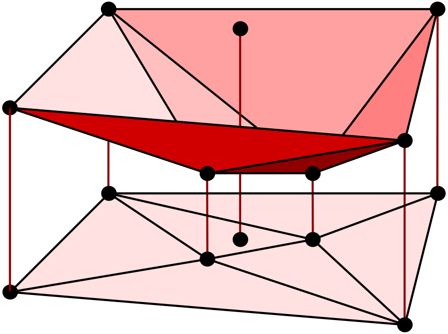
Consider the *lifted* configuration $\tilde{A} = \begin{Bmatrix} a_1 & \cdots & a_n \\ h_1 & \cdots & h_n \end{Bmatrix} \subset \mathbb{R}^{d+1}$.



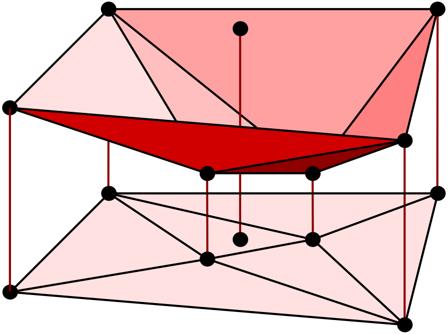
Compute the *lower envelope* of $\mathrm{conv}(\tilde{A})$



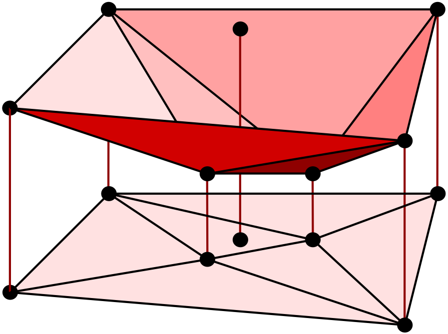
Projected down to $\mathbb{R}^d$. Projected faces form a *subdivision* of A.



If the vector h is "generic" then it forms a triangulation!

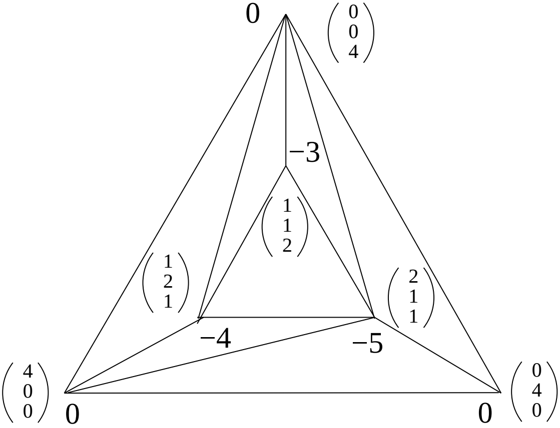


Remark: Different h's may provide different triangulations. But, for some A's, **not all triangulations can be obtained in this way.**



THE Example:

$$h = (\ 0,\ 0,\ 0, -5, -4, -3), \qquad A = \begin{bmatrix} 4 & 0 & 0 & 2 & 1 & 1 \\ 0 & 4 & 0 & 1 & 2 & 1 \\ 0 & 0 & 4 & 1 & 1 & 2 \end{bmatrix},$$



Another example:

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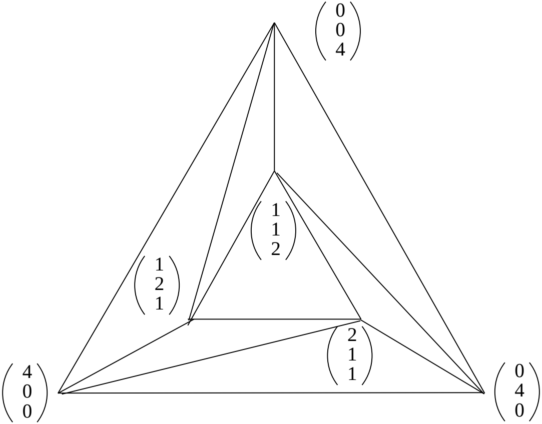
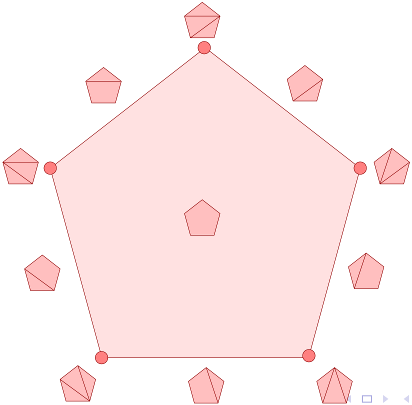


The secondary polytope

Theorem [Gelfand-Kapranov-Zelevinskii, 1990]
The *poset* (*p*artially *o*rdered *set*) of **regular** polyhedral subdivisions of a point configuration A equals the face poset of a certain polytope of dimension $n - d - 1$ ($n =$ number of points, $d =$ dimension).



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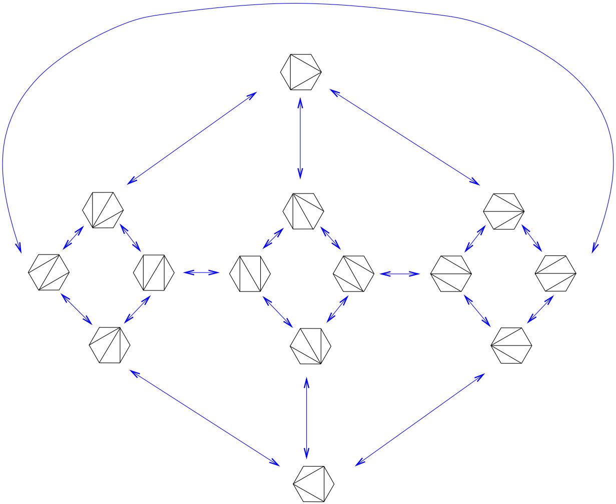
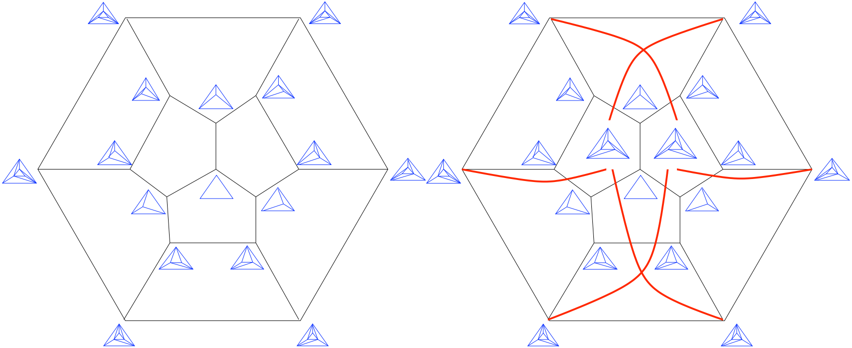


Secondary polytope of with non-regular triangulations



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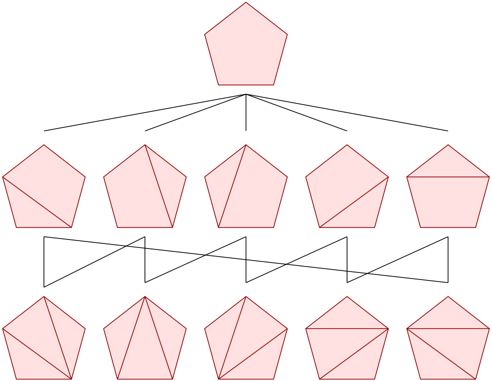
poset of subdivisions of a pentagon



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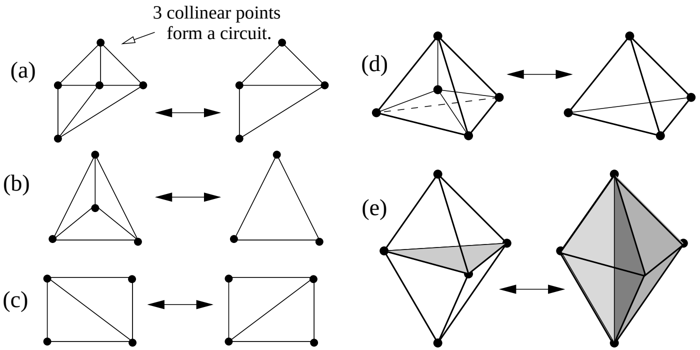

Detect via "minimal affine dependences in the point configuration"

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- In **dimension 6**, there are triangulations with arbitrarily large n and **ZERO flips** *[Santos, 2000].*

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- All edges in the secondary polytope correspond to flips between regular triangulations. **how about the converse??** **Proposition** *[JDL, Santos, Rambau 2010]* **False!** There are regular triangulations connected by flips, yet their flip is not an edge in secondary.

More questions!

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More questions!

- **Observation** Flips can be used to count all **regular** triangulations, but cannot be used to generate all triangulations, lack of connectivity!!
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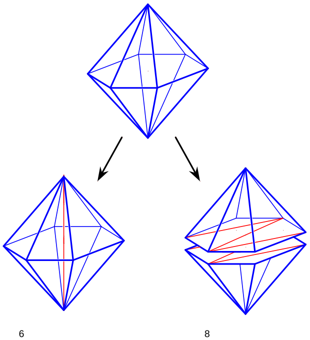
Sizes of Triangulations

Triangulations of d-polytopes come in different sizes!!



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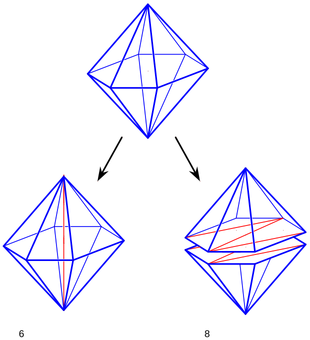

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Remark: Even for the $0/1$ cube, we do not know the smallest size, for dim ≥ 8.

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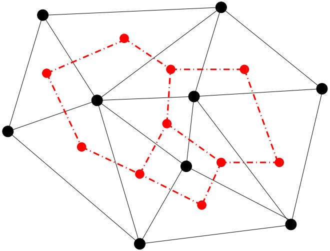
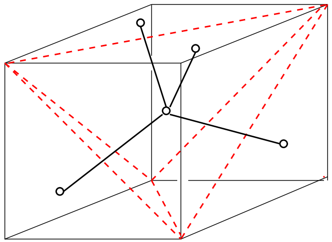
  OPEN PROBLEM What is the complexity of deciding when such triangulations exist?

The Diameter and Hamiltonicity of a triangulation

The dual graph of a triangulation: it has one vertex for each simplex and an edge joining two such vertices if the two simplices share a triangle:





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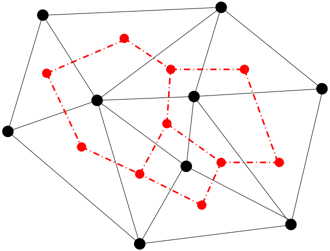
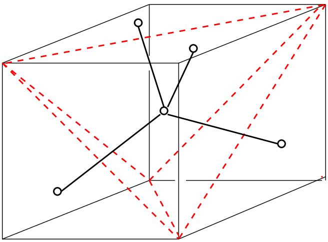


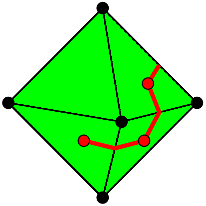


Open Problem Is it true that every 3-dimensional polyhedron has a triangulation whose dual graph is Hamiltonian?

Diameters of Simplicial Complexes

Definition

- The **distance** between two facets, F_1, F_2, is the length k of the shortest simplicial path $F_1 = f_0, f_1, \ldots, f_k = F_2$.
- The **diameter** of a triangulation is the maximum over all distances between all pairs of vertices.



QUESTION: What are the best bounds for the diameter of a triangulation?

A case study of application: Algebraic Geometry

• To every monomial $x_1^{a_1} \dots x_n^{a_n}$ we associate its *exponent vector* $(a_1, \dots, a_n)$.

• To a polynomial $f(x_1, \dots, x_n) = \sum c_i \, \mathbf{x}^{\mathbf{a}_i}$ we associate the corresponding *integer point set*. Its convex hull is the *Newton polytope* of f, $N(f)$.

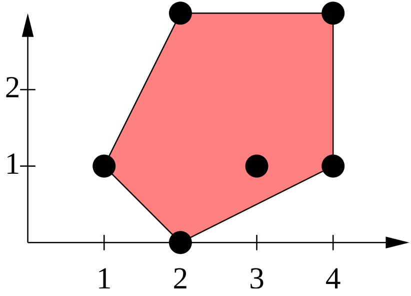
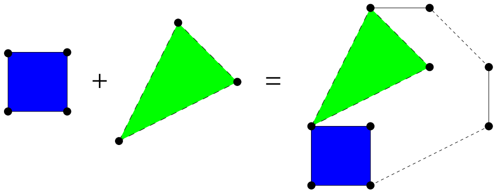


Figure: The Newton polytope for $x^2 + xy + x^3y + x^4y + x^2y^3 + x^4y^3$

Bernstein's Theorem

Theorem (Bernstein, 1975) Let $f_1, \ldots, f_n$ be n polynomials in n variables with "generic" coefficients. The number of common zeroes of them in $(\mathbb{C}^*)^n$ is either infinite or bounded above by the *mixed volume* of the n polytopes $N(f_1), \ldots, N(f_n)$.

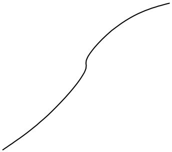


Computing the mixed volume boils down to computing a regular subdivision!!

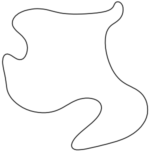
Hilbert's sixteenth problem (1900)

"What are the possible (topological) types of smooth real algebraic curves of a given degree *d*?"

Observation: Each connected component is either a *pseudo-line* or an *oval*. A curve contains one or zero pseudo-lines depending in its parity.



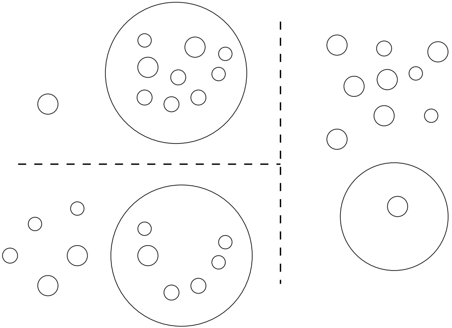
A pseudoline. Its complement has one component, homeomorphic to an open circle. The picture only shows the "affine part"; think the two ends as meeting at infinity.



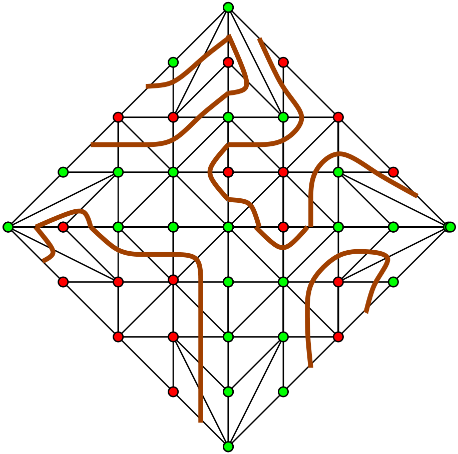
An oval. Its interior is a (topological) circle and its exterior is a Möbius band.

Example:

The classification of *non-singular real algebraic curves of degree six* was completed the 1960's [Gudkov]. There are 56 types degree six curves, only three with 11 ovals:



Viro's Theorem:



Construct the algebraic curve as a simplicial transversal curve of a regular triangulation!!