1 Goermans-Williamson Algorithm for the maxcut problem

This is a probabilistic approach to choosing a CUT

- Step 1: Solve the relax problem. Let the result be $\mathbf{X}$ and it can be factorized as

$$\mathbf{X} = \mathbf{V}^T\mathbf{V} \tag{1}$$

where $\mathbf{V} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}$.

- Step 2: Generate a random uniform vector $\mathbf{r}$ on the unit n-sphere $\mathcal{S}^n$.
- Step 3: A CUT which approximates the MAXCUT is

$$y_i = \text{sign}\left(\mathbf{r}^T\mathbf{v}\right) = \begin{cases} 1 & \text{if } \mathbf{r}^T\mathbf{v} \geq \mathbf{0} \\ -1 & \text{if } \mathbf{r}^T\mathbf{v} < \mathbf{0} \end{cases} \tag{2}$$

Since this CUT is randomly generated, we are interested in its expected value.

Lemma 1

$$Pr\left[\text{sign}\left(\mathbf{r}^T\mathbf{v}_i\right) \neq \text{sign}\left(\mathbf{r}^T\mathbf{v}_j\right)\right] = \frac{\arccos\left(\mathbf{v}_i^T\mathbf{v}_j\right)}{\pi}. \tag{3}$$

Theorem 2 *Expected value of* CUT is not less than .87856 the MAXCUT.

Proof The expected value of CUT is

$$\begin{aligned}
E\left(\text{CUT}\right) &= \frac{1}{2}\sum_{i,j\in E} w_{ij}\text{Pr}\left[1 - y_iy_j = 2\right] \\
&= \frac{1}{2}\sum_{i,j\in E} w_{ij}\text{Pr}\left[\text{sign}\left(\mathbf{r}^T\mathbf{v}_i\right) \neq \text{sign}\left(\mathbf{r}^T\mathbf{v}_j\right)\right] \\
&= \frac{1}{2}\sum_{i,j\in E} w_{ij}\frac{\arccos\left(\mathbf{v}_i^T\mathbf{v}_j\right)}{\pi} \\
&= \frac{1}{2}\sum_{i,j\in E} w_{ij}\frac{\arccos\left(\mathbf{X}_{ij}\right)}{\pi} \\
&= \frac{1}{4}\sum_{i,j\in E} w_{ij}\left(1 - \mathbf{X}_{ij}\right)\frac{2}{\pi}\frac{\arccos\left(\mathbf{X}_{ij}\right)}{\left(1 - \mathbf{X}_{ij}\right)} \\
&\geq \frac{1}{4}\sum_{i,j\in E} w_{ij}\left(1 - \mathbf{X}_{ij}\right)\left(\min_{-1\leq t\leq 1}\frac{2}{\pi}\frac{\arccos\left(t\right)}{\left(1 - t\right)}\right) \\
&= \text{RELAX} \times \left(\min_{0\leq\theta\leq\pi}\frac{2}{\pi}\frac{\theta}{\left(1 - \cos\left(\theta\right)\right)}\right) \\
&\geq 0.87856 \times \text{RELAX}
\end{aligned} \tag{4}$$

■

2 Global optimization using SDP

So far we have seen linear programs and Semidefinite programs are very useful and can be solved efficiently. They will be used quite a bit later. We will consider now situations where

$$\begin{array}{lll} \text{min/max} & f(\mathbf{x}) & \\ \text{subject to} & g_1(\mathbf{x}) = 0 & h_1(\mathbf{x}) \geq 0 \\ & g_2(\mathbf{x}) = 0 & h_2(\mathbf{x}) \geq 0 \\ & \vdots & \vdots \\ & g_k(\mathbf{x}) = 0 & h_s(\mathbf{x}) \geq 0 \end{array} \tag{5}$$

There is no integrality restriction except f, g_i, h_j are all polynomials.

2.1 Univariate polynomial

Let us start understanding such problem for one variable.
Let $f(x)$ be a polynomial of variable x as

$$f(x) = p_n x^n + p_{n-1} x^{n-1} + \cdots p_1 x + p_0 \tag{6}$$

We wish to find the global minimum of $f(x)$

$$\min f(x) \tag{7}$$

This problem is equivalent to find

$$\begin{array}{ll} \max & \gamma \\ \text{s.t.} & f(x) - \gamma \geq 0 \text{ for all } x. \end{array} \tag{8}$$

First note: $f(x) - \gamma$ must be of even degree.

Lemma 3 *A univariate polynomial $p(x)$ is non-negative iff $p(x)$ is a sum of square (SOS).*

Proof It is obvious to see if $p(x)$ is a SOS then $p(x)$ is non-negative.

Now, given non-negative polynomial $p(x)$ of order n and coefficients $p_j, (j = 1..n)$ we proof that $p(x)$ is also SOS.
According to the fundamental theorem of algebra, $p(x)$ can be factorized as

$$p(x) = p_n \prod_{n_j} (x - r_j)^{n_j} \prod_{m_k} (x - a_k + ib_k)^{m_k} (x - a_k - ib_k)^{m_k} \tag{9}$$

where r_j and $a_k \pm ib_k$ are the real and complex roots of $p(x)$, respectively.

Because $p \geq 0$, n_j are even,i.e. $n_j = 2s_j$, and $p_n \geq 0$. Moreover, $(x - a_k + ib_k)(x - a_k - ib_k) = (x - a_k)^2 + b_k^2$.

Therefore

$$p(x) = p_n \prod_{s_j} (x - r_j)^{2s_j} \prod_{m_k} \left((x - a_k)^2 + b_k^2 \right)^{m_k}. \quad (10)$$

The first product of $p(x)$ in (9) is a product of square terms.

Note that

$$(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2, \quad (11)$$

we prove that $p(x)$ is a SOS.■

Corollary 4 *Any non-negative polynomial in $\mathbb{R}$ is a sum of 2 squares.*

$$p(x) = q_1^2(x) + q_2^2(x) \quad (12)$$

The next question is how to check quickly whether $p(x)$ is a SOS.

Theorem 5 *A univariate polynomial $p(x)$ of even degree $2d$ is a SOS iff there exits a positive semidefinite matrix $\mathbf{Q}$ such that*

$$p(x) = \mathbf{x}_d^T \mathbf{Q} \mathbf{x}_d \quad (13)$$

where $\mathbf{x}_d = \begin{bmatrix} 1 & x & \cdots & x^d \end{bmatrix}^T$.

Proof We write $p(x)$ in SOS form

$$p(x) = \sum_{k=1}^{m} q_k^2(x) \quad (14)$$

Define $\mathbf{q} = \begin{bmatrix} q_1(x) & q_2(x) & \cdots & q_m(x) \end{bmatrix}^T$, we have

$$\mathbf{q} = \mathbf{V}\mathbf{x}_d, \quad (15)$$

where k-th row of $\mathbf{V}$ is made of coefficients of $q_m(x)$. Therefore

$$\begin{aligned} p(x) &= \mathbf{q}^T \mathbf{q} \\ &= \mathbf{x}_d^T \mathbf{V}^T \mathbf{V} \mathbf{x}_d \\ &= \mathbf{x}_d^T \mathbf{Q} \mathbf{x}_d \end{aligned} \quad (16)$$

where $\mathbf{Q} = \mathbf{V}^T\mathbf{V}$. By definition $\mathbf{Q} \succeq 0$. ■

The coefficients of $p(x)$ has the following relation with elements of $\mathbf{Q}$

$$p_i = \sum_{j+k=i} Q_{jk} \quad (17)$$

Proof We have

$$\begin{aligned} p(x) &= \mathbf{x}_d^T \mathbf{Q} \mathbf{x}_d \\ &= \sum_{j=0}^{d} \sum_{j=0}^{d} Q_{jk} x^{j+k} \\ &= \sum_{i=0}^{2d} \left(\sum_{j+k=i} Q_{jk} \right) x^i \\ &= \sum_{i=0}^{2d} p_i x^i \end{aligned} \tag{18}$$

■ Conclusion: SDP can solve (7) globally.

2.2 Multivariate polynomial

We wish to find a global minimum of a multivariate polynomial $f(\mathbf{x})$

$$\begin{aligned} &\min f(\mathbf{x}) \\ &\mathbf{x} \in \mathbb{R}^n \end{aligned} \tag{19}$$

This problem is equivalent to find

$$\begin{aligned} \max \quad & \gamma \\ \text{subject to} \quad & f(\mathbf{x}) - \gamma \geq 0 \text{ for all } \mathbf{x} \in \mathbb{R}^n \end{aligned} \tag{20}$$

The question is whether we can apply the same procedure as in the univariate polynomial case. The answer is no because we do not know if $f(\mathbf{x})$ is a SOS for $\mathbf{x} \in \mathbb{R}^{\ltimes}$. In fact Motzkin and Robinson (1950) has shown that the polynomial $x^4y^2 + x^2y^4 + 1 - 3x^2y^2$ can not be written as a SOS.

3 Motzkin function

The multivariate polynomial $f(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$ of even degree $\deg(f(\mathbf{x})) = 2d$ is a non-negative polynomial if and only if it satisfies:

$$f(\mathbf{x}) \geq 0, \forall \mathbf{x} \in \mathbb{R}^n. \tag{21}$$

Clearly, a sufficient condition in order for $f(\mathbf{x})$ to be non-negative is:

$$f(\mathbf{x}) = \sum_i q_i^2(\mathbf{x}), \tag{22}$$

that is, if $f(\mathbf{x})$ is reducible to a sum of squares (SOS), it is a non-negative polynomial. In general, condition (22) is only sufficient. Hilbert proved that it becomes also necessary if:

$$n = 1. \tag{23}$$

$$d = 1. \tag{24}$$

$$n = 2 \text{ and } d = 2. \tag{25}$$

Motzkin showed an example of a non-negative polynomial with $n = 2$ and $d = 3$ that cannot be reduced to a sum of squares:

$$f(x, y) = 1 + x^4y^2 + x^2y^4 - 3x^2y^2. \tag{26}$$

It is easy to show that $\forall(x, y) \in \mathbb{R}^2 \; f(x, y) \geq 0$:
Proof

$$1 + x^4y^2 + x^2y^4 \quad \geq \quad 3x^2y^2 \tag{27}$$

true, because:

$$\frac{a_1 + \ldots + a_n}{n} \geq \sqrt[n]{a_1 \cdot a_2 \cdot \ldots a_n}. \tag{28}$$

and:

$$1 + x^4y^2 + x^2y^4 \geq 3\sqrt[3]{x^4y^2(x^2y^4)} = 3x^2y^2. \tag{29}$$

■

The proof that the non-negative polynomial $f(x, y)$ cannot be reduced to a sum of squares is as follows:
Proof
Suppose:

$$f = \sum s_i^2, s_i \in \mathbb{R}[x, y]. \tag{30}$$

We note that each polynomial s_i must have Newton polytope contained in the triangle $(0, 0), (1, 2), (2, 1)$, therefore s_i is a linear combination of $1, xy^2, x^2y, xy$, but s_i^2 contains x^2y^2 and it must appear with a positive coefficient ($s_i = a + bx^2y + cxy^2 + dxy$ implies that x^2y^2 has coefficient $d^2 \geq 0$), but this is in contradiction with the fact that x^2y^2 appears with coefficient -3 in $f(x, y)$. ■

Lemma 6 *If $p(\mathbf{x}) = \sum q_i^2(\mathbf{x})$, then Newton($q_i$)$\subseteq \frac{1}{2}$Newton($p(\mathbf{x})$).*

This reduces the size of the compounding PSD matrix $\mathbf{Q}$. In fact, recall that semi-definite programming can be used to prove whether a given polynomial is a sum of squares. A polynomial $f(\mathbf{x}) = p_\alpha \mathbf{x}^\alpha$ can be written as:

$$f(\mathbf{x}) = [\mathbf{x}]_d^T \mathbf{Q} [\mathbf{x}]_d \tag{31}$$

with $[\mathbf{x}]_d$ vector of all monomials of n variables and degree $\deg([x_i]_d) \leq d$, if:

$$\mathbf{Q} \succeq 0, \tag{32}$$

$$p_\alpha = \sum_{\alpha = \beta + \delta} Q_{\beta\delta}. \tag{33}$$

3.1 Example

$$f(x, y, z, \omega) = (x^4 + 1)(y^4 + 1)(z^4 + 1)(\omega^4 + 1) + 3x + 4y + 5z + 2\omega. \tag{34}$$

$\deg(f(x, y, z, \omega)) = 16$, the total number of variables necessary to solve this problem is:

$$\begin{pmatrix} 4 + 8 \\ 8 \end{pmatrix} = 495. \tag{35}$$

Such a large number of variables is not efficiently handled on most of current computers. Recalling that $Newton(f(\mathbf{x}))$ is a cube, we can solve the problem using only $3^4 = 81$ monomials, which is duable on most of current computers.

4 Optimization problem

We wish to:

$$\begin{array}{rll} \text{maximize:} & f(\mathbf{x}) & \\ \text{subject to:} & g_1(\mathbf{x}) \geq 0 & h_1(\mathbf{x}) = 0, \\ & \vdots & \vdots \\ & g_m(\mathbf{x}) \geq 0 & h_k(\mathbf{x}) = 0. \end{array} \tag{36}$$

where functions $f(\cdot), g_j(\cdot)_{(j\in\{1,\dots,m\})}, h_i(\cdot)_{(i\in\{1,\dots,k\})}$ are polynomials with coefficients in $\mathbb{R}$. In order to tackle the solution of problem 36, let us consider the set of all polynomials in n variables with coefficients on a field $K \in \{\mathbb{C}, \mathbb{R}, \mathbb{Z}_p\}$, $K[x_1, \dots, x_n] = K[\mathbf{x}]$ and define an "ideal" $I \subseteq K[\mathbf{x}x]$ such that:

(a) if $f_1, f_2 \in I \rightarrow f_1 + f_2 \in I$,

(b) if $f \in I, g \in k[\mathbf{x}] \rightarrow f \cdot g \in I$.

Examples:

(1) Let $S \subseteq \mathbb{C}^n$, then:

$$I(S) = \{f(\mathbf{x}) | f(s) = 0, \forall s \in S\}. \tag{37}$$

$I(S)$ is defined as the vanishing ideal.

(2) Let $g_1, g_2, \dots, g_s$ be a series of polynomials; the ideal generated by $g_1, \dots, g_s$ is defined as:

$$< g_1, \dots, g_s >= \{f | f = \lambda_1 g_1 + \lambda_2 g_2 + \dots + \lambda_s g_s, \text{ with } \lambda_i \in K[\mathbf{x}]\}. \tag{38}$$

Exercise: show that in $\mathbb{C}[x_1]$ every ideal is principal. Theorem: every polynomial ideal is finitely generated (HILBERT basis theorem). Given a set of polynomials $h_1, \dots, h_k$ in $K[x_1, \dots, x_n]$, we define a variety as:

$$V(h_1, h_2, \dots, h_k) = \{s \in K^n | h_i(s) = 0, \ \forall i = 1, 2, \dots, k\}. \tag{39}$$

Example: $V_{\mathbb{R}^2}(x^2y^2)$ is the set of the x and y axis.

We define a basic semi-algebraic set as a subset of $\mathbb{R}^n$ that satisfies:

$$\{\mathbf{x} \in \mathbb{R}^n | g_1(x) \geq 0, \dots, g_m(x) \geq 0, \ h_1(x) \geq 0, \dots, h_k(x) \geq 0\}. \tag{40}$$

Examples:

(1) $x \in [0, 1]$ is a semi-algebraic set of $\mathbb{R}$ that can be written as:

$$\{x \in \mathbb{R} | -x + 1 \geq 0, x \geq 0\} \tag{41}$$

(2) The set of all symmetric $n \times n$ matrices that are positive semidefinite is a basic semi-algebraic set: **Proof** $A \succeq 0 \iff$ all 2^{n-1} principal minors of A are non-negative. ■

Exercise: find fewer than 2^{n-1} inequalities that define the same semi-algebraic set.

Given a set of polynomials $g_1, \dots, g_m$ in $K[\mathbf{x}]$, a cone is defined as the set of all linear combination of all products of subsets of g_i:

$$\text{cone}(g_1, \dots, g_m) = \{ \sum_{I \subseteq \{1,\dots,m\}} s_I \pi_{i \in I} g_i | s_I \text{ an SOS}\}. \tag{42}$$