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Polyhedra with the Integer Carathéodory Property

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ABSTRACT

A polyhedron P has the *Integer Carathéodory Property* if the following holds. For any positive integer k and any integer vector $w \in kP$, there exist affinely independent integer vectors $x_1, \dots, x_t \in P$ and positive integers $n_1, \dots, n_t$ such that $n_1 + \cdots + n_t = k$ and $w = n_1 x_1 + \cdots + n_t x_t$.

In this paper we prove that if P is a (poly)matroid base polytope or if P is defined by a totally unimodular matrix, then P and projections of P have the Integer Carathéodory Property. For the matroid base polytope this answers a question by Cunningham from 1984.

1. Introduction

A polyhedron $P \subseteq \mathbb{R}^n$ has the *integer decomposition property*, introduced by Baum and Trotter [1], if for every positive integer k, every integer vector in kP is the sum of k integer vectors in P. Equivalently, every $\frac{1}{k}$-integer vector $x \in P$ is a convex combination

$$x = \lambda_1 x_1 + \cdots + \lambda_t x_t, \quad x_i \in P \cap \mathbb{Z}^n, \ \lambda_i \in \frac{1}{k}\mathbb{Z}. \tag{1}$$

Examples of such polyhedra include: stable set polytopes of perfect graphs, polyhedra defined by totally unimodular matrices and matroid base polytopes.

It is worth pointing out the relation with Hilbert bases. Recall that a finite set of integer vectors H is called a *Hilbert base* if every integer vector in the convex cone generated by H, is an integer sum of elements from H. Hence if P is an integer polytope and $H := \{\binom{1}{x} \mid x \in P \text{ integer}\}$, then P has the integer decomposition property, if and only if H is a Hilbert base.

Let P be a polyhedron with the integer decomposition property. It is natural to ask for the smallest number T, such that we can take $t \leqslant T$ in (1) for every k and every $\frac{1}{k}$-integer vector $x \in P$. We denote

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this number by $\mathrm{cr}(P)$, the *Carathéodory rank* of P. Clearly, if P is a polytope, $\mathrm{cr}(P) \geqslant \dim(P)+1$ holds, since P is not contained in the union of the finitely many affine spaces spanned by at most $\dim(P)$ integer vectors in P.

Cook et al. [5] showed that when H is a Hilbert base generating a pointed cone C of dimension $n \geqslant 1$, every integer vector in C is the integer linear combination of at most $2n-1$ different elements from H. For $n > 1$, this bound was improved to $2n-2$ by Sebő [15]. By the above remark, this implies that $\mathrm{cr}(P) \leqslant 2\dim(P)$ holds for any polytope P of positive dimension.

Bruns et al. [2] gave an example of a Hilbert base H generating a pointed cone C of dimension 6, together with an integer vector in C that cannot be written as a nonnegative integer combination of less than 7 elements from H. Their example yields a 0–1 polytope with the integer decomposition property of dimension 5 but with Carathéodory rank 7, showing that $\mathrm{cr}(P) = \dim(P) + 1$ does not always hold. The vertices of the polytope are given by the columns of the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}. \tag{2}$$

In this paper we prove that if P is a (poly)matroid base polytope or if P is a polyhedron defined by a totally unimodular matrix, then P and projections of P satisfy the inequality $\mathrm{cr}(P) \leqslant \dim(P)+1$. For matroid base polytopes this answers a question of Cunningham [6] whether a sum of bases in a matroid can always be written as a sum using at most n bases, where n is the cardinality of the ground set (see also [15,8]).

In our proof we use the following strengthening of the integer decomposition property, inspired by Carathéodory's theorem from convex geometry. We say that a polyhedron $P \subseteq \mathbb{R}^n$ has the *Integer Carathéodory Property* (notation: ICP) if for every positive integer k and every integer vector $w \in kP$ there exist affinely independent $x_1, \dots, x_t \in P \cap \mathbb{Z}^n$ and $n_1, \dots, n_t \in \mathbb{Z}_{\geqslant 0}$ such that $n_1 + \cdots + n_t = k$ and $w = \sum_i n_i x_i$. Equivalently, the vectors x_i in (1) can be taken to be affinely independent. In particular, if P has the ICP, then $\mathrm{cr}(P) \leqslant \dim P + 1$.

It is implicit in [5,15] that the stable set polytope of a perfect graph has the ICP since a 'greedy' decomposition can be found, where the x_i are in the interior of faces of strictly decreasing dimension, and hence are affinely independent.

The organization of the paper is as follows. In Section 2 we introduce an abstract class of polyhedra and show that they have the ICP.

In Section 3 we apply this result to show that polyhedra defined by (nearly) totally unimodular matrices, and their projections, have the ICP.

Section 4 deals with applications to (poly)matroid base polytopes and the intersections of two gammoid base polytopes, showing that these all have the ICP. We conclude by stating two open problems related to matroid intersection.

2. A class of polyhedra having the ICP

In this section we give a sufficient condition for a polyhedron $P \subseteq \mathbb{R}^n$ to have the Integer Carathéodory Property. This condition is closely related to the *middle integral decomposition condition* introduced by McDiarmid in [11]. First we introduce some notation and definitions.

Throughout this paper we set $\bar{\mathbb{Z}} := \mathbb{Z} \cup \{-\infty, +\infty\}$. For a vector $x \in \mathbb{R}^n$ we denote the i-th entry of x by $x(i)$. Recall that a polyhedron P is called an *integer polyhedron* if every face of P contains an integer vector. The polyhedron P is called *box-integer* if for every pair of vectors $c \leqslant d \in \bar{\mathbb{Z}}^n$, the set $\{x \in P \mid c \leqslant x \leqslant d\}$ is an integer polyhedron.

Let $\mathcal{P}$ be the set of rational polyhedra $P \subseteq \mathbb{R}^n$ (for some n) satisfying the following condition:

$$\text{For any } k \in \mathbb{Z}_{\geqslant 0},\ r \in \{0, \dots, k\} \text{ and } w \in \mathbb{Z}^n \\ \text{the intersection } rP \cap \big(w - (k-r)P\big) \text{ is box-integer.} \tag{3}$$

Theorem 1. *If $P \in \mathcal{P}$, then P has the Integer Carathéodory Property.*

Before we prove this theorem we first need a few results describing some properties of $\mathcal{P}$.

Lemma 2. *Every $P \in \mathcal{P}$ has the integer decomposition property.*

Proof. Let $P \subseteq \mathbb{R}^n$ be in $\mathcal{P}$, let k be a positive integer and let $w \in kP \cap \mathbb{Z}^n$. Note that $P \cap (w-(k-1)P)$ is nonempty since it contains $\frac{1}{k}w = w - \frac{(k-1)w}{k}$.

Since $P \in \mathcal{P}$, the intersection $P \cap (w-(k-1)P)$ is box-integer. Take any integer vector $x_k \in P \cap (w-(k-1)P)$ and note that $w - x_k \in (k-1)P \cap \mathbb{Z}^n$. So by induction we can write $w = (x_1 + \cdots + x_{k-1}) + x_k$ with $x_i \in P \cap \mathbb{Z}^n$ for all i. □

As a consequence of Lemma 2, every $P \in \mathcal{P}$ is an integer polyhedron. Indeed, let F be a face of P and $x \in F$ a rational vector. Take $k \in \mathbb{Z}_{\geqslant 0}$ such that $kx \in \mathbb{Z}^n$. By Lemma 2 we can write $kx = \sum_{i=1}^k x_i$ with $x_i \in P \cap \mathbb{Z}^n$. Clearly, $x_1, \ldots, x_k \in F$, hence F contains an integer vector.

Lemma 3. *The collection $\mathcal{P}$ is closed under taking faces and intersections with a box.*

Proof. First note that if P_1 and P_2 are two polyhedra and $F_i \subseteq P_i$ are faces, then either $F_1 \cap F_2 = \emptyset$ or $F_1 \cap F_2$ is a face of $P_1 \cap P_2$.

Now let $P \in \mathcal{P}$ and let F be a face of P. To see that F satisfies (3), let $k \in \mathbb{Z}_{\geqslant 0}$, $r \in \{0, \ldots, k\}$ and let $w \in \mathbb{Z}^n$. If $rF \cap (w-(k-r)F)$ is empty there is nothing to prove. Otherwise, it is a face of $rP \cap (w-(k-r)P)$ and hence box-integer.

To see the second assertion, let $c \leqslant d \in \bar{\mathbb{Z}}^n$ and consider the polyhedron $P' := \{x \in P \mid c \leqslant x \leqslant d\}$. Let $w \in \mathbb{Z}^n$, $k \in \mathbb{Z}_{\geqslant 0}$ and $r \in \{0, \ldots, k\}$. Note that $rP' \cap (w-(k-r)P')$ is equal to

$$\bigl\{x \in rP \cap \bigl(w-(k-r)P\bigr) \bigm| rc \leqslant x \leqslant rd,\ w-(k-r)d \leqslant x \leqslant w-(k-r)c\bigr\}. \tag{4}$$

Hence $rP' \cap (w-(k-r)P')$ is the intersection of the box-integer polyhedron $rP \cap (w-(k-r)P)$ with a box, which is again box-integer. □

Note that Lemma 3 together with the observation below Lemma 2 imply the following.

Proposition 4. *Every $P \in \mathcal{P}$ is box-integer.*

We are now ready to prove Theorem 1.

Proof of Theorem 1. Let $P \subseteq \mathbb{R}^n$ be a polyhedron in $\mathcal{P}$. The proof is by induction on $\dim(P)$.

The case $\dim(P) = 0$ is clear, so we may assume $\dim(P) \geqslant 1$. Let k be a positive integer and let $w \in kP \cap \mathbb{Z}^n$. We may assume that P is a polytope by replacing P by

$$\left\{x \in P \;\middle|\; \left\lfloor \frac{1}{k} w(i) \right\rfloor \leqslant x(i) \leqslant \left\lceil \frac{1}{k} w(i) \right\rceil \text{ for all } i\right\}. \tag{5}$$

If $w(i)$ is a multiple of k for each $i = 1, \ldots, n$, we write $w = k \cdot \frac{1}{k} w$ and we are done. We may therefore assume that k does not divide $w(n)$ and write $w(n) = kq + r$ with $q \in \mathbb{Z}$ and $r \in \{1, \ldots, k-1\}$.

Note that $w \in k(\{x \in P \mid q \leqslant x(n) \leqslant q+1\})$. So by Lemmas 2 and 3 we can write $w = x_1 + \cdots + x_k$ with $x_i \in P \cap \mathbb{Z}^n$ and with $q \leqslant x_i(n) \leqslant q+1$ for all i. We may assume that $x_i(n) = q+1$ for $i = 1, \ldots, r$ and $x_i(n) = q$ for $i = r+1, \ldots, k$.

We denote $P_1 := \{x \in P \mid x(n) = q+1\}$ and $P_2 := \{x \in P \mid x(n) = q\}$. Set $w' := x_1 + \cdots + x_r$. This gives a decomposition of w into two integer vectors

$$\begin{aligned} &w' \in rP_1, \\ &w - w' = x_{r+1} + \cdots + x_k \in (k-r)P_2. \end{aligned} \tag{6}$$

Define

$$Q := rP_1 \cap \bigl(w-(k-r)P_2\bigr) = \bigl(rP \cap \bigl(w-(k-r)P\bigr)\bigr) \cap \bigl\{x \bigm| x(n) = r(q+1)\bigr\} \tag{7}$$

and note that Q is nonempty as it contains w'. Let $y \in Q$ be an integral vertex. Such a y exists because $rP \cap (w-(k-r)P)$ is box-integer by assumption. Let F_1 be the inclusionwise minimal face of rP_1 containing y and let F_2 be the inclusionwise minimal face of $w-(k-r)P_2$ containing y. Let $H_i = \text{aff.hull}(F_i)$. Then

$$H_1 \cap H_2 = \{y\}. \tag{8}$$

Indeed, every supporting hyperplane of rP_1 containing y should also contain F_1, by minimality of F_1 hence it contains H_1. Similarly, every supporting hyperplane of $w-(k-r)P_2$ containing y also contains H_2. Since y is a vertex of Q, it is the intersection of the supporting hyperplanes of the two polytopes containing y and the claim follows.

Let F_i' be the face of P_i corresponding to F_i $(i=1,2)$. That is: $F_1 = rF_1'$ and $F_2 = w-(k-r)F_2'$. Since $\dim F_i' \leqslant \dim P_i \leqslant \dim P - 1$, we inductively obtain integer decompositions

$$y = m_1x_1 + \cdots + m_sx_s, \qquad w-y = n_1y_1 + \cdots + n_ty_t, \tag{9}$$

where $x_1, \dots, x_s \in F_1'$ are affinely independent integer vectors, $y_1, \dots, y_t \in F_2'$ are affinely independent integer vectors and $m_1 + \cdots + m_s = r$, $n_1 + \cdots + n_t = k-r$.

To complete the proof, we show that $x_1, \dots, x_s, y_1, \dots, y_t$ are affinely independent. Suppose there is an affine dependence

$$\sum_{i=1}^{s} \lambda_i x_i + \sum_{i=1}^{t} \mu_i y_i = 0, \qquad \sum_i \lambda_i + \sum_i \mu_i = 0. \tag{10}$$

We need to show that all λ_i and all μ_i are zero.

By considering the last coordinate, we see that $(q+1)\sum_i \lambda_i + q\sum_i \mu_i = 0$ and hence $\sum_i \lambda_i = \sum_i \mu_i = 0$.

Since $y, rx_1, \dots, rx_s \in F_1$ and $\sum_i \frac{\lambda_i}{r} = 0$, it follows from

$$y + \sum_i \lambda_i x_i = y + \sum_i \frac{\lambda_i}{r}(rx_i) \tag{11}$$

that $y + \sum_i \lambda_i x_i$ is in the affine hull H_1 of F_1. Similarly, $y + \sum_i \lambda_i x_i$ is in the affine hull H_2 of F_2, since $y, w-(k-r)y_1, \dots, w-(k-r)y_t \in F_2$ and

$$y + \sum_i \lambda_i x_i = y - \sum_i \mu_i y_i = y + \sum_i \frac{\mu_i}{k-r}\bigl(w-(k-r)y_i\bigr). \tag{12}$$

It follows by (8) that $y + \sum_i \lambda_i x_i = y$. By affine independence of the x_i, this implies that $\lambda_1 = \cdots = \lambda_s = 0$. Hence $\sum_i \mu_i y_i = 0$, which implies by affine independence of the y_i that $\mu_1 = \cdots = \mu_t = 0$. $\square$

We end this section by showing that projections of polyhedra in $\mathcal{P}$ also have the ICP.

Theorem 5. *Let $m \leqslant n$ and let $\pi : \mathbb{R}^n \to \mathbb{R}^m$ be the projection onto the first m coordinates. If $P \subset \mathbb{R}^n$ and $P \in \mathcal{P}$, then $\pi(P)$ has the ICP.*

Proof. Define $Q := \pi(P)$. Let k be a positive integer and let $w \in kQ \cap \mathbb{Z}^m$. We may assume that P is bounded. Indeed, taking $N \in \mathbb{Z}_{\geqslant 0}$ large enough such that $\pi^{-1}(\{w\}) \cap [-kN, kN]^n$ is not empty. We can replace P by

$$P \cap [-N, N]^n, \tag{13}$$

and replace Q by $\pi(P \cap [-N, N]^n)$.

Let $F \subseteq kP$ be an inclusionwise minimal face intersecting $\pi^{-1}(\{w\})$. Then

$$\pi|_F \text{ is injective.} \tag{14}$$

Indeed, suppose that $\pi(a) = \pi(b)$ for distinct $a, b \in F$. Let $x \in F \cap \pi^{-1}(\{w\})$. Then since F is bounded, the line $x + \mathbb{R}(b - a)$ intersects F in a smaller face, contradicting the minimality of F.

Now note that $F \cap \pi^{-1}(\{w\})$ is the intersection of F with the box

$$\{x \in \mathbb{R}^n \mid x(i) = w(i),\ i = 1, \ldots, m\}. \tag{15}$$

Since $P \in \mathcal{P}$, also $kP \in \mathcal{P}$ and so kP is box-integer by Proposition 4. This in turn implies that F is box-integer. Hence we can lift w to an integer vector $\hat{w} \in F \cap \pi^{-1}(\{w\})$. By Theorem 1 we can find affinely independent integer vectors $x_1, \ldots, x_t$ in $\frac{1}{k}F$ and positive integers $n_1, \ldots, n_t$ such that $n_1 + \cdots + n_t = k$ and

$$\hat{w} = \sum_{i=1}^{t} n_i x_i. \tag{16}$$

Since $\pi|_F$ is injective, $\pi(x_1), \ldots, \pi(x_t)$ are also affinely independent. Hence

$$w = \sum_{i=1}^{t} n_i \pi(x_i) \tag{17}$$

is the desired decomposition of w. □

3. Polyhedra defined by totally unimodular matrices

In this section we prove that polyhedra defined by (nearly) totally unimodular matrices have the ICP. Recall that a matrix A is called *totally unimodular* (notation: TU) if for each square submatrix C of A, $\det(C) \in \{-1, 0, 1\}$. For details on TU matrices we refer to [13].

Theorem 6. *Let $P := \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$, where A is an $m \times n$ TU matrix and $b \in \mathbb{Z}^m$. Then $P \in \mathcal{P}$. In particular, every projection of P has the ICP.*

Proof. Let $k \in \mathbb{Z}_{\geqslant 0}$, $r \in \{0, \ldots, k\}$ and $w \in \mathbb{Z}^n$. Define

$$A' = \begin{bmatrix} A \\ -A \end{bmatrix}, \qquad b' := \begin{bmatrix} rb \\ (k-r)b - Aw \end{bmatrix}. \tag{18}$$

Then $rP \cap (w - (k-r)P) = \{x \mid A'x \leqslant b'\}$. Since A' is totally unimodular and b' is integral, it follows that $\{x \mid A'x \leqslant b'\}$ is a box-integer polyhedron (see for example Chapter 19.1 in [13]). It follows that $P \in \mathcal{P}$.

Theorem 5 now implies that every projection of P has the ICP. □

A consequence of Theorem 6 is that co-flow polyhedra, introduced by Cameron and Edmonds in [3], have the ICP since they are projections of TU polyhedra (see also [16]).

We end this section with an extension of Theorem 6 to so-called nearly totally unimodular matrices. In [10] a matrix A is called *nearly totally unimodular* (notation: NTU) if there exists a TU matrix $\hat{A}$ a row a of $\hat{A}$ and an integer vector c such that $A = \hat{A} + ca^{\mathsf{T}}$. For an $m \times n$ NTU matrix A and an integer vector b the integer polyhedron $P_{A,b}$ is defined by

$$P_{A,b} := \text{conv.hull}\big(\{x \in \mathbb{Z}^n \mid Ax \leqslant b\}\big). \tag{19}$$

Note that

$$P_{A,b} = \text{conv.hull}\left(\bigcup_{s\in\mathbb{Z}}\{y \mid \hat{A}y \leqslant b - sc,\ a^{\mathsf{T}}y = s\}\right). \tag{20}$$

In order to show $P_{A,b}$ has the ICP, we will use the following theorem from [10].

Theorem 7. *Let $\hat{A}$ be an $m \times n$ TU matrix let a be a row of $\hat{A}$, let $b, c \in \mathbb{Z}^m$ and define $A := \hat{A} + ca^{\mathsf{T}}$. Let k be a nonnegative integer and let $w \in \mathbb{Z}^n$. Write $a^{\mathsf{T}}w = qk + r$ with $q, r \in \mathbb{Z}$ and with $0 \leqslant r \leqslant k-1$. Equivalent are*:

(i) $w \in kP_{A,b}$;
(ii) *the system*

$$\begin{aligned} &\hat{A}y \leqslant r(b - (q+1)c), \\ &\hat{A}y \geqslant \hat{A}w + (k-r)(qc - b), \\ &a^{\mathsf{T}}y = r(q+1) \end{aligned} \tag{21}$$

is feasible.

Theorem 8. *Let $\hat{A}$ be an $m \times n$ TU matrix let a be a row of $\hat{A}$, let $b, c \in \mathbb{Z}^m$ and define $A := \hat{A} + ca^{\mathsf{T}}$. Then $P_{A,b}$ has the ICP.*

Proof. Let k be a positive integer and let $w \in kP_{A,b}$. Write $a^{\mathsf{T}}w = qk + r$ with $q, r \in \mathbb{Z}$ and with $0 \leqslant r \leqslant k-1$. We may assume that $P_{A,b}$ is bounded. Indeed, by Theorem 7 we can take a solution y of (21) and let $l, u \in \mathbb{Z}^n$ be such that $rl \leqslant y \leqslant ru$ and $w - (k-r)u \leqslant y \leqslant w - (k-r)l$. Define the NTU matrix A' and the integer vector b' by

$$A' := \begin{bmatrix} A \\ I \\ -I \end{bmatrix}, \qquad b' := \begin{bmatrix} b \\ u \\ -l \end{bmatrix}. \tag{22}$$

By Theorem 7 it follows that $w \in kP_{A',b'}$. Since $P_{A',b'} \subseteq P_{A,b}$ is bounded, we can replace $P_{A,b}$ by $P_{A',b'}$.

Define polyhedra $P_i \subseteq P_{A,b}$ by

$$\begin{aligned} P_1 &:= \{y \in \mathbb{R}^n \mid \hat{A}y \leqslant b - (q+1)c,\ a^{\mathsf{T}}y = q+1\}, \\ P_2 &:= \{y \in \mathbb{R}^n \mid \hat{A}y \leqslant b - qc,\ a^{\mathsf{T}}y = q\}. \end{aligned} \tag{23}$$

If $r = 0$ then $w \in kP_2$ and then the claim follows directly from Theorem 6. So we may assume $r > 0$.

Note that the polyhedron defined by (21) is equal to $rP_1 \cap (w - (k-r)P_2)$ and is nonempty by Theorem 7. Let y be a vertex of $rP_1 \cap (w - (k-r)P_2)$. Then y is an integral vector because (21) is defined by a TU matrix. Let $F_1 \subseteq rP_1$ and $F_2 \subseteq w - (k-r)P_2$ be the inclusionwise minimal faces containing y.

So we now have a decomposition of $w = y + (w - y)$ with $y \in F_1$ and $w - y \in w - F_2$. Since P_1 and P_2 are polytopes defined by TU matrices, Theorem 6 implies that we can find a positive integer decomposition of y into affinely independent integer vectors $x_1, \ldots, x_t$ from $\frac{1}{r}F_1$ and of $w - y$ into affinely independent integer vectors $y_1, \ldots, y_s$ from $\frac{1}{k-r}(w - F_2)$.

Completely similar to the proof of Theorem 1 it follows that $x_1, \ldots, x_t, y_1, \ldots, y_s$ are affinely independent. Hence combining the decompositions for y and $w - y$ gives the desired decomposition for w. □

Interestingly enough, not every polytope defined by an NTU matrix is contained in $\mathcal{P}$. Consider the following example. Let $P := \{x \in \mathbb{R}^2 \mid x \geqslant 0,\ x_1 + 2x_2 \leqslant 2\}$. This is an integer polytope, but not box-integer (take intersection with $x_1 \leqslant 1$). But P is defined by an NTU matrix. Namely, define

$$A := \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 2 \end{bmatrix}, \qquad b := \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \tag{24}$$

then $P = P_{A,b}$.

This shows that there exist polytopes having the ICP, which are not projections of polytopes in $\mathcal{P}$, as box-integrality is maintained under projections.

4. The (poly)matroid base polytope

In his paper on testing membership in matroid polyhedra, Cunningham [6] asked for an upper bound on the number of different bases needed in a representation of a vector as a nonnegative integer sum of bases. It follows from Edmonds' matroid partitioning theorem [7] that the incidence vectors of matroid bases form a Hilbert base for the pointed cone they generate. Hence denoting by n the size of the ground set of the matroid, the upper bound of $2n-2$ applies by Sebő [15]. This bound was improved by de Pina and Soares [12] to $n+r-1$, where r is the rank of the matroid. Chaourar [4] showed that an upper bound of n holds for a certain minor closed class of matroids.

In this section we show that the (poly)matroid base polytope has the ICP. This in particularly implies that the upper bound of n holds for all matroids. Furthermore, we show that the intersection of any two gammoid base polytopes has the ICP.

First we introduce the basic notions concerning submodular functions. For background and more details, we refer the reader to [9,14].

Let E be a finite set and denote its power set by 2^E. For any $x : E \to \mathbb{R}$, and any $U \subseteq E$ we write $x(U) := \sum_{i \in U} x(i)$.

A function $f : 2^E \to \mathbb{Z}$ is called *submodular* if for any $A, B \subseteq E$ the inequality $f(A) + f(B) \geqslant f(A \cup B) + f(A \cap B)$ holds. A function $g : 2^E \to \mathbb{Z}$ is called *supermodular* if $-g$ is submodular.

Consider the following polyhedra

$$\begin{aligned} EP_f &:= \{x \in \mathbb{R}^E \mid x(U) \leqslant f(U) \text{ for all } U \subseteq E\}, \\ P_f &:= \{x \in EP_f \mid x(U) \geqslant 0 \text{ for all } U \subseteq E\}, \\ B_f &:= \{x \in EP_f \mid x(E) = f(E)\}. \end{aligned} \tag{25}$$

The polyhedron EP_f is called the *extended polymatroid* associated to f, P_f is called the *polymatroid* associated to f and B_f is called the *base polytope* of f. Observe that B_f is indeed a polytope, since for $x \in B_f$ and $e \in E$, the inequalities $f(E) - f(E - e) \leqslant x(e) \leqslant f(\{e\})$ hold, showing that B_f is bounded.

A submodular function $f : 2^E \to \mathbb{Z}$ is the rank function of a matroid M on E if and only if f is nonnegative, nondecreasing and $f(U) \leqslant |U|$ for every set $U \subseteq E$. In that case, B_f is the convex hull of the incidence vectors of the bases of M.

Our main tool for proving that EP_f has the ICP is the following result from [14], which is similar to Edmonds' polymatroid intersection theorem [7].

Theorem 9. *Let $f, g : 2^E \to \mathbb{Z}$ be two set functions. If f is submodular and g is supermodular, then*

$$\{x \in \mathbb{R}^E \mid g(U) \leqslant x(U) \leqslant f(U), \text{ for all } U \subseteq E\} \tag{26}$$

is box-integer.

Theorem 9 implies that the extended polymatroid is an element of $\mathcal{P}$ and hence has the ICP.

Theorem 10. *Let E be a finite set and let $f : 2^E \to \mathbb{Z}$ be a submodular function, then $EP_f, P_f, B_f \in \mathcal{P}$. In particular, each of these polyhedra and their projections have the ICP.*

Proof. By Theorem 5, it suffices to prove the first part of the theorem. Furthermore, since B_f is a face of EP_f and P_f is the intersection of EP_f with a box, it suffices by Lemma 3 to prove that $EP_f \in \mathcal{P}$.

Let $k \in \mathbb{Z}_{\geqslant 0}$, $r \in \{0, \dots, r\}$ and $w \in \mathbb{Z}^E$. First note that $rEP_f = EP_{rf}$, with rf submodular again. Secondly, let $g := -(k-r)f + w$ and note that g is supermodular. Observe that for $x \in \mathbb{R}^E$ we have $x \in w - (k-r)EP_f$ if and only if $x(U) \geqslant g(U)$ for all $U \subseteq E$. Hence $rEP_f \cap (w - (k-r)EP_f)$ is box-integer by Theorem 9. So indeed, $EP_f \in \mathcal{P}$. □

Note that Theorem 10 implies that generalized polymatroid base polytopes also have the ICP, as they are projections of base polytopes of polymatroids. See [9] for more details on generalized polymatroids.

Below we show that if P is the intersection of two base polytopes of gammoids, then P has the ICP.

Given a digraph $D = (V, A)$ and subsets U, S of V, one can define a matroid on the set S as follows. A subset $I \subseteq S$ is independent if there exists $I' \subseteq U$ with $|I| = |I'|$ and if there are $|I|$ vertex-disjoint (directed) paths from I' to I. A matroid isomorphic to a matroid defined in this way is called a *gammoid*. Equivalently, gammoids are restrictions of duals of transversal matroids. See [14] for more details on gammoids. We have the following theorem.

Theorem 11. *Let P_1 and P_2 be the base polytopes of two gammoids M_1 and M_2 of rank k defined on the same ground set S. Let $P := P_1 \cap P_2$, then P has the ICP.*

Proof. For $i = 1, 2$, let M_i be associated to digraph $D_i = (V_i, A_i)$ induced by sets U_i, S_i. We may assume that V_1 and V_2 are disjoint. We may further assume that $S = S_1$ and denote by $\varphi : S_1 \to S_2$ the bijection corresponding to the identification of S_2 and S.

We define a new digraph by glueing D_1 to the reverse of D_2 using the bijection φ and splitting each node v into a source node v^{out} and a sink node v^{in}. More precisely, define the digraph $D = (V, A)$ as follows.

$$\begin{aligned} V &:= \{v^{\mathrm{in}}, v^{\mathrm{out}} \mid v \in V_1 \cup V_2\}, \\ A &:= \{(v^{\mathrm{in}}, v^{\mathrm{out}}) \mid v \in V_1 \cup V_2\} \cup \{(s^{\mathrm{out}}, \varphi(s)^{\mathrm{in}}) \mid s \in S\} \\ &\quad \cup \{(u^{\mathrm{out}}, v^{\mathrm{in}}) \mid (u, v) \in A_1\} \cup \{(u^{\mathrm{out}}, v^{\mathrm{in}}) \mid (v, u) \in A_2\}. \end{aligned} \tag{27}$$

Identifying each element $s \in S$ with the corresponding arc $(s^{\mathrm{out}}, \varphi(s)^{\mathrm{in}})$, we have

$$\begin{aligned} &I \subset S \text{ is a common base of } M_1 \text{ and } M_2 \text{ if and only if there} \\ &\text{exists } k \text{ arc disjoint paths from } U_1^{\mathrm{in}} \text{ to } U_2^{\mathrm{out}} \text{ in } D \text{ passing through } I. \end{aligned} \tag{28}$$

Extend D with two extra vertices r (source) and s (sink), and arcs (r, u^{in}) for each $u \in U_1$, arcs (u^{out}, s) for each $u \in U_2$ and finally the arc (s, r). Let X be the incidence matrix of the resulting digraph $D' = (V', A')$. Define the flow polytope

$$Q := \{f \in \mathbb{R}^{A'} \mid Xf = 0,\ 0 \leqslant f(a) \leqslant 1,\ \forall a \in A' \setminus \{(s, r)\},\ f((s, r)) = k\}. \tag{29}$$

Since X is totally unimodular and $\mathcal{P}$ is closed under intersection with a box, Q belongs to $\mathcal{P}$ by Theorem 6. As $P = P_1 \cap P_2$ is the projection of Q onto the coordinates indexed by S, we conclude that P has the ICP. □

We end this section with some (open) questions concerning possible extensions of Theorem 11.

Gammoids form a subclass of so-called strongly base orderable matroids. It is known that for any two strongly base orderable matroids, the common base polytope has the integer decomposition property (see [14]).

Question 1. Does the intersection of two base polytopes of strongly base orderable matroids have the ICP?

In [15] Sebő asks whether the Carathéodory rank of the r-arborescence polytope can be bounded by the cardinality of the ground set. An r-arborescence is a common base of a partition matroid and a graphic matroid.

Question 2. Does the r-arborescence polytope have the ICP?

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