A GENERALIZATION OF BROUWER'S FIXED POINT THEOREM

BY SHIZUO IAKUTANI

The purpose of the present paper is to give a generalization of Brouwer's fixed point theorem (see [1]), and to show that this generalized theorem implies the theorems of J. von Neumann ([2], [3]) obtained by him in connection with the theory of games and mathematical economics.

1. The fixed point theorem of Brouwer reads as follows: if x \rightarrow \varphi(x) is a continuous point-to-point mapping of an r-dimensional closed simplex S into itself, then there exists an x_0 \in S such that x_0 = \varphi(x_0).

This theorem can be generalized in the following way: Let \mathcal{R}(S) be the family of all closed convex subsets of S. A point-to-set mapping x \rightarrow \Phi(x) \in \mathcal{R}(S) of S into \mathcal{R}(S) is called upper semi-continuous if x_\rightarrow \rightarrow \rightarrow x_0, y_n \in \Phi(x_n) and y_n \rightarrow y_0 imply y_0 \in \Phi(x_0). It is easy to see that this condition is equivalent to saying that the graph of \Phi(x): \sum_{x \in S} x \times \Phi(x) is a closed subset of S \times S, where \times denotes a Cartesian product. Then the generalized fixed point theorem may be stated as follows:

**Theorem 1.** If x \rightarrow \Phi(x) is an upper semi-continuous point-to-set mapping of an r-dimensional closed simplex S into \mathcal{R}(S), then there exists an x_0 \in S such that x_0 = \Phi(x_0).

**Proof.** Let S^{(n)} be the n-th barycentric simplicial subdivision of S. For each vertex x^n of S^{(n)} take an arbitrary point y^n from \Phi(x^n). Then the mapping x^n \rightarrow y^n thus defined on all vertices of S^{(n)} will define, if it is extended linearly inside each simplex of S^{(n)}, a continuous point-to-point mapping x \rightarrow \varphi_n(x) of S into itself. Consequently, by Brouwer's fixed point theorem, there exists an x_n \in S such that x_n = \varphi_n(x_n). If we now take a subsequence \{x_{n_1}, x_{n_2}, \ldots\} of \{x_n\} (n = 1, 2, \ldots) which converges to a point x_0 \in S, then this x_0 is a required point.

In order to prove this, let \Delta_n be an r-dimensional simplex of S^{(n)} which contains the point x_n. (If x_n lies on the lower-dimensional simplex of S^{(n)}, then \Delta_n is not uniquely determined. In this case, let \Delta_n be any one of these simplexes.) Let x_0^n, x_1^n, \ldots, x_r^n be the vertices of \Delta_n. Then it is clear that the sequence \{x_i^n\} (i = 0, 1, \ldots, r) converges to x_0 for i = 0, 1, \ldots, r, and we have x_n = \sum_{i=0}^{r} \lambda_i^n x_i^n for suitable \{\lambda_i^n\} (i = 0, 1, \ldots, r; n = 1, 2, \ldots) with \lambda_i^n \geq 0 and \sum_{i=0}^{r} \lambda_i^n = 1. Let us further put y_i^n = \varphi_n(x_i^n) (i = 0, 1, \ldots, r);

Received January 21, 1941.
n = 1, 2, \ldots). Then we have $y_i^n \in \Phi(x_i^n)$ and $x_n = \varphi_n(x_n) = \sum_{i=0}^{r} \lambda_i^n y_i^n$ for $n = 1, 2, \ldots$. Let us now take a further subsequence $\{n_k\}$ ($k = 1, 2, \ldots$) of $\{n_n\}$ ($n = 1, 2, \ldots$) such that $\{y_i^{n_k}\}$ and $\{\lambda_i^{n_k}\}$ converge for $i = 0, 1, \ldots, r$, and let us put $\lim_{k \to \infty} y_i^{n_k} = y_i^0$ and $\lim_{k \to \infty} \lambda_i^{n_k} = \lambda_i^0$ for $i = 0, 1, \ldots, r$.

Then we have clearly $\lambda_0^0 \geq 0$, $\sum_{i=0}^{n} \lambda_i^0 = 1$ and $x_0 = \sum_{i=0}^{r} \lambda_i^0 y_i^0$. Since $x_i^{n_k} \to x_0$, $y_i^{n_k} \in \Phi(x_i^{n_k})$ and $y_i^{n_k} \to y_i^0$ for $i = 0, 1, \ldots, r$, we must have, by the upper semi-continuity of $\Phi(x)$, $y_i^0 \in \Phi(x_0)$ for $i = 0, 1, \ldots, r$, and this implies, by the convexity of $\Phi(x_0)$, that $x_0 = \sum_{i=0}^{r} \lambda_i^0 y_i^0 \in \Phi(x_0)$. Thus the proof of Theorem 1 is completed.

**Remark.** It is easy to see that Brouwer's fixed point theorem is a special case of Theorem 1 when each $\Phi(x)$ consists only of one point $\varphi(x)$. In this case, the upper semi-continuity of $\Phi(x)$ is nothing but the continuity of $\varphi(x)$.

As an immediate consequence of Theorem 1 we have

**Corollary.** Theorem 1 is also valid even if $S$ is an arbitrary bounded closed convex set in a Euclidean space.

**Proof.** Take a closed simplex $S'$ which contains $S$ as a subset, and consider a continuous retracting point-to-set mapping $x \mapsto \psi(x)$ of $S'$ onto $S$. ($\psi(x) = x$ for any $x \in S$ and $\psi(x) \in S$ for any $x \in S'$.) Then $x \mapsto \Phi(\psi(x))$ is clearly an upper semi-continuous point-to-set mapping of $S'$ into $\Phi(S)$. Hence, by Theorem 1, there exists an $x_0 \in S'$ such that $x_0 \in \Phi(\psi(x_0))$. Since $\Phi(\psi(x_0)) \subseteq S$, we must have $x_0 \in S$ and consequently, by the retracting property of $\psi(x)$, $x_0 \in \Phi(x_0) \subseteq S$. This completes the proof of the corollary.

2. **Theorem 2.** Let $K$ and $L$ be two bounded closed convex sets in the Euclidean spaces $R^m$ and $R^n$ respectively, and let us consider their Cartesian product $K \times L$ in $R^{m+n}$. Let $U$ and $V$ be two closed subsets of $K \times L$ such that for any $x_0 \in K$ the set $U_{x_0}$ of all $y \in L$ such that $(x_0, y) \in U$, is non-empty, closed and convex, and such that for any $y_0 \in L$ the set $V_{y_0}$, of all $x \in K$ such that $(x, y_0) \in V$, is non-empty, closed and convex. Under these assumptions, $U$ and $V$ have a common point.

**Proof.** Put $S = K \times L$, and let us define a point-to-set mapping $z \mapsto \Phi(z)$ of $S$ into $\Phi(S)$ as follows: $\Phi(z) = V_y \times U_x$ if $z = (x, y)$. Since $U$ and $V$ are both closed by assumption, $\Phi(z)$ is clearly upper semi-continuous. Hence, by the corollary of Theorem 1, there exists a point $z_0 \in K \times L$ such that $z_0 \in \Phi(z_0)$. In other words, there exists a pair of points $x_0$ and $y_0$, $x_0 \in K$, $y_0 \in L$ such that $(x_0, y_0) \in V_{y_0} \times U_{x_0}$ or equivalently, $x_0 \in V_{y_0}$ and $y_0 \in U_{x_0}$. This means that $z_0 = (x_0, y_0) \in U \cdot V$, and the proof of Theorem 2 is completed.

**Remark:** Theorem 2 is due to J. von Neumann [3], who proved this by using a notion of integral in Euclidean spaces. The proof given above is simpler.
This theorem has applications to the problems of mathematical economics as was shown by J. von Neumann.

**Theorem 3.** Let \( f(x, y) \) be a continuous real-valued function defined for \( x \in K \) and \( y \in L \), where \( K \) and \( L \) are arbitrary bounded closed convex sets in two Euclidean spaces \( \mathbb{R}^m \) and \( \mathbb{R}^n \). If for every \( x_0 \in K \) and for every real number \( \alpha \), the set of all \( y \in L \) such that \( f(x_0, y) \leq \alpha \) is convex, and if for every \( y_0 \in L \) and for every real number \( \beta \), the set of all \( x \in K \) such that \( f(x, y_0) \geq \beta \) is convex, then we have

\[
\max_{x \in K} \min_{y \in L} f(x, y) = \min_{y \in L} \max_{x \in K} f(x, y).
\]

**Proof.** Let \( U \) and \( V \) be the sets of all \( z_0 = (x_0, y_0) \in K \times L \) such that \( f(x_0, y_0) = \min_{y \in L} f(x_0, y) \) and \( f(x_0, y_0) = \max_{x \in K} f(x, y_0) \) respectively. Then it is easy to see that both \( U \) and \( V \) satisfy the conditions of Theorem 2. Hence, by Theorem 2, there exists a point \( z_0 = (x_0, y_0) \in K \times L \) such that \( z_0 \in U \cap V \) or equivalently, \( f(x_0, y_0) = \min_{y \in L} f(x_0, y) = \max_{x \in K} f(x, y_0) \). Consequently, we have \( \min_{y \in L} \max_{x \in K} f(x, y) \leq \max_{y \in L} f(x_0, y) = \min_{x \in K} f(x_0, y) \leq \max_{y \in L} \min_{x \in K} f(x, y) \).

Since it is clear that we have \( \min_{y \in L} \max_{x \in K} f(x, y) \geq \max_{y \in L} \min_{x \in K} f(x, y) \), the proof of Theorem 3 is completed.

**Remark.** Theorem 3 is one of the fundamental theorems in the theory of games developed by J. von Neumann [2].

In concluding this paper I should like to express my hearty thanks to Dr. A. D. Wallace for his kind discussions on this problem. He has also obtained analogous results for trees. (A. D. Wallace [4].)

**References**