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SOME APPLICATIONS OF A THEOREM ON CONVEX FUNCTIONS

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§1. Introduction

This note presents several applications of the theory developed elsewhere by the authors and H. F. Bohnenblust [1]. The results established here depend upon a fundamental theorem on convex functions, previously used in relation to the Theory of Games. Certain extensions of Helly's theorem (§2), approximation and fitting results (§3), and covering theorems for the n dimensional unit sphere (§4) are obtained. All these are intrinsically connected with one another. The authors believe they possess independent interest.

§2. Convex sets

For later use, we state the theorem referred to above ([1], Theorem 1):

THEOREM 1. *If A is a convex compact set lying in n dimensional space, and if $\mathfrak{A} = \{\phi_\alpha\}$ is a family of continuous convex functions defined over A, with*

$$\inf_{x \in A} \sup_{\alpha} \phi_\alpha(x) > 0,$$

then there exists a convex combination of at most $n + 1$ of the functions which is positive over A. That is, there exist $\phi_i \in \mathfrak{A}$ and $\xi_i \geqq 0, i = 1, \cdots, n+1$, with $\sum_1^{n+1} \xi_i = 1$ and

$$\inf \sum_{i=1}^{n+1} \xi_i \varphi_i(x) > 0.$$

First, this can be used to give a simple proof of the well known theorem of Helly on the intersection of convex sets:

LEMMA 1. *Let $\mathfrak{A}$ be a family of convex closed bounded sets Γ_α in n dimensiona Euclidean space E_n. If every $n + 1$ members of $\mathfrak{A}$ intersect, then $\bigcap_\alpha \Gamma_\alpha$ is non-empty.*

PROOF. It is sufficient to show that any finite number of sets of $\mathfrak{A}$ intersect, for then compactness will yield the general result if we restrict ourselves, as we may, to a bounded portion of the space. Let $\{\Gamma_1, \cdots, \Gamma_m\}$ be any finite subfamily of $\mathfrak{A}$, and let A be a convex, compact region containing them. Let $\phi_i(x)$ be the distance from a point x to Γ_i, then ϕ_i is a convex function. If the Γ_i do not all intersect, then every point of A is outside some Γ_i, and hence

$$\inf_{x \in A} \sup_{i} \phi_i(x) > 0.$$

We apply now Theorem 1, and obtain the existence of a convex combination of $n + 1$ functions ϕ_i with $\sum \xi_i \phi_i(x) > 0$ for every x in A. This easily yields a contradiction of hypothesis.

LEMMA 2. *If $\mathfrak{A}$ is a family of closed bounded convex sets Γ_α in E_n, and if every n sets intersect, then there exists a line through the origin which intersects every member of $\mathfrak{A}$.*

PROOF. It is sufficient to prove the lemma for finite number of sets Γ_i. A simple compactness argument then yields the conclusion for the general case, as follows: Let s_α denote the set of points on the projective sphere corresponding to the direction of the lines through the origin which intersect Γ_α. If we prove that every finite sub-family of the s_α intersect, then, the projective space being compact, the same conclusion will apply to the entire family.

We enlarge every convex set Γ_i by describing an ϵ sphere about each point of Γ_i and forming the convex closure of the resulting set. Denote the obtained sets by $\Gamma_i(\epsilon)$. Thus each set is now n dimensional. It is established now that there exists a line through the origin intersecting every $\Gamma_i(\epsilon)$. A direct compactness argument will then give the same conclusion for Γ_i.

To this end, consider the unit sphere and for any direction which corresponds to a point x or its antipodal point $-x$, construct the orthogonal linear space L_x (a hyperplane through the origin whose normal has direction numbers proportional to x). We project the Γ_i perpendicularly on L_x. The resulting convex sets satisfy the hypothesis of Lemma 1 in L_x. Thus they intersect in a convex set C_x. Hence, in terms of the projections of $\Gamma_i(\epsilon)$, the intersection $C_x(\epsilon)$ must be $n - 1$ dimensional.

Now, if $C_x(\epsilon)$ contains the origin Θ, the line through Θ and x will intersect each of the $\Gamma_i(\epsilon)$ and the lemma is established. Suppose this not to be the case for any x. It is shown now that the closest point of $C_x(\epsilon)$ to Θ varies continuously with x. Indeed, let x_n converge to x_0. For each $C_{x_n}(\epsilon)$ consider the closest point a_n to the origin and project back perpendicularly. Each direction x_n through a_n intersects every $\Gamma_i(\epsilon)$ and hence the limiting direction x_0 intersects all $\Gamma_i(\epsilon)$. Let a_0 denote the point where this line pierces L_{x_0}. If a_0 were not the closest point of $C_{x_0}(\epsilon)$ to the origin, then there exists a point b_0 at least η closer to the origin. Since $C_{x_0}(\epsilon)$ is $n - 1$ dimensional, choose a point on the interios of $C_{x_0}(\epsilon)$ a distance of $\bar{\eta}$ from b_0. Projecting back along the direction x_0, this line must pierce every $\Gamma_i(\epsilon)$ in an interior point. This follows directly because of the fact that $\Gamma_i(\epsilon)$ are n dimensional and convex. Consequently by varying the direction sufficiently little from x_0 every line must pierce each $\Gamma_i(\epsilon)$. Thus, the projections in these respective hyperplanes L_x where x is near x_0 possess a point in $C_x(\epsilon)$ such that the distance from the origin is the distance b_0 from origin $\pm\eta$ at most. Here η can be arbitrarily small with η. But this contradicts the fact that the x_n directions had the closest points different from a_0 as small as one wishes. This contradiction shows that a_0 is the closest point. Hence the function mapping the direction x into the closest point of $C_x(\epsilon)$ from the origin is a closed mapping and hence continuous.

Let $f(x)$ denote the closest point of $C_x(\epsilon)$ to the origin, projected radially onto the unit sphere. This defines a continuous function of the unit sphere into itself, with the properties, for all x,

(a) $$(f(x), x) = 0,$$

(b) $$f(x) = f(-x).$$

But (a) and (b) are inconsistent. The former implies that f is a map of odd degree, since an obvious deformation takes it into the indentity map. The latter implies that the degree of f is even, since, if A and A' are symmetrically defined chains on a hemisphere and its complement (so that $A + A'$ is the oriented unit sphere), then

$$f(A + A') = 2f(A) \text{ or } 0$$

according as n is even or odd. (Actually (a) is possible only for n even.) This inconsistency confirms the lemma.

The last remarks are essentially a proof of the theorem that there is no non-vanishing tangential vector field on a sphere, of any dimension, such that the vectors at antipodal points are parallel (with the same sense).

THEOREM 2. *Let $\mathfrak{A}$ be a family of closed bounded convex sets in E_n. Let L be an $n - r$ dimensional linear manifold. If the intersection of every r members of $\mathfrak{A}$ is non-empty, then there exists an $n - r + 1$ dimensional linear manifold in E_n containing L and intersecting every member of $\mathfrak{A}$.*

This theorem was obtained by Horn in 1948 ([2]).

PROOF. Choose an origin in L and project E_n on the (r dimensional) orthogonal complement of L. Then apply Lemma 2.

It is to be remarked that neither Lemma 1 nor Theorem 2 remains valid for closed convex sets that are not bounded.

§3. A fitting theorem

Suppose that m points in the plane: (x_i, y_i), $i = 1, \cdots, m$, are given. We shall determine conditions on fitting the points by functions of the form

$$y = \phi(x) = \sum_{j=1}^{n} a_j \phi_j(x) \tag{1}$$

where $\phi_j(x)$ are arbitrary functions. We say that ϕ approximates (x_i, y_i) within δ if $|\phi(x_i) - y_i| \leqq \delta$.

LEMMA 3. *If every $n + 1$ points of $\{(x_i, y_i)\}$ can be approximated within δ by a function of the form* (1) *then there exists a function of that form which approximates within δ all the points.*

PROOF. The set $a = (a_1, \cdots, a_n, -1)$ form an n dimensional subset L of E_{n+1}. Each point (x_i, y_i) generates a linear function g_i defined over L as follows:

$$g_i(a) = \sum_{j=1}^{n} \phi_j(x_i)a_j + (-1)y_i.$$

The hypothesis states that every $n + 1$ such linear functions possess a common "root" a in the sense that $|g_i(a)| \leqq \delta$ for these functions.

It is clear, since there are only a finite number of points, that we may assume that all these functions and their linear combinations possess roots a with $|a_j| \leqq M$ for some uniform bound M. Let A be the n dimensional convex bounded set of all points a with $|a_j| \leqq M$, and let $\mathfrak{F}$ be the totality of all g_i and $-g_i$ arising from the given points (x_i, y_i). Being linear, they are trivially convex. If they do not all possess a common root in the sense described above, then for every point $a \epsilon A$ we may find a function $f_a \epsilon \mathfrak{F}$ with $f_a(a) > \delta$. By Theorem 1 there exists a convex combination of $n + 1$ functions which is greater than δ for all a. This contradicts the hypothesis and establishes the result.

It is to be remarked that the lemma can also be proved by a reduction to Helly's Theorem.

The same result can be concluded for an infinite number of points (x_i, y_i), provided we assume that the convex set, A, of those a which approximate some pair of points, say (x_1, y_1) and (x_2, y_2), is bounded. (This condition will be satisfied in most applications.) For, by Lemma 3, we can fit any finite number of points within δ. Moreover, every finite set containing the two points (x_1, y_1) and (x_2, y_2) can be approximated by an a lying in the bounded region A. By compactness, the infinite set can also be so approximated. Thus, under the assumption of the existence of two points having the property stated above, we have shown:

THEOREM 3. *If every $n + 1$ of an infinite collection $\{(x_\alpha, y_\alpha)\}$ of points in the plane can be approximated within δ by a function of the form* (1), *then there exists a function of the same form which approximates simultaneously within δ all the points (x_α, y_α).*

In the following examples all the hypotheses are easily seen to be fulfilled:

EXAMPLE 1. (Take $\phi_j(x) = x^{j-1}$.) If every $n + 1$ points (x_α, y_α) of a prescribed collection can be fitted within δ by a polynomial of degree $n - 1$, then the entire set $\{(x_\alpha, y_\alpha)\}$ can be fitted by a polynomial of the same degree.

EXAMPLE 2. (Take $\phi_{2k+1}(x) = \cos kx$, $\phi_{2k}(x) = \sin kx$, where $k = 0, 1, \cdots, r$.) If every $2r + 2$ points of a given collection $\{(x_\alpha, y_\alpha)\}$ can be approached within δ by a trigonometric polynomial of degree r, then the same can be accomplished for all (x_α, y_α).

Finally, we remark that the requirement that the points lie in two dimensional space is not essential. Any finite dimension can be considered for x, with y serving as the dependent variable (i.e., the approximation being measured in the y direction). However, the analogous theorem, which uses the geometric distance from point to curve (or hypersurface) as the measure of approximation, does not hold. For example, consider a regular polygon of $2r$ sides inscribed in a circle of unit radius. There is a line whose distance to all but one of the vertices is at most $\delta = (1 + \cos \pi/r)/2$. However, no line passes that close to all the vertices.

It is to be emphasized that the result imposes no restriction whatever on the component functions $\phi_j(x)$.

§4. A covering theorem

In this section, we present a result on coverings of the surface of a n-sphere by closed hemispheres. Despite its intimate connection with the foregoing, it is more convenient to give an independent proof. We reproduce the following lemma from [1], [3]:

LEMMA 4. *Let A be a convex set in E_n spanned by points p_i, $i = 1, \cdots, m$.*

Every point in A can be represented as a convex combination of at most $n + 1$ points p_i.

PROOF. We consider only the case $m > n + 1$. Take a simplex S_m in E_{m-1} and let T be a linear transformation mapping it on the given convex A in an obvious manner. The inverse transformation takes a given point of A into a plane of dimension at least $m - n - 1$. This plane intersects S_m and therefore must intersect some face of dimension n or less. The vertices of this face correspond to the desired subset of $\{p_i\}$.

THEOREM 4. *Let the surface of a sphere in E_n be covered by a compact family of closed hemispheres, then there exists $n + 1$ members of the family which cover the surface.*

REMARK. A family of hemispheres is compact if the unit vectors normal to the hyperplanes bounding the hemispheres (directed into the hemispheres) constitute a compact family.

PROOF. Let l_α denote the unit normal to the hemisphere H_α in the sense described in the remark. A point x on the surface of the sphere is covered by H_α if and only if $(l_\alpha, x) \geqq 0$. We consider a countable set $\{l_i\}$ dense in $\{l_\alpha\}$. Let Γ_i be the convex set spanned within the unit sphere by $l_1, \cdots, l_i$. We wish to show that, for m sufficiently large Γ_m is arbitrarily close to the origin, Θ. If the contrary, then for some ϵ the distance $\rho(\Theta, \Gamma_i)$ exceeds ϵ for all i. By the choice of $\{l_i\}$ this implies that $\rho(\Theta, \Gamma) \geqq \epsilon$, where Γ is the convex spanned by all the l_α. Take a plane through the origin which does not pass within ϵ of Γ, and let x_0 denote its unit normal, directed away from Γ. Then $(l_\alpha, x_0) \leqq -\epsilon$ for all l_α and hence x_0 is not covered by $\{H_\alpha\}$. This contradiction implies that for any k there exists a $m(k)$ with $\rho(\Theta, \Gamma_{m(k)}) < 1/k$. Let $x^{(k)}$ be a point of $\Gamma_{m(k)}$ of distance less than $1/k$ from the origin. By Lemma 4, we have a convex representation:

$$x^{(k)} = \sum_{i=1}^{n+1} \xi_i^{(k)} l_i^{(k)}.$$

Since $n + 1$ is fixed and $l_i^{(k)}$ and $\xi_i^{(k)}$ are drawn from compact sets, we may pass to the limit and obtain a representation:

$$\Theta = \sum_{i=1}^{n+1} \xi_i l_i.$$

It is clear that $\sum \xi_i = 1$ and that all ξ_i are non-negative. The hemispheres H_i corresponding to the l_i of this representation, $i = 1, 2, \cdots, n + 1$, must cover the full sphere.

We remark that the theorem is not true if the compactness requirement is removed. For example, consider the family of hemispheres on a sphere in E_2 described by the angles $\pi, 1, 1/2, 1/3, \cdots, 1/m, \cdots$.

It is interesting to observe that the finite covering given by Theorem 4 may be made to contain one hemisphere specified at pleasure. The following is an equivalent statement of this stronger result:

COROLLARY. *Let a given hemisphere H on the surface of a sphere in E_n be covered by a compact family of closed hemispheres. Then there exist n members of the family which cover H.*

PROOF. The given family, together with the closed complement H_0 of H, cover the sphere. Theorem 4 provides an $n + 1$-member sub-family of the augmented family which also covers the sphere. If this sub-family does not include H_0, consider the convex C spanned within the unit sphere by the unit normals l_i to the sub-family. C contains the origin Θ. Let l_0 denote the unit normal to H_0, and y_0 the intersection of the radius $[\Theta, -l_0]$ with the boundary of C. Then y_0 is a convex combination of n (or fewer) of the l_i, and Θ is a convex combination of l_0 and y_0. (If y_0 and Θ happen to coincide, then l_0 will appear vacuously.) It follows that an $n + 1$-member sub-family containing H_0 and covering the sphere can always be found. The closed complement of H_0—which is the hemisphere originally given—is necessarily covered by the other n members of any such sub-family.

The direct relation between this section and the earlier sections becomes immediately clear when we write Theorem 4 in its contrapositive form: "If every $n + 1$-member sub-family fails to cover, then the full family does not cover." Theorem 1 could not be applied directly because the spherical distance to a spherical convex set is not a convex function.

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