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Note

A note on kernels and Sperner's Lemma☆

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ABSTRACT

The kernel-solvability of perfect graphs was first proved by Boros and Gurvich, and later Aharoni and Holzman gave a shorter proof. Both proofs were based on Scarf's Lemma. In this note we show that a very simple proof can be given using a polyhedral version of Sperner's Lemma. In addition, we extend the Boros–Gurvich theorem to h-perfect graphs and to a more general setting.

1. Introduction

In a directed graph $D = (V, A)$, a stable set $S \subseteq V$ is said to be a *kernel* if from every node of $V \setminus S$ there is an arc to S. Kernels have several applications in combinatorics and game theory, and there has been extensive work on the characterization of digraphs that have kernels. See [5] for a survey on the topic.

One approach to characterize the existence of kernels has been to identify undirected graphs for which every "nice" orientation has a kernel. This led to the introduction of the notion of kernel-solvability by Berge and Duchet [3].

Definition 1. Let $G = (V, E)$ be an undirected graph. A *superorientation* of G is a directed graph $\vec{G}$ obtained by replacing each edge uv of G by an arc uv or an arc vu or both. A *proper directed cycle* in a superorientation is a directed cycle consisting of arcs that are not present reversed in the digraph. In this article we define a *source node* of an induced subdigraph $\vec{G}[U]$ as a node in U from which there are arcs to all of its neighbours in $G[U]$. A superorientation is *clique-acyclic* if no clique contains a proper directed cycle (equivalently, if every clique contains a source node). A graph G is *kernel solvable* if every clique-acyclic superorientation of G has a kernel.

Berge and Duchet [3] conjectured that the kernel solvable graphs are exactly the perfect graphs. The kernel-solvability of perfect graphs was proved by Boros and Gurvich [4], and later a shorter proof was given by Aharoni and Holzman [1]. A concise version of the proof can be found in Schrijver's book [14]. A common feature of these proofs is that they use Scarf's Lemma [13], a result originating in game theory. We should also mention that the other direction of the Berge–Duchet conjecture follows from the Strong Perfect Graph Theorem [7], and no other proof is known.

One contribution of the present paper is a simple proof of the kernel-solvability of perfect graphs based on Sperner's Lemma instead of Scarf's Lemma. Since Sperner's Lemma is more widely known and conceptually simpler, this might have some interest. We note that the methods presented here can also be used to derive Scarf's Lemma from Sperner's Lemma, see [10]. In Section 2, we prove the polyhedral version of Sperner's Lemma that we use in the rest of the paper, and the new proof of the Boros–Gurvich Theorem is presented in Section 3.

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In the remaining sections we present new results on kernels in superorientations of non-perfect graphs. Since these graphs are not kernel solvable, we have to make additional restrictions on the superorientation. In Section 4 we show a result where these restrictions depend on the facets of STAB(G) (the convex hull of the characteristic vectors of the stable sets of G). The result is specialized to h-perfect graphs in Section 5, where it is shown that every clique-acyclic and odd-hole-acyclic superorientation of an h-perfect graph has a kernel. The reverse implication is not true here, but the result can be slightly strengthened (Theorem 9), and we conjecture that this stronger property characterizes h-perfect graphs.

2. Polyhedral versions of Sperner's Lemma

In this section we present versions of Sperner's Lemma that deal with colourings of vertices and facets of polytopes and polyhedra.

Definition 2. For a colouring of the vertices of a polytope P, a facet of P is *multicoloured* if it contains vertices of every colour. For a colouring of the facets of P, a vertex of P is *multicoloured* if it lies on facets of every colour.

The following theorem is a variant of the multidimensional Sperner's Lemma, applied to the Schlegel diagram of a polytope (see [16] for a reference on Schlegel diagrams). We present a simple direct proof.

Theorem 1. *Let P be an n-dimensional polytope, with a simplex facet F_0. Suppose we have a colouring of the vertices of P with n colours such that F_0 is multicoloured. Then there is another multicoloured facet.*

Proof. Let us divide the non-simplex facets of P into simplices. We need to show that there is a multicoloured simplex. Let C be the set of all colours and let red be one of them.

Define a graph whose nodes are the simplices in the division and there is an edge between two simplices if and only if they share an $(n-2)$-dimensional facet whose vertices use each colour in $C \setminus \{\text{red}\}$ exactly once. It is easy to see that in this graph the multicoloured simplices are of degree one, while the simplices which use one colour in $C \setminus \{\text{red}\}$ twice, and the others once, are of degree two. The other simplices are of degree zero, so the graph is the disjoint union of paths and cycles (and isolated vertices). The assumption implies that F_0 is a node of degree one, so there has to be another node of degree one which gives a multicoloured simplex. □

By polarity, the following theorem is also true.

Theorem 2. *Let P be an n-dimensional polytope, with a simplicial vertex v_0. Suppose we have a colouring of the facets of P with n colours such that v_0 is multicoloured. Then there is another multicoloured vertex.*

The above results can be generalized to unbounded pointed polyhedra, which is the form that we will use later. First, we have to extend the notion of vertices.

Definition 3. The *ends* of a pointed polyhedron P are its vertices and its extreme directions (an *extreme direction* of a polyhedron is an extreme ray of its characteristic cone).

We extend also the incidences between facets and vertices to ends in the natural way: a facet of P contains an extreme direction of P if it is also an extreme direction of the facet. In addition, if a pointed polyhedron's characteristic cone is full-dimensional, then we consider the extreme directions as being on a "facet at infinity".

Definition 4. Two polyhedra are called *combinatorially equivalent* if there is a bijection between their facets (including the "facet at infinity") and their ends which preserves the incidences. Two polyhedra are called *combinatorially polar* if there is a bijection between the facets of the one and the ends of the other and vice versa which reverses the inclusion relation.

We claim that if P is a pointed full-dimensional polyhedron then there exists a polytope which is combinatorially equivalent to it. This is because if we move P so that the origin is in its interior and then take its polar, it will be a polytope which is combinatorially polar to P; moreover, it will be full-dimensional since P is pointed. If we do the same a second time, we get a polytope which is combinatorially equivalent to P. Now we can state the analogues of Theorems 1 and 2 for unbounded polyhedra.

Corollary 3. *Let P be an n-dimensional pointed polyhedron whose characteristic cone is generated by n linearly independent vectors. If the vertices and extreme directions of P are coloured with n colours such that the extreme directions receive different colours, then there is a multicoloured facet.*

Proof. Let us take a polytope P' which is combinatorially equivalent to P, and let F be the facet of P' which corresponds to the infinite facet of P. So F is a multicoloured simplex facet. We can apply Theorem 1. □

Corollary 4. *Let P be an n-dimensional pointed polyhedron whose characteristic cone is generated by n linearly independent vectors. If the facets of the polyhedron are coloured with n colours such that facets containing the ith extreme direction do not get colour i, then there is a multicoloured vertex.*

Proof. Let us take again a polytope P' which is combinatorially equivalent to P, and let F be the simplex facet of P' which corresponds to the infinite facet of P. Let us attach a (sufficiently flat) simplex to P' on facet F, and colour the new facets so that the facet opposite (in the simplex) to the vertex corresponding to the ith extreme direction gets colour i. Applying Theorem 2 we get that there is another multicoloured vertex of P' (besides the new vertex of the simplex) and from the assumption it follows that this cannot correspond to an extreme direction, so it corresponds to a vertex of P. $\square$

We will apply this corollary to polyhedra of the form $P = Q - \mathbb{R}^n_+$ where Q is a bounded polytope. We use the notation $[n] = \{1, \dots, n\}$. If $a \in \mathbb{R}^n_+$ and $J \subseteq [n]$, then we denote by a^J the vector whose jth coordinate is

$$a^J(j) = \begin{cases} a(j) & \text{if } j \in J \\ 0 & \text{if } j \notin J. \end{cases}$$

We will need the following lemma:

Lemma 5. *If $P = Q - \mathbb{R}^n_+$ where $Q = \{x \in \mathbb{R}^n_+ : Ax \le b\}$ is a bounded polytope and A and b are non-negative, then P is described by inequalities of the form $a_i^J x \le b_i$, where a_i is the ith row of A, b_i is the ith coordinate of b, and $\emptyset \ne J \subseteq \mathrm{supp}(a_i)$. The extreme directions of P are $-e_j$ $(j = 1, \dots n)$.*

Proof. It is clear that the vectors in Q satisfy the inequalities. If $p \in P$, then there is a vector in Q such that $p \le q$, so p also satisfies the inequalities since their coefficients are non-negative.

For the other direction, let x be a vector that satisfies all inequalities. Let $y \ge x$ be a maximal vector with the same property. Suppose that $y_j < 0$; then $j \notin J$ whenever y satisfies the inequality $a_i^J y \le b_i$ with equality, since otherwise $a_i^{J \setminus \{j\}} y > b_i$. This means that y_j could be increased without violating the conditions, which contradicts the maximality of y. We can conclude that $y \ge 0$, so $y \in Q$, which means that $x \in P$. $\square$

3. Kernel-solvability of perfect graphs

Boros and Gurvich [4] proved the following conjecture of Berge and Duchet, using game theoretic results that rely on Scarf's Lemma.

Theorem 6 ([4]). *Every perfect graph is kernel solvable.*

Note that it follows easily from the Strong Perfect Graph Theorem [7] that non-perfect graphs are not kernel solvable. One needs the observations that odd holes and odd antiholes are not kernel solvable, and that induced subgraphs of kernel solvable graphs are kernel solvable. However, no proof is known that does not rely on the SPGT.

Aharoni and Holzman [1] gave a shorter proof of Theorem 6 using Scarf's Lemma directly. In this section we present a similarly short proof that relies on the more familiar Sperner's Lemma instead.

Proof of Theorem 6. Let $G = (V, E)$ be a perfect graph, with $V = [n]$, and let $\vec{G}$ be a clique-acyclic superorientation of G. Let $\mathcal{C}$ denote the set of all (not necessarily maximal) cliques of G. We consider the polyhedron $P := \mathrm{STAB}(G) - \mathbb{R}^n_+$. By the well known results of Fulkerson [8] and Lovász [11], STAB(G) is described by the clique-inequalities, so Lemma 5 implies that

$$P = \{x \in \mathbb{R}^n : x(C) \le 1 \text{ for every } C \in \mathcal{C}\},$$

and the extreme directions of P are $-e_j$ $(j = 1, \dots n)$.

Let the colour of a facet $\{x \in P : x(C) = 1\}$ be a source node j of clique C. Clearly the extreme direction $-e_j$ does not belong to a facet of colour j, so applying Corollary 4 we get that there exists a multicoloured vertex x^* of P. By the definition of P, $x^* = \chi_S$ for a maximal stable set S.

Since x^* is multicoloured, for each node j of V, there is a clique C such that the facet $\{x \in P : x(C) = 1\}$ contains x^* and it has colour j. This means that $|C \cap S| = 1$ and j is a source node of C. Thus from each node $j \notin S$ there is an arc to S, so S is a kernel. $\square$

4. Generalization based on the facets of STAB(G)

In this section we extend Theorem 6 to arbitrary undirected graphs, provided some conditions hold that depend on the facets of STAB(G). We say that a superorientation $\vec{G}$ of $G = (V, E)$ is *acyclic* in $U \subseteq V$ if there is no proper directed cycle in $\vec{G}[U]$.

Theorem 7. *If $\mathrm{STAB}(G) = \{x \in \mathbb{R}^n_+ : Ax \le b\}$ (where A and b are non-negative) and $\vec{G}$ is a superorientation of G which is acyclic in $\mathrm{supp}(a)$ for every row a of A, then there is a kernel in $\vec{G}$.*

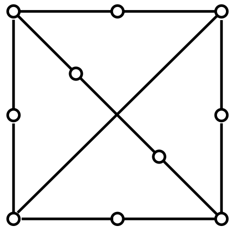

Fig. 1. A non-h-perfect graph whose clique- and odd-hole-acyclic superorientations all have kernels.

Proof. Let $P = \mathrm{STAB}(G) - \mathbb{R}^n_+$. By Lemma 5, P is described by the inequalities of the form $a_i^J x \leq b_i$, where a_i is the ith row of A, b_i is the ith coordinate of b, and $\emptyset \neq J \subseteq \mathrm{supp}(a_i)$.

Let the colour of a facet of the form $P \cap \{x : a_i^J x = b_i\}$ be a source node of the subdigraph of $\vec{G}$ induced by J. Such a source node exists because $\vec{G}$ is acyclic in $\mathrm{supp}(a_i)$. In order to apply Corollary 4, we have to show that a facet containing the jth extreme direction does not have colour j. This is true because in this case $j \notin J$. Thus Corollary 4 implies that P has a multicoloured vertex $x^* = \chi_S$ for a maximal stable set S.

For every node j, there is a facet F of colour j containing x^*. Let F be $P \cap \{x : a_i^J x = b_i\}$. Then $S \cap J$ is a maximal stable set in $G[J]$ because $a_i^J x^* = b_i$ and $\mathrm{supp}(a_i^J) = J$. This and $j \in J$ imply that either $j \in S$ or j has a neighbour in $S \cap J$, in which case j has an out-neighbour in $S \cap J$, since j is a source node of J. We proved that S is a kernel of $\vec{G}$. $\square$

5. Kernels in *h*-perfect graphs

Sbihi and Uhri [12] introduced the class of *h-perfect graphs* as the graphs for which the stable set polytope is described by the following set of inequalities:

$$x_v \geq 0 \quad \text{for every } v \in V, \tag{1}$$

$$x(C) \leq 1 \quad \text{for every maximal clique } C, \tag{2}$$

$$x(Z) \leq \frac{|Z| - 1}{2} \quad \text{for every odd hole } Z. \tag{3}$$

The class of h-perfect graphs contains some interesting graph types beyond perfect graphs; for examples see [6,9,15].

To apply Theorem 7 to h-perfect graphs, let us call a superorientation of a graph *odd-hole-acyclic* if no oriented odd hole is a proper directed cycle. Our result is as follows.

Theorem 8. *If G is an h-perfect graph then every clique-acyclic and odd-hole-acyclic superorientation of G has a kernel.*

Proof. Directly follows from Theorem 7. $\square$

Obviously a superorientation of a perfect graph is always odd-hole-acyclic, thus Theorem 8 is an extension of Theorem 6. However, while the reverse implication is also true in case of Theorem 6, the same does not hold for Theorem 8, and a counterexample is given here. The graph on Fig. 1 is not h-perfect (this follows from the results of Barahona and Mahjoub [2]), but it can be seen by case analysis that every clique- and odd-hole-acyclic superorientation of it has a kernel.

Nevertheless, one may hope for a stronger theorem where the reverse direction also holds. We give here a less elegant but stronger theorem for which we conjecture that this is the case.

Let G be an h-perfect graph, and let $\vec{G}$ be a clique-acyclic superorientation of G. Some odd holes of G may become proper directed cycles; let us denote these by $Z_1, \ldots, Z_k$. Let us select nodes $v_1, \ldots, v_k$ such that $v_i \in Z_i$ for $i = 1, \ldots, k$ (the selected nodes need not be distinct). We call this a *superorientation with special nodes*. An *almost-kernel* for a superorientation with special nodes is a stable set S with the following property:

If a node $v \notin S$ has no outgoing arc into S, then $v = v_i$ for some i and $|Z_i \cap S| = (|Z_i| - 1)/2$.

Theorem 9. *If G is an h-perfect graph then every clique-acyclic superorientation with special nodes has an almost-kernel.*

Proof. The proof is almost the same as the proof of Theorem 7. Let $P = \mathrm{STAB}(G) - \mathbb{R}^n_+$. Here again we colour a facet of the form $P \cap \{x : a_i^J x = b_i\}$ with a source node of the subdigraph of $\vec{G}$ induced by J, if there is such a node. If not, then a_i^J is the characteristic vector of a proper directed odd hole Z_l; in this case, let its colour be the selected v_l.

As in the proof of Theorem 7, P has a multicoloured vertex $x^* = \chi_S$ for a maximal stable set S, so for every node j, there is a facet $F = P \cap \{x : a_i^J x = b_i\}$ of colour j containing x^*. If j is a source node of the subdigraph of $\vec{G}$ induced by J, we obtain (by the same argument as in the proof of Theorem 7) that either $j \in S$ or there is an arc from j to S. Otherwise a_i^J is the characteristic vector of a proper directed odd hole Z_l, $j = v_l$, and $|Z_l \cap S| = (|Z_l| - 1)/2$ since $a_i^J x^* = b_i$. Thus S is an almost-kernel. $\square$

Note that this theorem is stronger than Theorem 8 since every almost-kernel in a clique-acyclic and odd-hole-acyclic orientation is a kernel. We conjecture that here the converse also holds:

Conjecture 10. *A graph G is h-perfect if and only if every clique-acyclic superorientation with special nodes has an almost-kernel.*

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