Polytopes and Algebraic Geometry

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Outline of the talk

1. Four classic results relating polytopes and algebraic geometry:

(A) Toric Geometry
(B) Viro's Theorem
(C) Bernstein's Theorem.
(D) The g-theorem.

Toric Varieties

Gröbner Bases of Toric Ideals

- We work over $K[x_1, \ldots, x_n]$ the polynomial ring in n variables over the field K. Hilbert proved its ideals are finitely generated.

- A Gröbner basis for an ideal I is a set of generators with great computational abilities!

- Gröbner bases are central for polynomial problems: Ideal membership, Elimination of variables, Polynomial system solving, etc.

- A monomial order $\preceq$ on the monomials of $\mathbf{K}[x_1, ..., x_n]$ is said to be $admissible$ if $1 \preceq m_1$ for all monomial m_1 and $m_1 \preceq m_2$ implies $m_1 m_3 \preceq m_2 m_3$ for all monomials m_1, m_2, m_3 in $\mathbf{K}[x_1, ..., x_n]$. The lexicographic order is an important example of admissible monomial order.

- The leading monomial of a polynomial f in $\mathbf{K}[x_1, ..., x_n]$ with respect to $\preceq$ will be denoted by $in_{\preceq}(f)$.

  For an ideal I in $\mathbf{K}[x_1, .., x_n]$ its initial ideal is the ideal $in_{\preceq}(I)$ generated by the leading monomials of all polynomials in I. The monomials $m \notin in_{\preceq}(I)$ are called standard.

- THEOREM: A Gröbner basis is a generating set for the ideal I.

- Given any polynomial p and a Gröbner bases G, we can "divide" p by G and obtain a UNIQUE remainder, its normal form.

- Given input (1) generators of an ideal and (2) a monomial order, a Gröbner bases can be computed using the famous Buchberger's algorithm.

  Sadly, the degree of GB elements can be doubly exponential on the degrees of the input generators. The size can explode! In practice things are often better.

- A Gröbner basis comes with a Division Algorithm that allow us to compute with residue classes of the quotient ring $K[x_1, x_2, \ldots, x_n]/I$.

  In general ideals, there are no bounds for the number of division steps that it takes to compute the normal form.

TORIC IDEALS

Key Definition

K denotes a field, most often $\mathbb{C}$, and $K[x_1, \ldots, x_n]$ the polynomial ring in n variables. $K[x_1^{-1}, x_1, x_2^{-1}, x_2, \ldots, x_n^{-1}, x_n]$ denotes the Laurent polynomial ring.

Given an integer $d \times n$ matrix A consider the kernel of the ring map ϕ_A induced by the columns of A:

$$\phi_A : K[x_1^{-1}, x_1, \ldots, x_n^{-1}, x_n] \to K[y_1^{-1}, y_1, \ldots, y_d^{-1}, y_d]$$

$$x_i \to y_1^{A[1,i]} y_2^{A[2,i]} \ldots y_d^{A[d,i]}$$

The kernel this map, denoted I_A, is the toric ideal for A.

EXAMPLE

Use the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

Some of the elements of the kernel of ϕ_A are

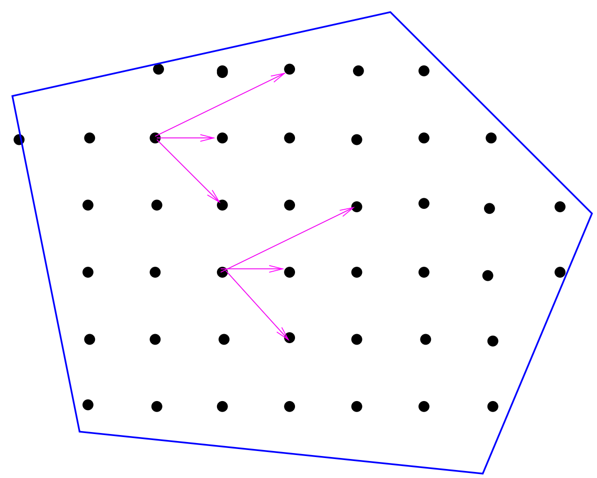
$$x_2^2 - x_1x_3, \quad -x_3^2 + x_2x_4, \quad -x_2x_3 + x_1x_4$$

In fact, they form a set of Generators for the ideal I_A.

Theorem The ideal I_A has always a generating set given by finitely many binomials. In fact, for any monomial order, the reduced Gröbner basis for the toric ideal I_A is generated by binomials $x^u - x^v$ where $Au = Av$.

Special Geometric Interpretation

Given any d-vector b, Visualize a Gröbner bases of I_A using the lattice points of the polyhedron $P(b) := \{x|\ Ax = b, x \geq 0,\ x \in \mathbb{Z}^n\}$. The Gröbner bases binomials are vectors departing from each lattice point $u \in P(b)$. The direction of arrows is given by the term order.



Theorem: The graph of lattice points and arrows is connected for all $P(b)$ if and only if the set of binomials generates the ideal I_A. It has a unique sink and if an only if the set of binomials is a Gröbner bases for I_A.

Theorem: The **Graver bases** of a matrix A is a Gröbner bases for the toric ideal associated to A, in fact, it is a true with respect to ANY monomial order.

Gröbner bases of Toric ideals ARE just Lattice points in POLYHEDRA

Lemma[Sturmfels] Let M be equal to $(d+1)(n-d)D(A)$, where A is a $d \times n$ integral matrix and $D(A)$ is the biggest $d \times d$ subdeterminantof A in absolute value. Any entry of an exponent vector of any reduced Gröbner basis for the toric ideal I_A is less than M.

Given an integer $d \times n$ matrix A the toric ideal I_A is the polynomial ideal generated by the binomials

$$B_A(M) = \{x^u - x^v | Au = Av \text{ and } 0 \leq u, v \leq M\}.$$

A-graded ideals

Let $A = \{a_1, \ldots, a_n\} \subset \mathbb{Z}^d$ be an acyclic integer vector configuration. Let K be a field. In the polynomial ring $K[x_1, \ldots, x_n]$ we consider the variable a_i to have (multi-)degree a_i.

Example: $A = \{1, \ldots, 1\}$ defines the standard grading.

An ideal $I \subset K[x_1, \ldots, x_n]$ is said to be A-homogeneous if it can be generated by polynomials with all its monomials of the same multi-degree.

The prototypical example of an A-homogeneous ideal is the toric ideal I_A:

$$I_A = \langle \mathbf{x}^c - \mathbf{x}^d : \quad c, d \in \mathbb{Z}^n_{\geq 0}, \quad \sum c_i a_i = \sum d_i a_i \rangle$$

Example: $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \subset \mathbb{Z}^2$.

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} = 2\begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow x_1x_3 - x_2^2 \in I_A$$

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 4 \\ 1 \end{pmatrix} = 2\begin{pmatrix} 3 \\ 1 \end{pmatrix} \Rightarrow x_2x_4 - x_3^2 \in I_A$$

$$2\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 4 \\ 1 \end{pmatrix} = 3\begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow x_1^2x_4 - x_2^4 \in I_A$$

Idea: linear relations among the a_i's produce binomials in I_A.

The multi-graded Hilbert function

Every A-homogeneous ideal I decomposes as

$$I = \bigoplus I_b,$$

where $b \in A(\mathbb{Z}^n_{\geq 0})$ ranges over all possible multidegrees.

The A-graded Hilbert function of I is the map

$$A(\mathbb{Z}^n_{\geq 0}) \to \mathbb{Z}_{\geq 0}$$

that sends each I_b to its linear dimension over K.

Remark: $\dim_K(I_b) \leq \#A^{-1}(b) = \#$ of monomials of degree b.

The Sturmfels Correspondence

In 1988, Bayer and Morrison and Mora and Robbiano defined the **State Polytope** of an ideal I. Its vertices are in bijection to the different Gröbner bases of I

In 1991, Sturmfels had proved:

Theorem: The secondary polytope of triangulations of A is a Minkowski summand of the state polytope of the toric ideal I_A).

In particular, there is a well-defined map

$$\text{initial ideals of } I_A \to \text{regular polyhedral subdivisions of } A$$

(the map sends monomial initial ideals to regular triangulations, and is surjective).

In 1995, he extended the map to

$$\Phi : A\text{-graded ideals} \rightarrow \text{polyhedral subdivisions of } A,$$

and the map sends monomial ideals to triangulations.

The map is **not surjective** [Peeva 1995], but its image contains all the unimodular triangulations of A [Sturmfels 1995] (moreover, each unimodular triangulation T is the image of a unique monomial A-graded ideal, namely the Stanley-Reisner ring of T).

Semigroups and Cones

- Recall **MAGIC SQUARES** form a convex polyhedral cones of the form $Ax = 0, x \geq 0$, where A is a matrix with $0, 1, -1$ entries.

- EXAMPLE: The cone C of 3×3 magic matrices is defined by the system of equations (i.e, row sums, column sums, and diagonal sums are equal).

  The cone C of 3×3 magic squares has dimension 3, it is a cone based on a quadrilateral, thus it has 4 rays.

1/3	0	2/3
2/3	1/3	0
0	2/3	1/3

2/3	0	1/3
0	1/3	2/3
1/3	2/3	0

0	2/3	1/3
2/3	1/3	0
1/3	0	2/3

1/3	2/3	0
0	1/3	2/3
2/3	0	1/3

Figure 1: The four RAYS the cone of 3×3 magic squares.

- For a cone C (e.g., magic squares) we are interested in $S_C = C \cap \mathbb{Z}^n$, the **semigroup of the cone** C. And the associated Rings. We can use Hilbert bases!!

- With any rational pointed polyhedral cone $C = \{Ax = 0, x \geq 0\}$ and a field k we associate a *semigroup ring*, $R_C = k[x^a : a \in S_C]$, where there is one monomial in the ring for each element of the semigroup S_C.

- R_C equals $k[x_1, x_2, \ldots, x_N]/I_C$ where I_C is an ideal generated by binomials and N is the number of Hilbert basis elements.

- For a graded k-algebra has decomposition

$$R_C = \bigoplus R_C(i)$$

,

each $R_C(i)$ collects all elements of degree i and it is a k-vector space (where $R_C(0) = k$).

- The function $H(R_C, i) = dim_k(R_C(i))$ is the *Hilbert function* of R_C. *Hilbert-Poincaré series* of R_C

$$H_{R_C}(t) = \sum_{i=0}^{\infty} H(R_C, i)t^i.$$

- **Lemma.** *Let R_C be the semigroup ring obtained from the Hilbert basis of a cone C. The number of elements of the semigroup that have degree s are equal to the value of the Hilbert function $H(R_C, s)$.*

- Example: The number of distinct magic arrays of magic constant s equals the value of the Hilbert function $H(R_C, s)$.

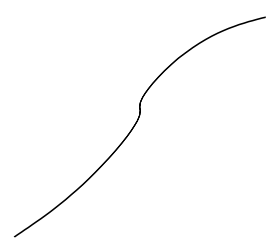
- The Hilbert-Poincaré series can be computed from the knowledge of the Gröbner bases of the Hilbert bases I_C.

Viro's Theorem

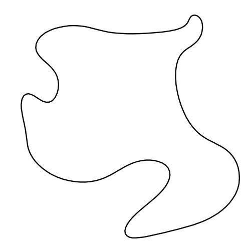
Hilbert's sixteenth problem (1900)

"What are the possible (topological) types of non-singular real algebraic curves of a given degree d?"

Observation: Each connected component is either a pseudo-line or an oval. A curve contains one or zero pseudo-lines depending in its parity.



A pseudoline. Its complement has one component, homeomorphic to an open circle. The picture only shows the "affine part"; think the two ends as meeting at infinity.

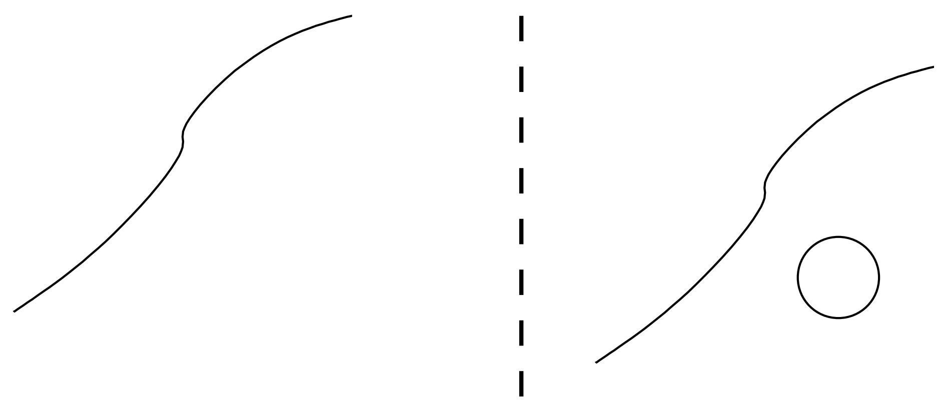


An oval. Its interior is a (topological) circle and and its exterior is a Möbius band.

Partial answers:

Bezout's Theorem: A curve of degree d cuts every line in at most d points. In particular, there cannot be nestings of depth greater than $\lfloor d/2 \rfloor$

Harnack's Theorem: A curve of degree d cannot have more than $\binom{d-1}{2} + 1$ connected components (recall that $\binom{d-1}{2}$= genus)



Two configurations are possible in degree 3

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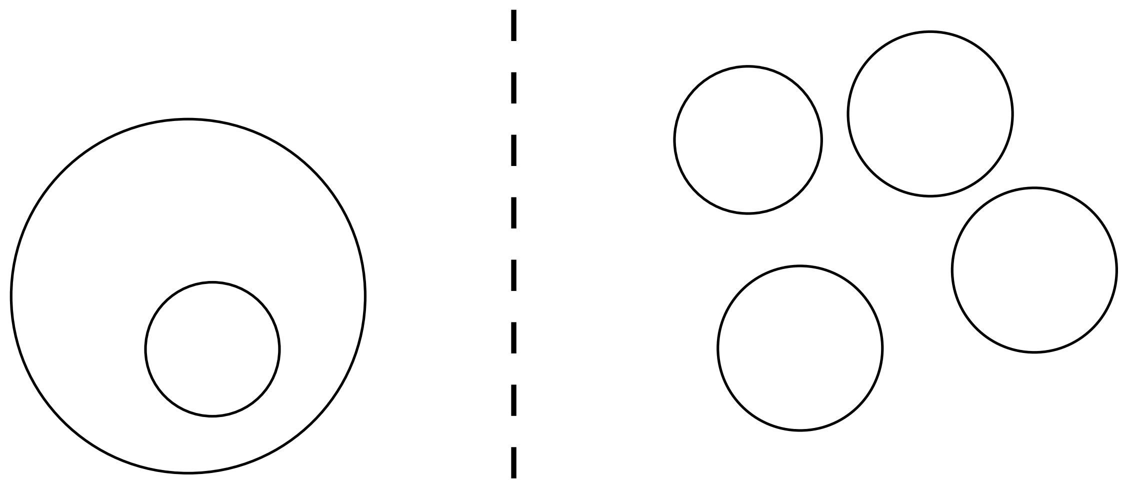
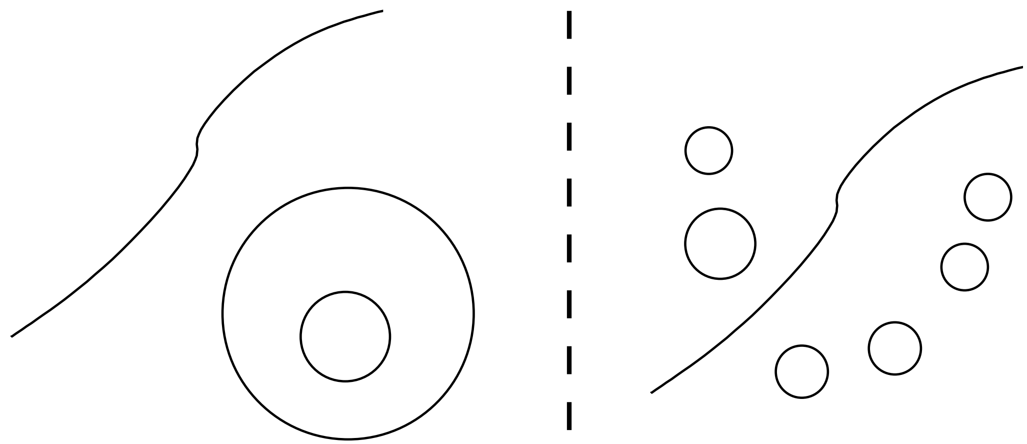


Six configurations are possible in degree 4. Only the maximal ones are shown.

Partial answers:

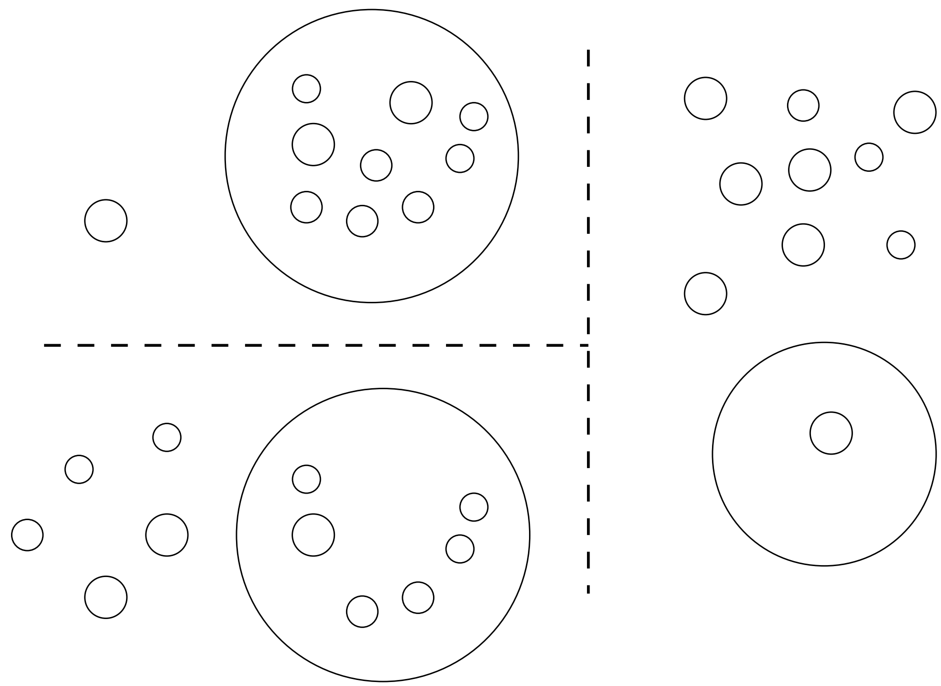
Bezout's Theorem: A curve of degree d cuts every line in at most d points. In particular, there cannot be nestings of depth greater than $\lfloor d/2 \rfloor$

Harnack's Theorem: A curve of degree d cannot have more than $\binom{d-1}{2} + 1$ connected components (recall that $\binom{d-1}{2}$= genus)

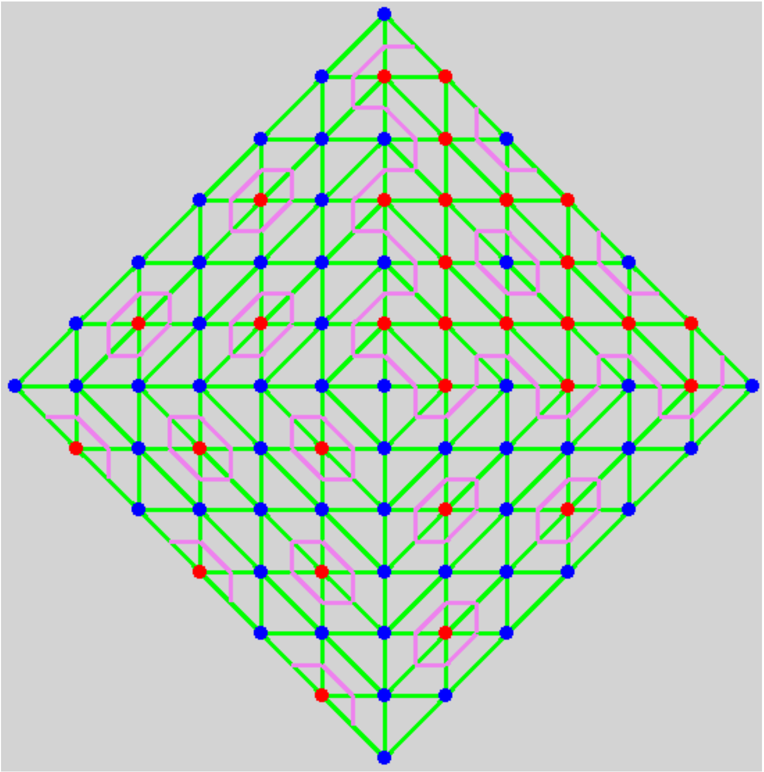


Eight configurations are possible in degree 5. Only the maximal ones are shown.

All that was known when Hilbert posed the problem, but the classification of non-singular real algebraic curves of degree six was not completed until the 1960's [Gudkov]. There are 56 types degree six curves, three with 11 ovals:

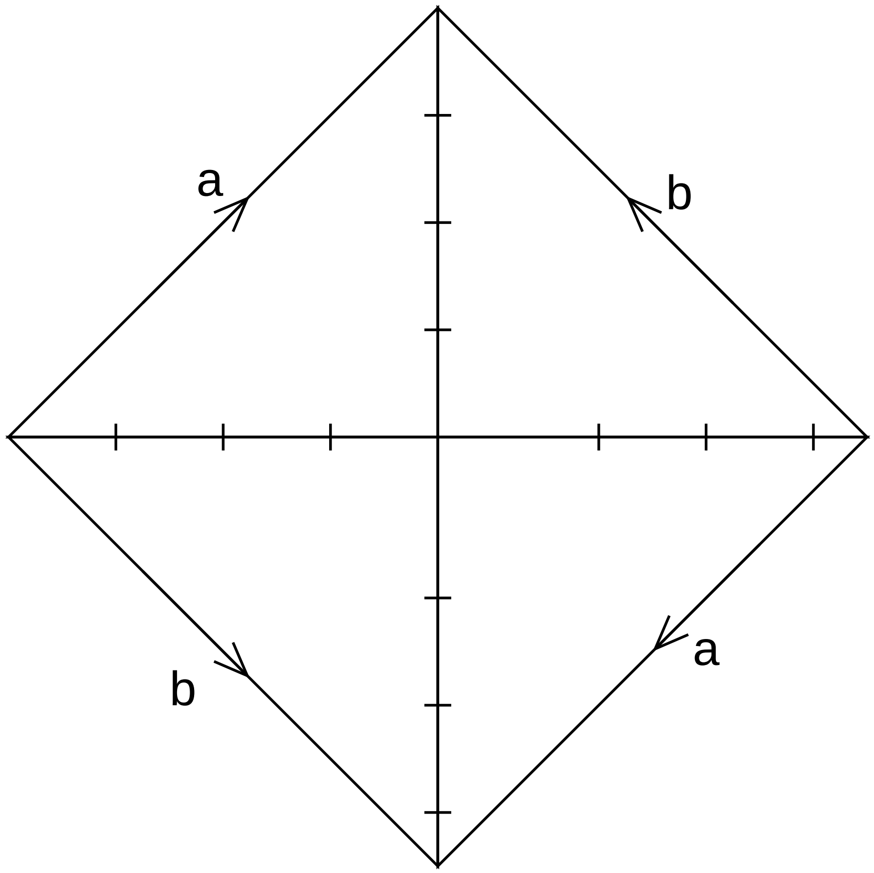


What about dimension 7? It was solved by **Viro**, in 1984 with a method that involves triangulations.



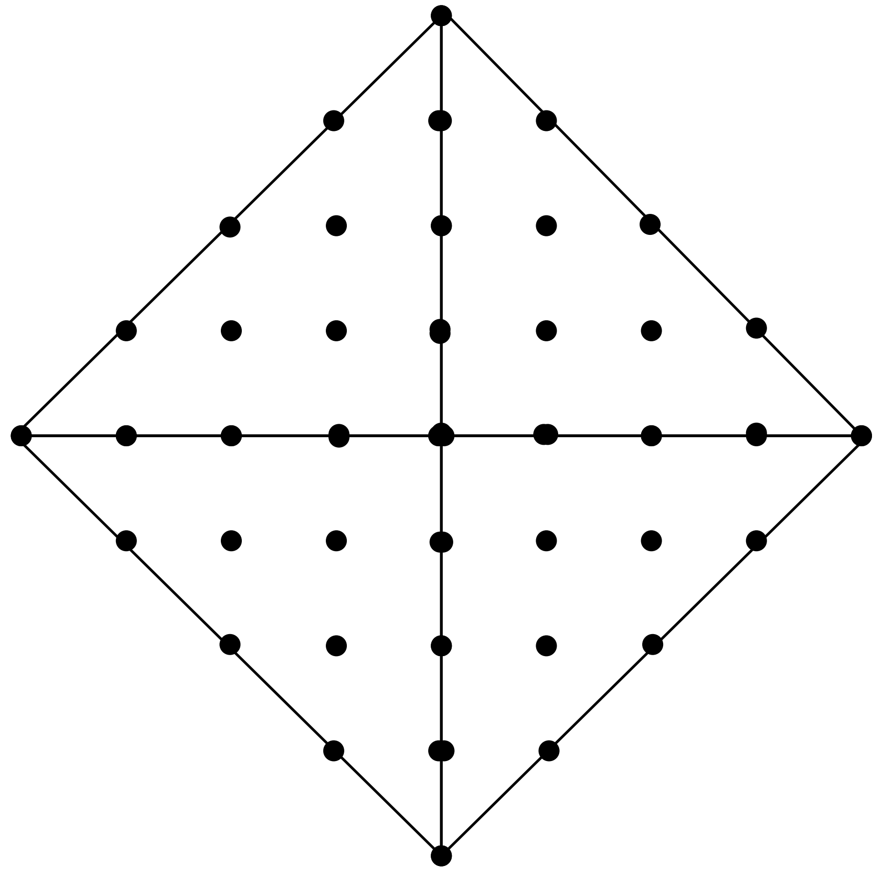
A curve of degree 6 constructed using Viro's method

Viro's method:



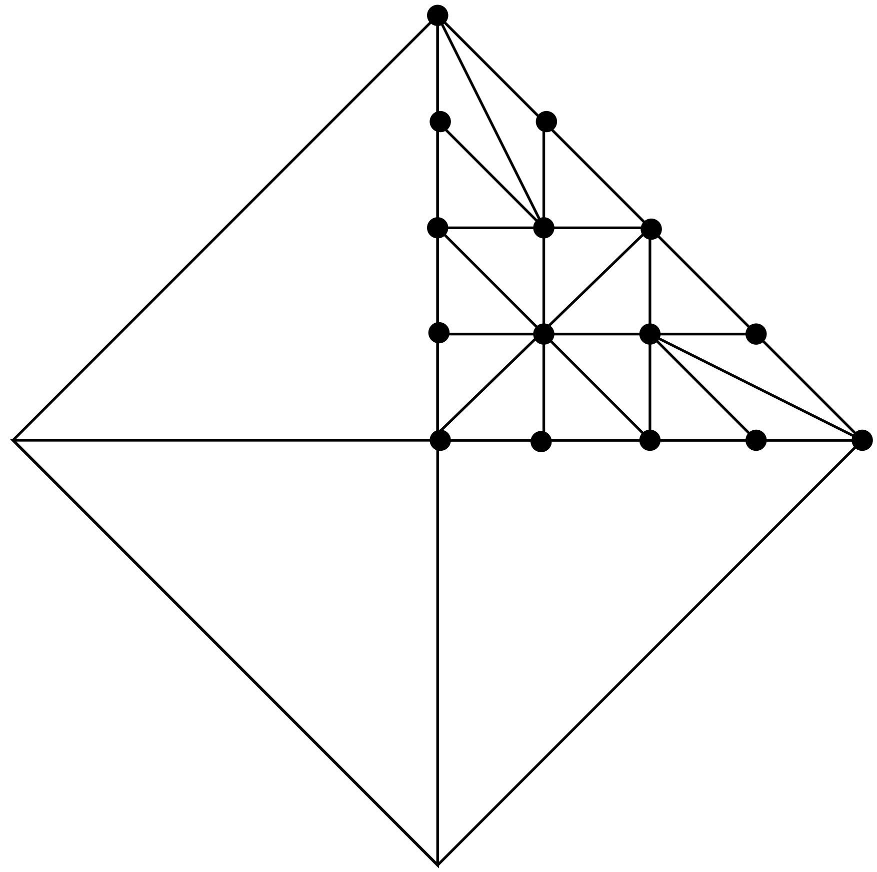
For any given d, construct a topological model of the projective plane by gluing the triangle $(0,0),(d,0),(0,d)$ and its symmetric copies in the other quadrants:

Viro's method:



Consider as point set all the integer points in your rhombus (remark: those in a particular orthant are related to the possible homogeneous monomials of degree d in three variables).

Viro's method:



Triangulate the positive orthant arbitrarily . . .

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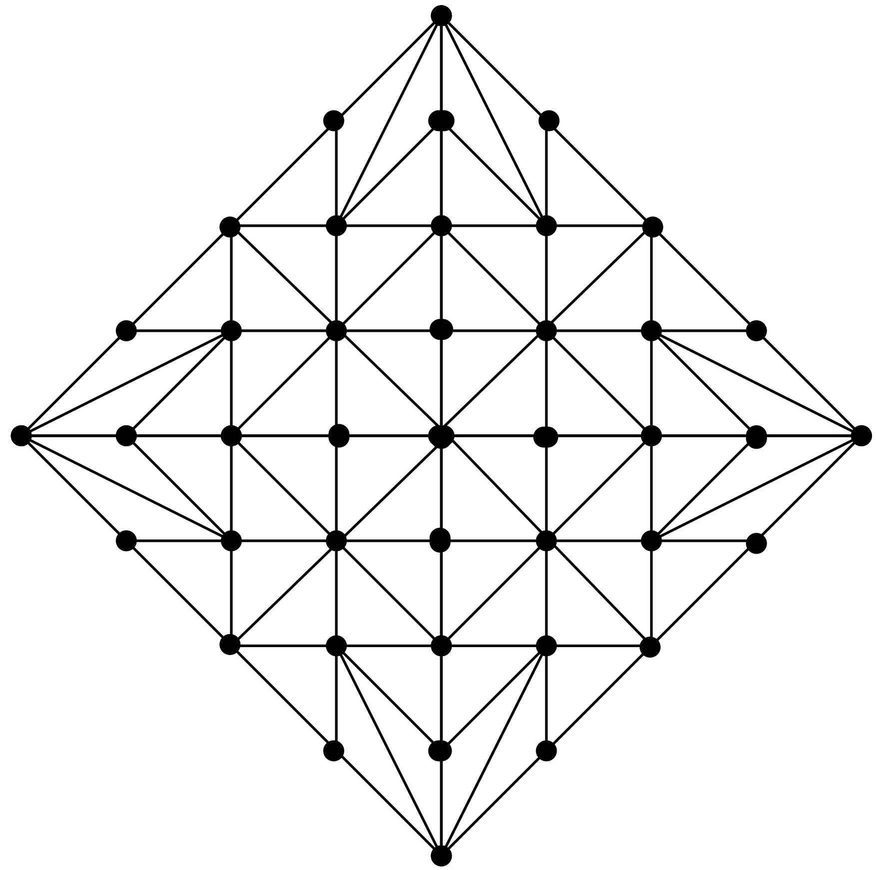
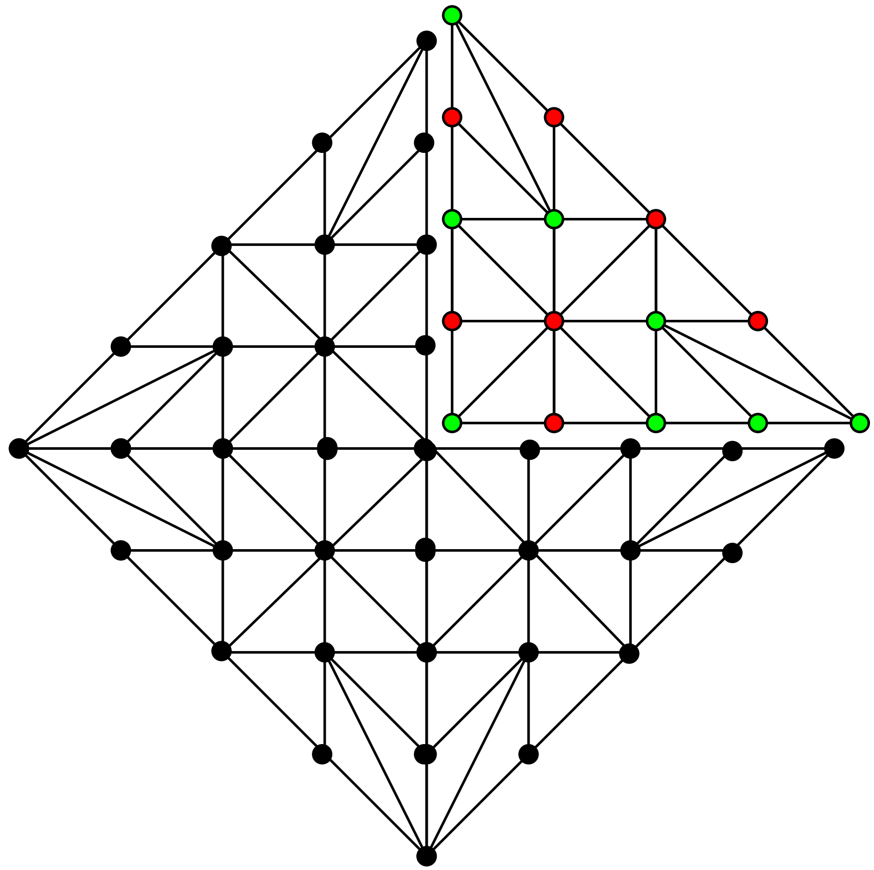


Triangulate the positive quadrant arbitrarily . . .

. . . and replicate the triangulation to the other three quadrants by reflection on the axes.

Viro's method:



Choose arbitrary signs for the points in the first quadrant

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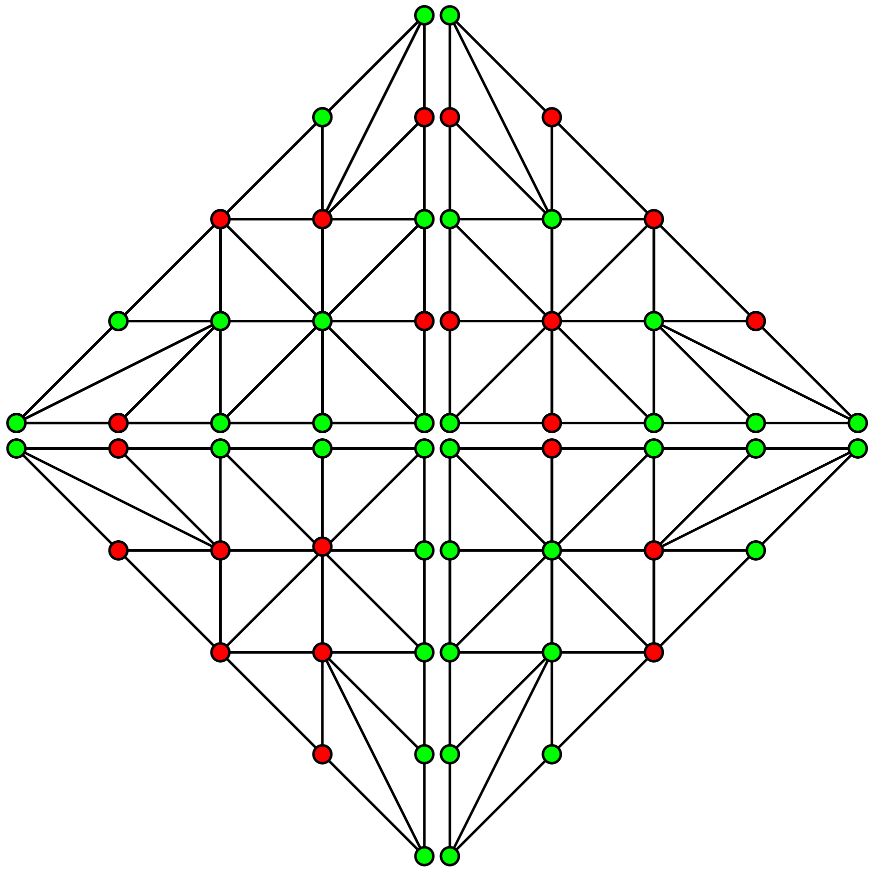
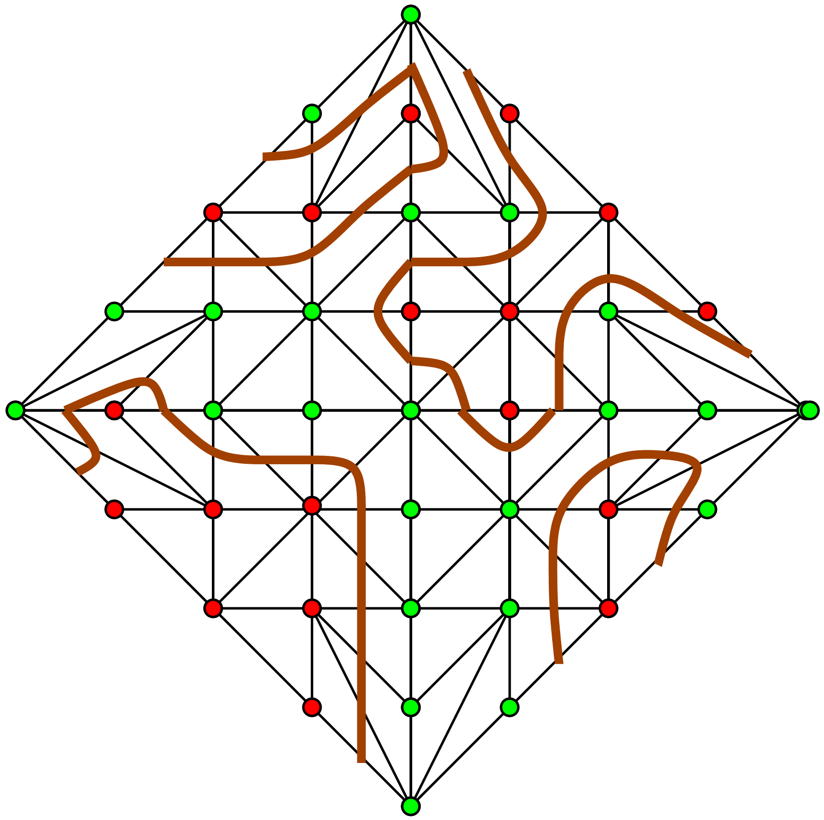


Choose arbitrary signs for the points in the first quadrant . . . and replicate them to the other three quadrants, taking parity of the corresponding coordinate into account.

Viro's method:



Finally draw your curve in such a way that it separates positive from negative points.

Viro's Theorem

Theorem (Viro, 1987) If the triangulation T chosen for the first quadrant is regular then there is a real algebraic non-singular projective curve f of degree d realizing exactly that topology.

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More precisely, let $w_{i,j}$ $(0 \leq i \leq i+j \leq d)$ denote "weights" ($\leftrightarrow$cost vector$\leftrightarrow$lifting function) producing your triangulation and let $c_{i,j}$ be any real numbers of the sign you've given to the point (i,j).

Then, the polynomial

$$f_t(x,y) = \sum c_{i,j} x^i y^j z^{d-i-j} t^{w(i,j)}$$

for any positive and sufficiently small t gives the curve you're looking for.

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• What happens if we do the construction with a non-regular triangulation?
Well, then the formula in the theorem cannot be applied (there is no possible choice of weights). But there is no known example of a curve constructed via Viro's method (with a non-regular triangulation) and which is not isotopic to a real algebraic curve of the corresponding degree. (There are examples of such curves in toric varieties other than the projective plane [Orevkov-Shustin, 2000]).

Viro's Theorem

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• What happens if we do the construction with a non-regular triangulation?
Well, then the formula in the theorem cannot be applied (there is no possible choice of weights).

• Still, the curves constructed with Viro's method (with non-regular triangulations) can be realized as **pseudo-holomorphic** curves in $\mathbb{CP}^2$ [Itenberg-Shustin, 2002].

Ragsdale's conjecture

Let $d = 2k$ be even, so that a non-singular curve of degree d consists only of ovals. An oval is called positive (or even) if it lies inside an even number of other ovals, and negative (or odd) otherwise.

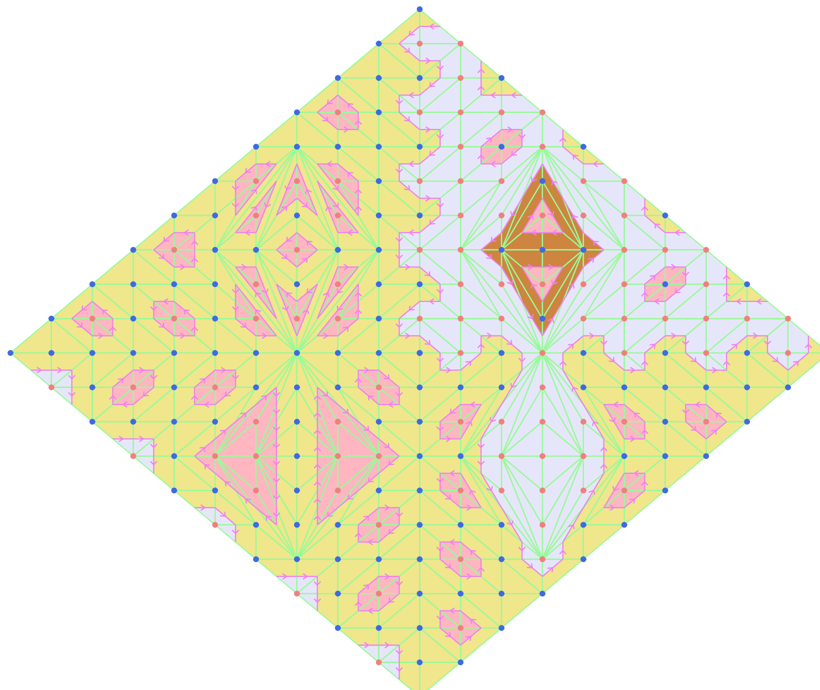
Harnack's inequality says that the total number of ovals cannot exceed $2k^2 \pm O(k)$. Virginia **Ragsdale** conjectured in 1906 (based on empirical evidence) that the numbers of positive ovals could not exceed $3k^2/2 \pm O(k)$.

In the 1930's, **Petrovskii** proved that the difference between the two numbers was bounded by

$$p - n \leq 3(k^2 - k)/2 + 1,$$

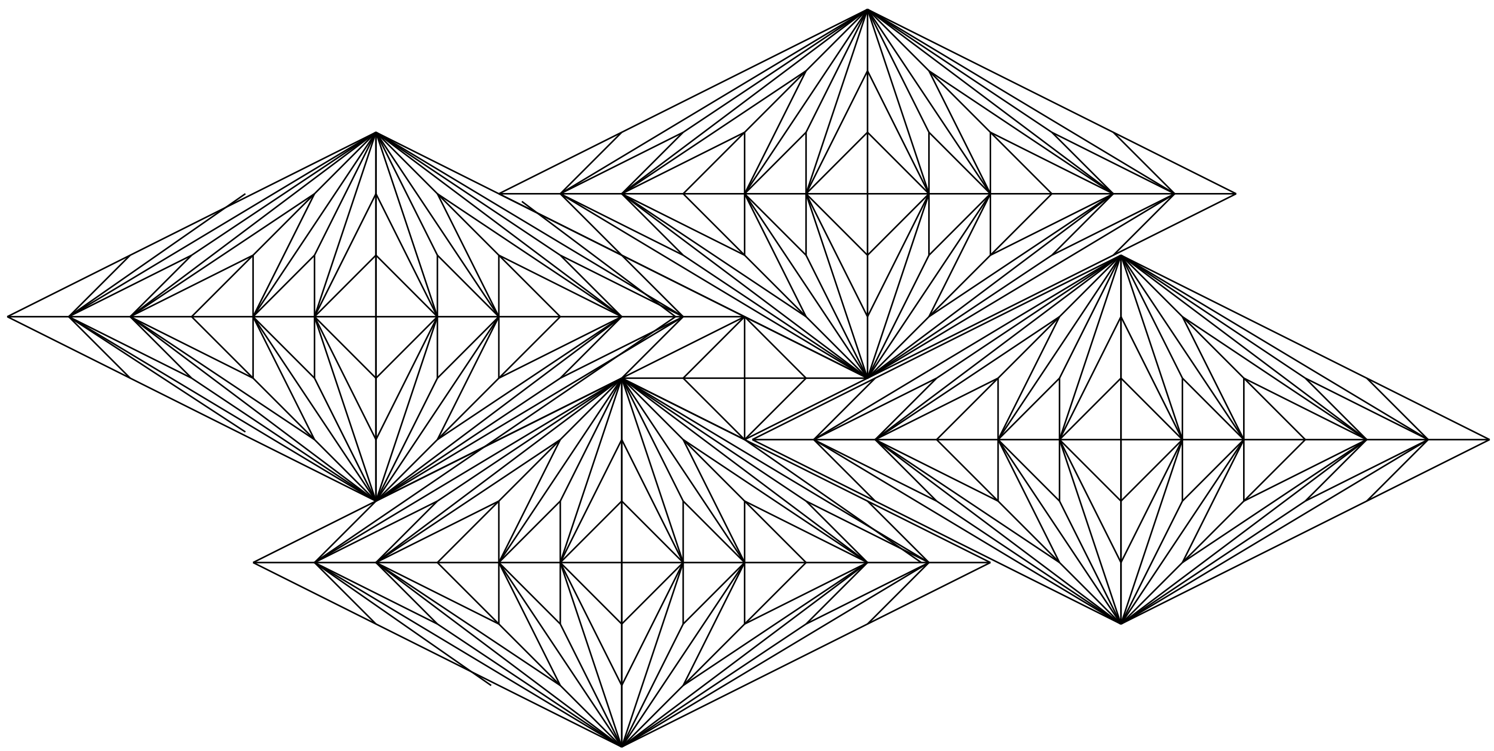
which implies $p \leq 7k^2/4 \pm O(k)$.

In 1993, **Itenberg** (using Viro's method) constructed curves having $13k^2/8 \pm O(k)$ positive ovals.



This was improved by B. **Haas** to $10k^2/6 \pm O(k)$.

Curiously enough, using **non-regular triangulations**, one can construct Viro curves with $17k^2/10 \pm O(k)$ positive ovals [Santos, 1995].



Are these curves realizable algebraically?

For comparison

Ragsdale's conjecture: 180 $k^2/120 \pm O(k)$.

Itenberg construction: 195 $k^2/120 \pm O(k)$.

Haas construction: 200 $k^2/120 \pm O(k)$.

Santos construction: 204 $k^2/120 \pm O(k)$.

Petrovskii inequality: 210 $k^2/120 \pm O(k)$.

Harnack inequality: 240 $k^2/120 \pm O(k)$.

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Remark: Petrovskii inequality is valid for pseudo-holomorphic curves (hence for Viro curves too)

The interaction between polyhedral geometry and algebraic geometry is a classical topic (Newton,. . .), but it got a big boost in the 1970's, with two results that used it in both directions:

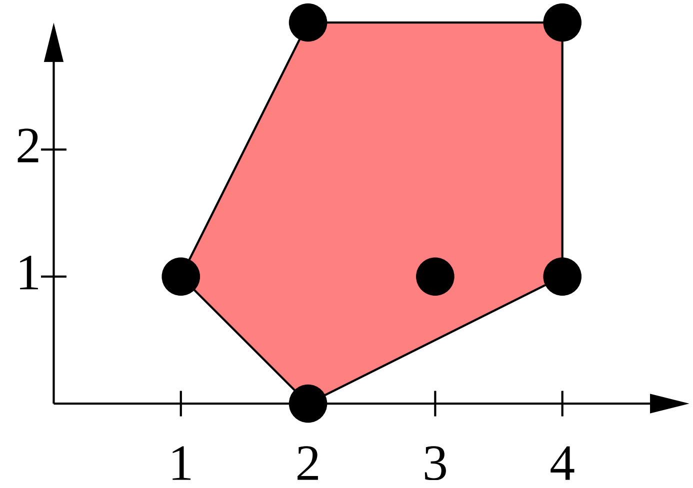
- The **Bernstein Theorem** on the number of roots of a zero-dimensional system of sparse polynomials, via mixed subdivisions of their Newton polytopes.
- The **Stanley proof** of the g-theorem on the numbers of faces of simplicial polytopes, via cohomology of toric varieties.

Bernstein's Theorem

Newton polytopes

- To every monomial $x_1^{a_1} \ldots x_n^{a_n}$ we associate its exponent vector $(a_1, \ldots, a_n)$.

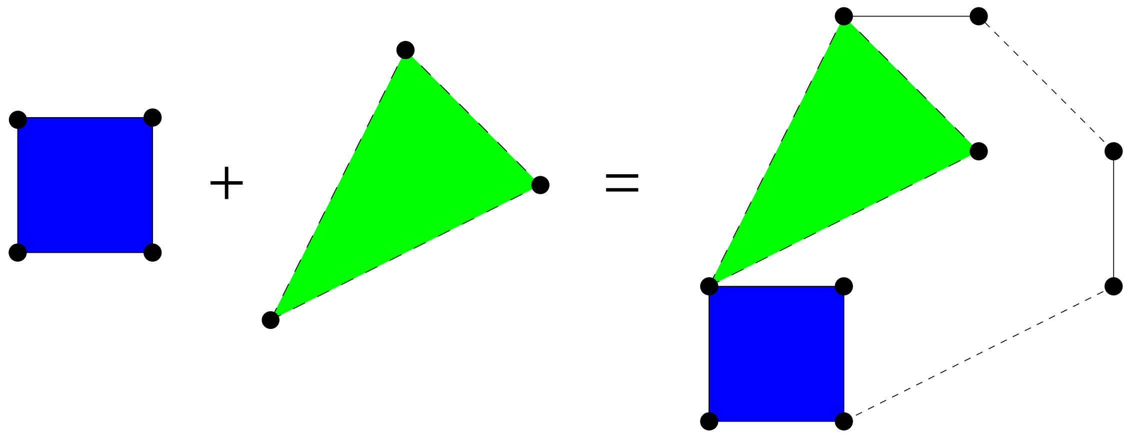
- To a polynomial $f(x_1, \ldots, x_n) = \sum c_i \mathbf{x}^{\mathbf{a}_i}$ we associate the corresponding integer point set. Its convex hull is the Newton polytope of f, $N(f)$.



The Newton polytope for the polynomial $x^2 + xy + x^3y + x^4y + x^2y^3 + x^4y^3$

Bernstein's Theorem

Theorem (Bernstein, 1975) Let $f_1, \ldots, f_n$ be n polynomials in n variables. The number of common zeroes of them in $(\mathbb{C}^*)^n$ is either infinite or bounded above by the mixed volume of the n polytopes $N(f_1), \ldots, N(f_n)$.



Mixed area of a triangle and a rectangle.

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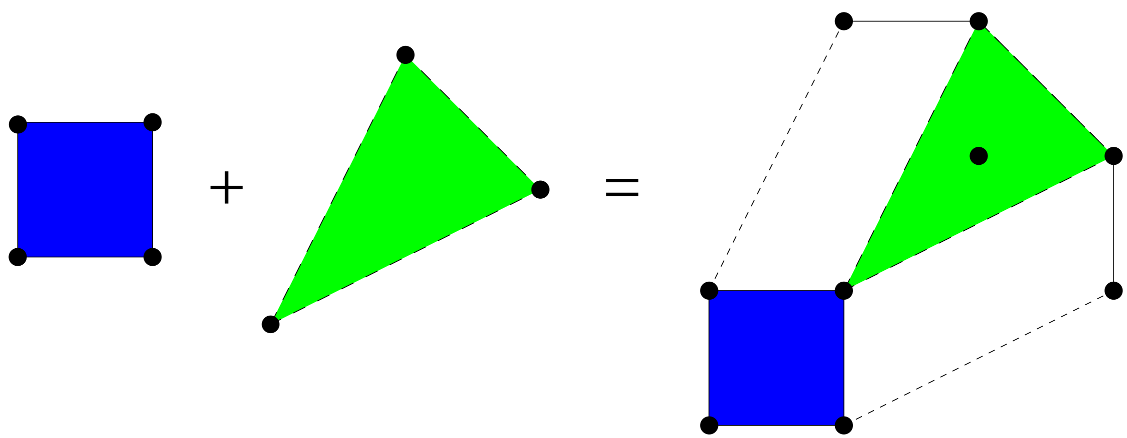

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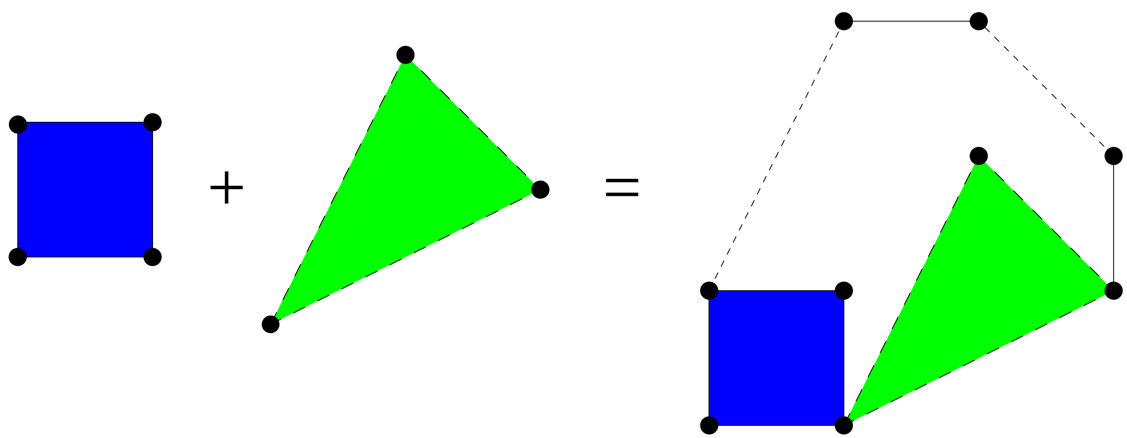

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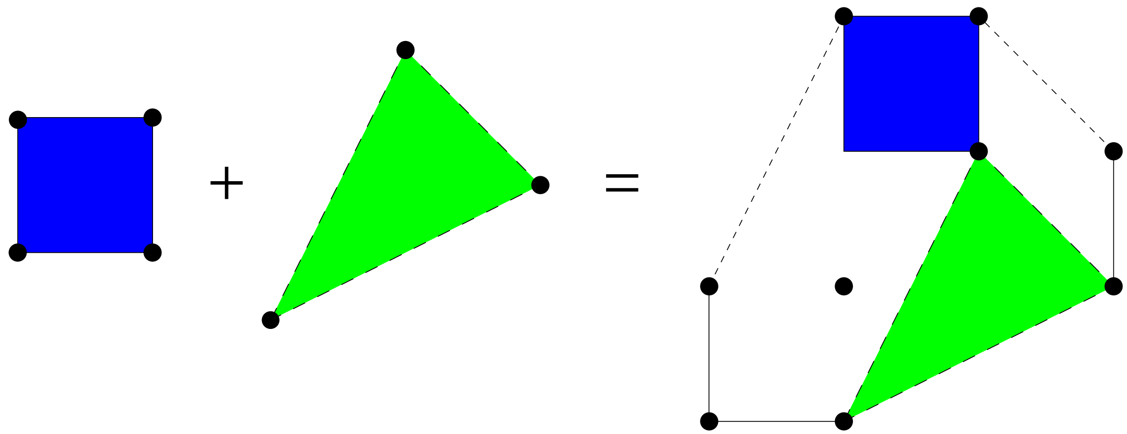

Mixed area of a triangle and a rectangle.

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Theorem (Bernstein, 1975) Let $f_1, \ldots, f_n$ be n polynomials in n variables. The number of common zeroes of them in $(\mathbb{C}^*)^n$ is either infinite or bounded above by the mixed volume of the n polytopes $N(f_1), \ldots, N(f_n)$...
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What is the mixed volume?

Mixed volume

Definition 1: Let $Q_1, Q_2, \ldots, Q_n$ be n polytopes in $\mathbb{R}^n$. Their mixed volume $\mu(Q_1, \ldots, Q_n)$ equals

$$\sum_{I \subset \{1,2,\ldots,n\}} (-1)^{|I|} \operatorname{vol}\left(\sum_{j \in I} Q_i\right).$$

Mixed volume

Definition 2: Let $Q_1, Q_2, \ldots, Q_n$ be n polytopes in $\mathbb{R}^n$. Their mixed volume $\mu(Q_1, \ldots, Q_n)$ equals

the coefficient of $\lambda_1 \lambda_2 \cdots \lambda_n$ in the homogeneous polynomial $\mathrm{vol}(\lambda_1 Q_1 + \cdots + \lambda_n Q_n)$.

Mixed volume

Definition 3: Let $Q_1, Q_2, \ldots, Q_n$ be n polytopes in $\mathbb{R}^n$. Their mixed volume $\mu(Q_1, \ldots, Q_n)$ equals

the sum of the volumes of the mixed cells in any fine mixed subdivision of $Q_1 + \cdots + Q_n$.

Mixed volume

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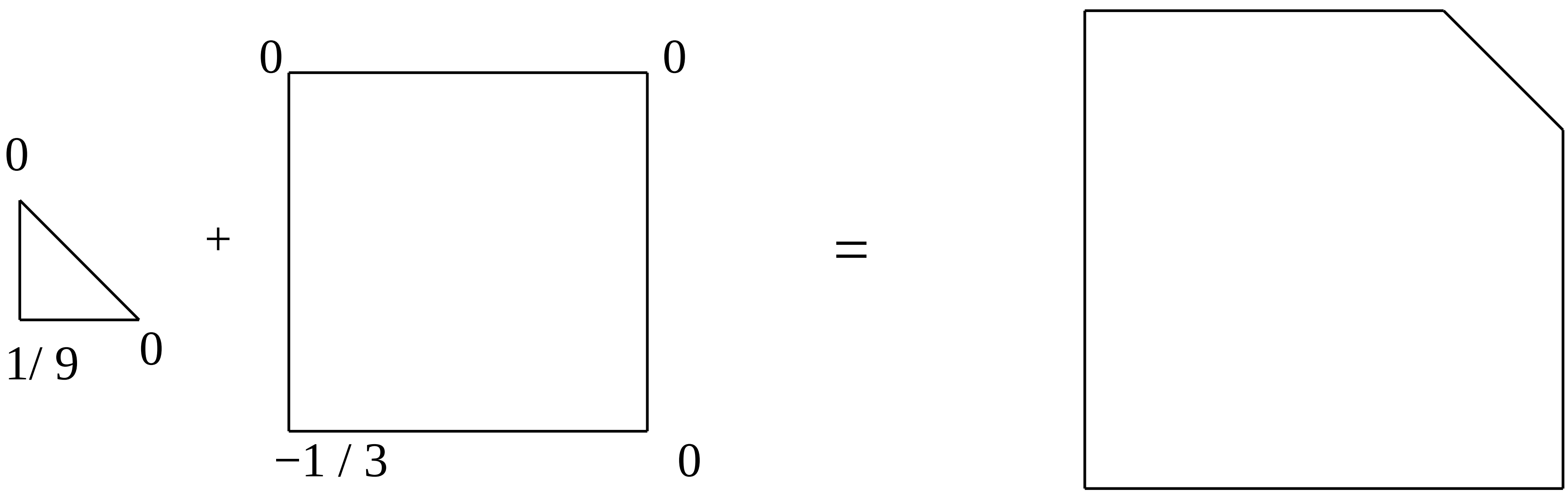
the sum of the volumes of the mixed cells in any fine mixed subdivision of $Q_1 + \cdots + Q_n$.

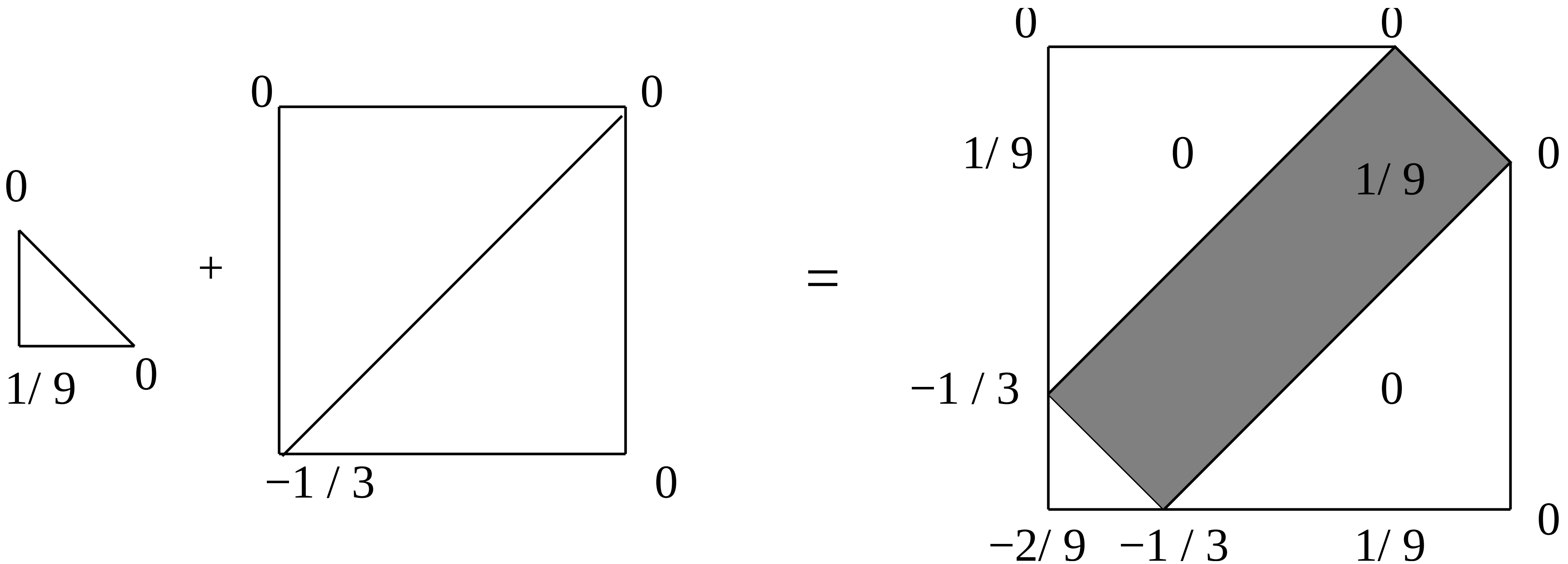
In particular, to compute the number of zeroes of a sparse system of polynomials $f_1, \ldots, f_n$ one only needs to compute a "fine mixed subdivision" of $N(f_1) + \cdots + N(f_n)$.

A cooking recipe for fine mixed subdivisions:



Choose sufficiently generic (e.g. random) numbers $w_a \in \mathbb{R}$, one for each a in each of the Q_i's

A cooking recipe for fine mixed subdivisions:



Use the numbers to lift the points of $Q_1 + \cdots + Q_n$ and compute the lower envelope of the lifted point configuration.

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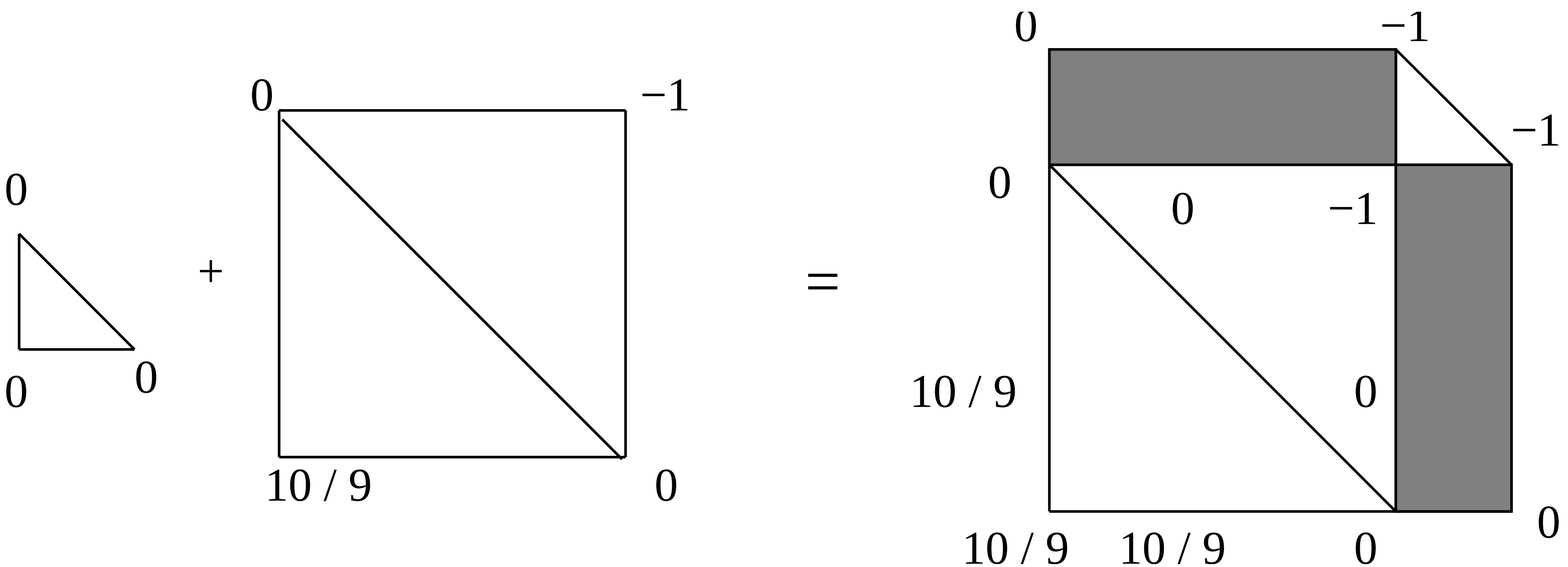

Use the numbers to lift the points of $Q_1 + \cdots + Q_n$ and compute the lower envelope of the lifted point configuration.

The polyhedral Cayley Trick

. . . as it turns out, for every family of polytopes $Q_1, \dots, Q_n$ in $\mathbb{R}^d$ there is another polytope $\mathcal{C}(Q_1, \dots, Q_n)$ in $\mathbb{R}^{n+d-1}$ such that

$$\text{mixed subdivisions of } Q_1, \dots, Q_n \leftrightarrow \text{subdivisions of } \mathcal{C}(Q_1, \dots, Q_n)$$

$$\text{fine mixed subdivisions of } Q_1, \dots, Q_n \leftrightarrow \text{triangulations of } \mathcal{C}(Q_1, \dots, Q_n)$$

That is to say, the number of roots of a sparse system of polynomials can be computed via triangulations.

How to compute the roots

- From (the proof of) Bernstein's theorem one gets more than the number of roots.
- Also, a germ at $t = 0$ of an algebraic curve $(x(t))$ such that $(x(1))$ is a root (roots are in bijection to the mixed cells in the mixed subdivision, counted with their volume; the germs are given by the slopes of mixed cells in the lifting that was used to construct the mixed subdivision).
- Using the germ, one can follow the curve numerically until reaching the solution

  These are the so-called homotopy methods or numerical continuation methods.

The g-theorem

Face numbers of polytopes

A polytope P of dimension d has faces of dimensions -1 to d. The f-vector of P is the vector $f = (f_{-1}, f_0, \ldots, f_d) \in \mathbb{N}^{d+2}$ where f_i is the number of faces of dimension i of P.

Some f-vectors:

segment: $(1,2,1)$

n-gon: $(1,n,n,1)$

cube: $(1,8,12,6,1)$

octahedron: $(1,6,12,8,1)$

dodecahedron: $(1,20,30,12,1)$

icosahedron: $(1,12,30,20,1)$

d-simplex: $\left(1, d+1, \binom{d+1}{2}, \ldots, \binom{d+1}{2}, d+1, 1\right)$.

Big question:

What are the possible f-vectors of polytopes?

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Definition: Let P be a polytope with f-vector $(f_{-1}, f_0, \ldots, f_d)$. For each $k = 0, \ldots, d$ let

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{d-k} f_{i-1}.$$

and $g_k = h_k - h_{k-1}$ for $k = 1, \ldots, d/2$. ($g_0 := h_0 = 1$).

These are called the h-vector and the g-vector of P.

g-theorem

Theorem [Billera-Lee 1981, Stanley 1980] A vector f of positive integer entries is the f-vector of a simplicial polytope if and only if the h-vector and g-vector obtained from it satisfy:

1. h is symmetric ("Dehn-Sommerville relations").

2. g is non-negative ("lower bound theorem").

3. g is an M-sequence (M is for Macaulay).

Comments

The sufficiency part of the g-theorem was proved by Billera and Lee via an explicit construction of a simplicial polytope with given f-vector.

The Dehn-Sommerville equations are a generalization of Euler's formula in two senses:

- $h_d = h_0$ is Euler's formula.

- The equations follow from applying Euler's formula to links of different dimensions in the complex (hence, the equations are valid for all homology simplicial spheres; this was the original proof by Sommerville).

From the algebraic point of view, Dehn-Sommerville is Poincaré duality: the h-vector of P is the vector of (even) Betti numbers of the toric variety V_P. If P is simplicial, V_P is (almost) non-singular, and the Betti numbers are symmetric.

Comments

In this same setting, the lower bound theorem is equivalent to the "hard Lefschetz Theorem" for the intersection homology of the toric variety V_P.

To prove the third condition in the g-theorem (the M-sequence part) Stanley (1980) used Cohen-Macaulayness of the cohomology ring of V_P (more precisely, the fact that the ring is generated by classes ofdegree one).

This cohomology ring can be directly described from the combinatorics of the polytope [Danilov 1978]: it equals a certain quotient of the "Stanley-Reisner ring" of the simplicial complex ∂P.

4. Summing up

(quick regular triangulations reminder)

A

Start with a point set A,

A **n,**

Start with a point set A, with n elements

A n, k

Start with a point set A, with n elements and rank k.

h A n, k

Start with a point set A, with n elements and rank k.

Then, every vector of heights $h : A \to \mathbb{R}$,

h A n, k S

Start with a point set A, with n elements and rank k.

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T h A n, k S

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Then, every vector of heights $h : A \longrightarrow \mathbb{R}$, defines a regular subdivision S.

If h is generic, S is actually a triangulation.

T h A n k S

T h A n k S

T h A n k S

T h A n k S

T h A   n k S

T h A n k S

T h A n k S

T h A n k S