Lab 1:

1. Compute a random 4-dimensional polytope P as the convex hull of 10 random points using `rand_sphere(4,10)`. Run `VISUAL` to see a Schlegel diagram. How many 3-dimensional polytopes do you see? How many facets does P have?

2. Using a for-loop in polymake, generate 100 random 3-dimensional polytopes and 5-dimensional polytopes using the `polymake` function `rand_sphere`. Compute their f-vectors. What pattern (called Euler's relation) do you notice?

3. The graph of a soccer ball (below) is 3-regular and has exactly 32 faces, some of which are pentagonal (5-gonal) and some of which are hexagonal (6-gonal). Let p_5 (respectively p_6) be the number of pentagonal (respectively hexagonal) faces. Write p_5 in terms of p_6 using Euler's relation. Then determine the values of p_5 and p_6.

4. For each $4 \geq d < n \leq 9$, examine the graph of the cyclic polytope $C_d(n)$. The function `cyclic(d,n)` constructs the polytope $C_d(n)$. Use the function `VISUAL_GRAPH` to view the graph. How many edges are in each graph?

5. Try finding coordinates for eight points $s_1, \ldots, s_8$ in $\mathbb{R}^3$ such that $\mathrm{conv}(s_1, \ldots, s_8)$ has the same face lattice as an honest cube, but it has fewer coplanarities. How well can you do? Can you make the only 4-point coplanarities on the six facets?

6. Prove that the regular unit cube (say 1cm=unit) of sufficiently high dimension can fit inside it the whole city of New York.

7. Either consult your favorite book on convexity OR (better yet) provide your very own proofs of Caratheodory's theorem, Radon's theorem, Helly's theorem, and Krein-Milman theorem.

8. Let P be the polytope of all 3×4 matrices whose row sums are $(4, 4, 4)$ and column sums are$(3, 3, 3, 3)$. How many vertices does P have? How many facets does P have?

Lab 2:

1. Use `polymake` to construct the d-simplex and d-cube for $d = 3, 4$. Visualize the face lattice and the Schlegel diagram of those polytopes. Draw the Schlegel diagram of some other cute polytopes (e.g. platonic solids?).

2. Starting in dimension three, construct a polytope as the convex hull of a collection of points using the `polymake` function `POINTS`. How many vertices (use `N_VERTICES`) can you get? Try to get the largest number possible. How many facets can you get by trying?

3. Construct the cyclic polytopes $C_3(9)$. Compute the dual of the cyclic polytope $C_3(9)$. What does its graph look like?

4. Construct an octahedron using `polymake` in at least two different ways!

5. We prove Farkas lemma, using Fourier-Motzkin elimination.

   Farkas lemma. For a matrix $A \in \mathbb{R}^{m\times d}$ and a vector $b \in \mathbb{R}^m$, exactly one of the following holds true:
   (a) There is a $x \in \mathbb{R}^d$ such that $Ax \leq b$;
   (b) There is a $z \in \mathbb{R}^m$ such that $z \geq 0, z^T A = 0$ and $z^T b < 0$.

   Prove another version of Farkas lemma: $\{x : Ax = b, x \geq 0\} = \emptyset \iff \{y : y^T A \geq 0, y^T b < 0\} \neq \emptyset$.

6. Consider the polytope P defined by the following system of inequalities:

$$\begin{aligned} -x - 4y &\leq 9 \\ -2x - y &\leq -4 \\ x - 2y &\leq 0 \\ x &\leq 4 \\ 2x + y &\leq 11 \\ -2x + 6y &\leq 17 \\ -6x - y &\leq -6. \end{aligned}$$

   Use Fourier-Motzkin elimination to eliminate the variable y. What are the smallest and largest values of x? Draw the polytope P to confirm this. Enter the polytope P into `polymake`. How do (the coordinates of) the vertices confirm this same information?

7. How many faces are there in a d-dimensional cube? That is, what is the value of $f_0 + f_1 + f_2 + \cdots + f_d$? How many faces are there in a d-dimensional cross-polytope? How many faces are there in a d-dimensional simplex? How about a cross-polytope?

8. Let

$$P = conv(\{(-1, 3, 1, 2), (-1, 3, -1, 1), (-1, -1, 1, 1), (-1, -1, -1, 0),$$

$$(3,-1,1,-3),(3,-1,-1,4)\}).$$

Use polymake to: Determine the vertices of the polar polytope (HINT: polar cone!) and draw the graphs of P and its polar. What are the facets? How many are there?

9. Construct two polytopes in 4-dimension that have the same graph, yet they do not have the same face lattice!! (This exercise is actually part of the Polymake web site!).

Lab 3

1. Suppose we consider the spanning trees of a graph as vectors of 0's or 1's. Each tree-vector has as many entries as the number of edges and there is a 1 in entry e if the tree contains edge e and 0 otherwise. Figure the set of inequalities of the Spanning tree polytope (i.e. the convex hull of those vectors.

2. Follow the Polymake tutorial investigating the matching polytopes of a graph. It is available at

   `http://polymake.org/doku.php/tutorial/matching_polytopes`

3. An $n \times n$ magic square with sum S is a filling of an $n \times n$ table with non-negative integers so that the n numbers in each row and the n numbers in each column add up to S. How many 4×4 magic squares with sum 20 are there?

4. Use LattE to compute the Ehrhart series and the Ehrhart polynomials of various lattice polytopes. Can you construct one that has negative coefficients?

5. Consider again P the polytope of all 3×4 matrices whose row sums are $(4, 4, 4)$ and column sums are$(3, 3, 3, 3)$. Using Polymake find the matrix X that minimizes the sum $2X_{11} + 7X_{22} + 3X_{34}$. Is this matrix integral?

   (information on how to do linear programs can be found at

   `http://www.polymake.org/doku.php/tutorial/ilp_and_hilbertbases?s[]=linearprogram`

6. Euro coins come in 1 cent, 2 cent, 5 cent, 10 cent, 20 cent, 50 cent, 1 euro, and 2 euro pieces. (100 cents is equal to 1 euro.) How many ways are there to split a 5 euro bill into these smaller pieces? (Hint: Use `LattE` and describe a 7-dimensional polytope in 8-dimensional space using eight inequalities and one equation.)

7. Consider the 3-dimensional cube C with vertices $(\pm 1, \pm 1, \pm 1)$. Use TOPCOM to consider the triangulations of C. How many triangulations are there? Do they all use the same number of tetrahedra? Pick one triangulation: Compute the volume of each tetrahedron in the triangulation. What is the sum of these volumes?

Lab 4

1. Consider the vector configuration $(1,0),(-1,0),(0,1),(0,-1),(1,1),(-1,-1)$. Use `topcom`'s command line program `points2triangs` to compute the number of triangulations. Draw the triangulations. How many are there? How many are regular?

2. How many different ways are there to triangulate a pentagon? How many different ways are there to triangulate a hexagon? Prove the formula we stated in the lecture!

3. Show that four points (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4) lie on a circle if and only if the four points of the form $(x_i, y_i, x_i^2 + y_i^2)$ lie on a plane.

4. Compute the Voronoi diagram of a set of points 8 random points in the plane. (YES, Polymake know how to do it!).

5. Experiment with polymake to find values for the numbers a and b to make the linear program

$$\begin{array}{ll} \text{maximize} & x_1 + x_2 \\ \text{subject to} & \begin{cases} ax_1 + bx_2 \leq 1 \\ x_1 \geq 0 \\ x_2 \geq 0 \end{cases} \end{array}$$

   (a) have an optimal solution.

   (b) be infeasible.

   (c) be unbounded.

6. Compute the volume of the polytope given by the five inequalities

$$\begin{aligned} -x_1 + x_2 &\leq 2 \\ x_2 &\leq 4 \\ 3x_1 + 2x_2 &\leq 15 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

   using the Lawrence method.

7. Prove that there are only five Platonic solids.

8. Determine the graph of a 3-polytope with 13 vertices and 25 edges. Can you find vertices for this polytope and verify your work in polymake?

9. The **order polytope** of a poset was introduced in the lecture. Prove that the vertices of an order polytope have always integer coordinates. Use Polymake to discover a theorem! Find out that what the vertices are in terms of the poset?

10. You are linear algebra savvy, you can solve $Ax = b$, but do you know how to solve $Ax = b$ with x integral vector? Great oral exam question for graduate students!