Computing with Lattice Points in Polyhedra

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Le Menu

PARAMETRIC POLYTOPES

MULTIVARIATE EHRHART's THEORY

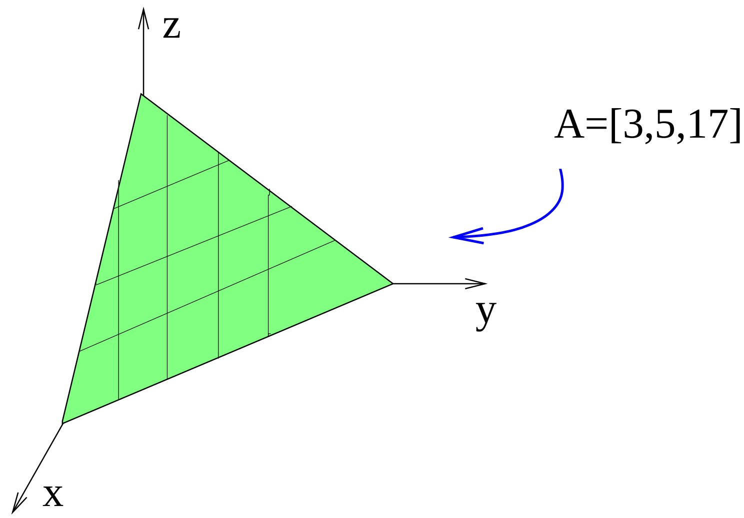
GENERALIZATIONS AND YOUR CREDIT CARD!

HILBERT and GRAVER BASES.

THE PROBLEM!!!

Given a polytope, $P = \{x | Ax = b,\ x \geq 0\}$,

COUNT HOW MANY LATTICE POINTS are inside P.



$$\phi_A(b) = \#\{(x, y, z) | 3x + 5y + 17z = b,\ x \geq 0, y \geq 0, z \geq 0\}$$

More general...

Let

$$\phi_A(b) = \#\{x : Ax = b, x \geq 0, \quad x \quad \text{integral}\}.$$

It counts **the number of lattice points inside convex polyhedra with fix matrix** A.

1. (APPLIED MATHEMATICIAN) Fast exact evaluation of $\phi_A(b)$ for fixed values of b. or compute a "short" representation of $\phi_A(b)$.

2. (PURE MATHEMATICIAN) To compute explicit exact formulas in terms of the parameters b_i.

EXAMPLE When $A = [3, 5, 17]$, a short formula for $\phi_A(b)$ would be a generating function!

$$\sum_{n=0}^{\infty} \phi_A(n) t^n = \frac{1}{(1 - t^{17})(1 - t^5)(1 - t^3)}.$$

From that, you can see that $\phi_A(100) = 25, \phi_A(1110) = 2471$, etc...

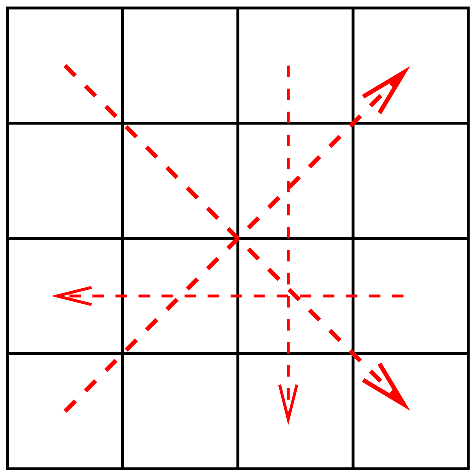
Disclaimers: Whenever I say counting, I mean **EXACT COUNTING**. There is a rich and exciting theory of estimation and approximation, but that is not us!

We really care to get this rational functions **In PRACTICE!!**

MOTIVATION

Combinatorics

Many discrete structures can be counted this way: e.g. matchings on graphs, Hamiltonian cycles, t-designs, linear extensions of posets, **MAGIC squares:**



12	0	5	7
0	12	7	5
7	5	0	12
5	7	12	0

5

QUESTION:**HOW MANY** 4×4 **magic squares with sum n are there?** Call this number $M_{4\times 4}(n)$.

?	?	?	?	24
?	?	?	?	24
?	?	?	?	24
?	?	?	?	24
24	24	24	24	24

The possible tables are non-negative integer solutions of the system of equations: Four equations, one for each row sum and column sum. For example,

$x_{11} + x_{12} + x_{13} + x_{14} = 24$, **first row**
$x_{13} + x_{23} + x_{33} + x_{43} = 24$, **third column**

Generating Function Formulas

The problem we have is equivalent to determining a short expression for $\sum_{n=0}^{\infty} M_{4\times 4}(n)t^n$.

Because we are dilating a polytope, as we increase the magic sum n, one can prove the following theorem:

Theorem The number of 4×4 magic squares with magic sum n has a **toric rational generating function**:

$$\frac{t^8 + 4\,t^7 + 18\,t^6 + 36\,t^5 + 50\,t^4 + 36\,t^3 + 18\,t^2 + 4\,t + 1}{(-1+t)^4\,(-1+t^2)^4}$$

Compiler Design

How often is a certain instruction I of the computer code executed?

Example:

```
void proc(int N, int M)
{
int i,j;
for (i=2N-M; i<= 4N+M-min(N,M), i++)
  for(j=0; j<N-2*i; j++)
    I;
}
```

$$\{(i,j) \in \mathbb{Z}^2 | i \geq 2N-M, i \leq 4N+M-min(N,M),\ j \geq 0, j-2i \leq N-1\}$$

Algebra and Number Theory

Number Theory Relations to the theory of partitions, Geometry of Numbers. For example, Frobenius problem: Given relatively prime $a_1, ..., a_n$ what is the highest value of N for which $a_1x_1 + \cdots + a_nx_n = N$, $x_i \geq 0$ is integral INFEASIBLE.

Representation Theory: The calculation of multiplicities and tensor product multiplicities for decomposition of representations into irreducible representations are given by Gelf'and-Tsetlin polytopes, Hive Polytopes (Knutson-Tao), Berenstein-Zelevinsky polytopes, Lattice-Path cones (Littelmann). Kostant's partition function for simple Lie algebras can be seen naturally as counting lattice points.

Commutative Algebra The Hilbert series of monomial algebras and Grobner bases of toric ideals can be seen as problems of counting lattice points in certains polytopes.

EHRHART's THEORY

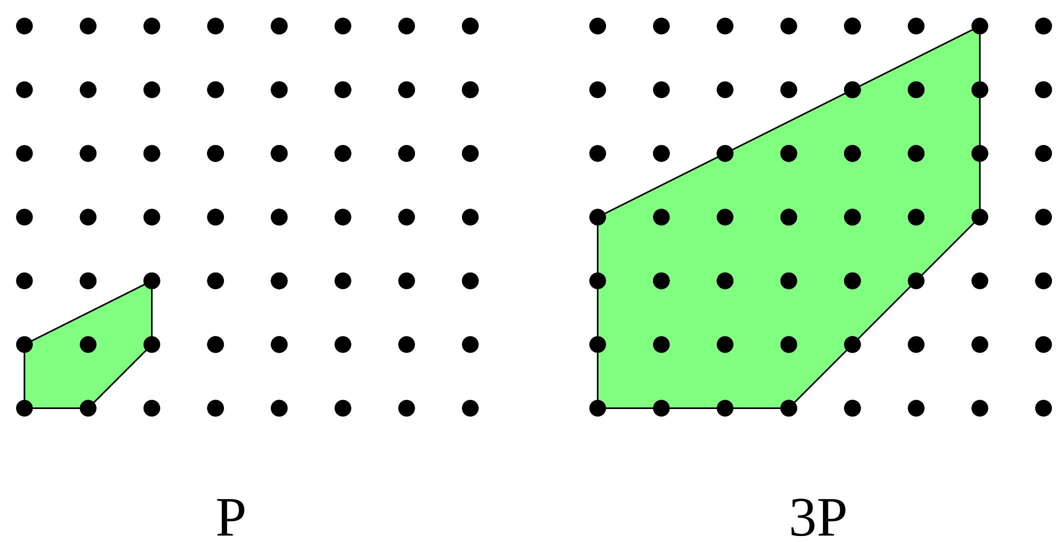
& THE DESCRIPTION OF

$$\phi_A(b)$$

Dilations of Polyhedra

Let P be a convex polytope in $\mathbb{R}^d$. For each integer $n \geq 1$, let

$$nP = \{nq | q \in P\}$$



Ehrhart Counting function

For P a d-polytope, let

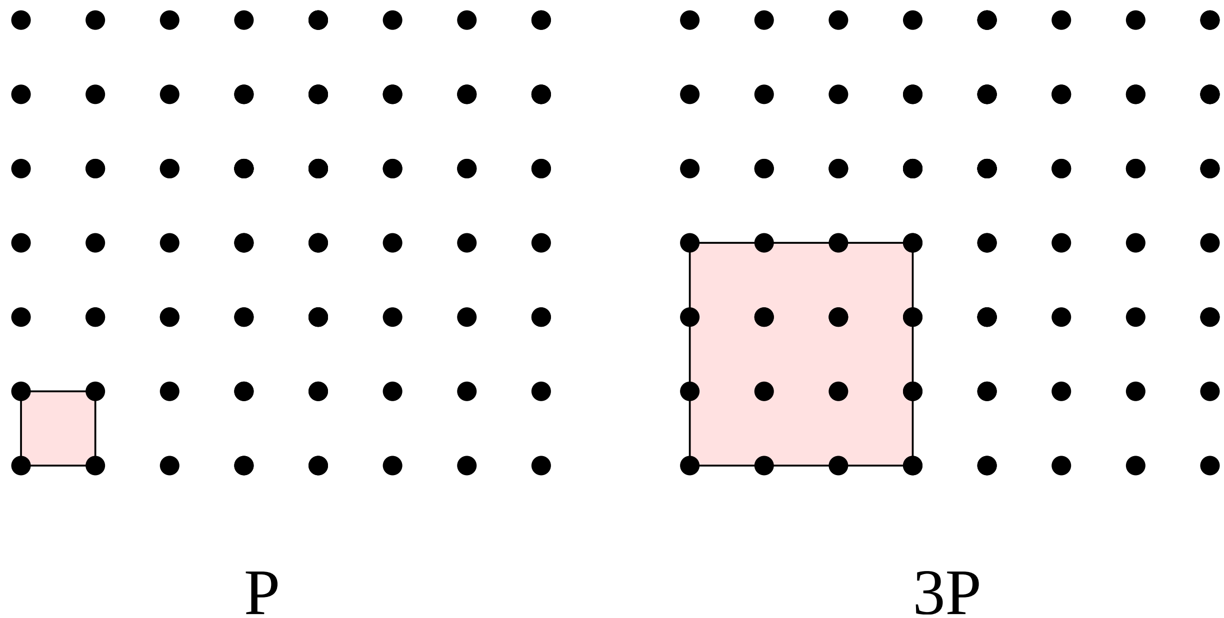
$$i(P,n) = \#(nP \cap \mathbb{Z}^d) = \#\{q \in P \ | nq \in \mathbb{Z}^d\}$$

This is the number of lattice points in the dilation nP.

Similarly if P° denotes the interior of P.

$$i(P^\circ, n) = \#\{q \in P - \partial P | \ nq \in \mathbb{Z}^d\}$$

Example 1: Cubes

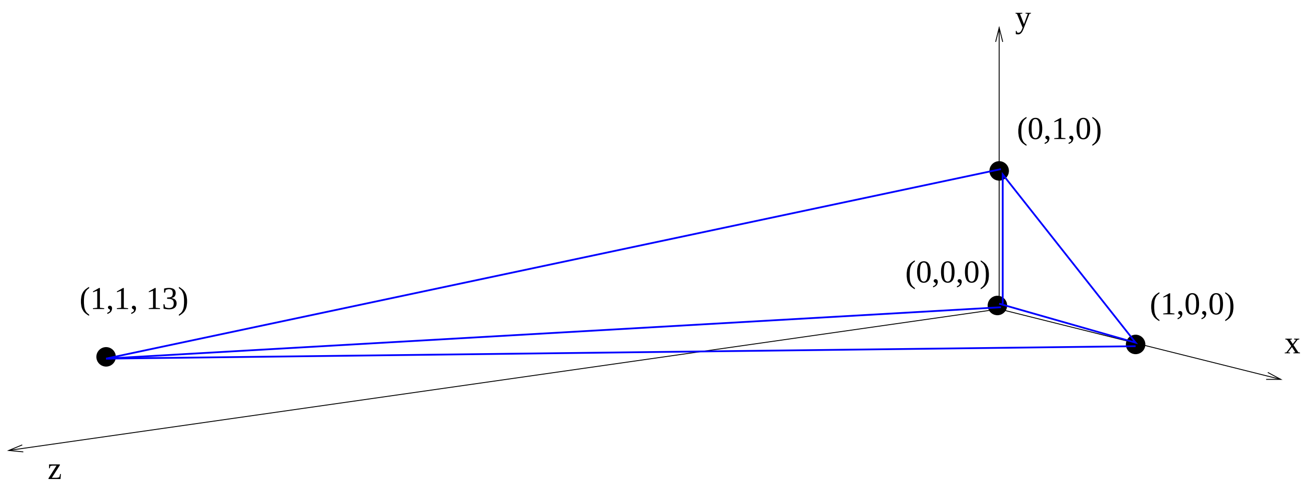


$$i(P, n) = (n+1)^2 \qquad i(P^\circ, n) = (n-1)^2$$

In general for a d-dimensional unit cube we have $i(P, n) = (n+1)^d$

Example 2

Let P be the tetrahedron



Then

$$i(P,n) = \frac{13}{6}n^3 + n^2 - \frac{1}{6}n + 1$$

WARNING: The coefficients of Ehrhart polynomials can be negative!

Example 3: MAGIC SQUARES polytopes

WARNING: The theory for polytopes with fractional vertices is more complicated.

We can consider the convex polytope inside $\mathbb{R}^{n^2}$ of magic $n \times n$ squares of magic sum 1. For example, for $n = 3$ the vertices are

1/3	0	2/3
2/3	1/3	0
0	2/3	1/3

2/3	0	1/3
0	1/3	2/3
1/3	2/3	0

0	2/3	1/3
2/3	1/3	0
1/3	0	2/3

1/3	2/3	0
0	1/3	2/3
2/3	0	1/3

In this case the Ehrhart counting function is not a polynomial, it is a *quasipolynomial!*

$$i(P, s) = \begin{cases} \frac{2}{9}s^2 + \frac{2}{3}s + 1 & \text{if } 3|s, \\ 0 & \text{otherwise,} \end{cases}$$

Ehrhart-Macdonald Theorem

Theorem (E. Ehrhart 1962, I. Macdonald 1963)

Let P be a full dimensional *rational polytope*. Then $i(P, n)$ is univariate quasipolynomial, the Ehrhart quasipolynomial of P, in the dilation variable n and of degree $dim(P)$ whose leading term on each quasipolynomial piece equals the volume of P.

Moreover, when the coordinates of the vertices of P are integers $i(P, n)$ is a polynomial.

A Generalized version

Theorem Let P be a convex rational d-polytope. Let f be any homogeneous polynomial function in $\mathbb{Z}[x_1, x_2, \ldots, x_d]$ of degree D. Then the counting function

$$i_{P,f}(n) = \sum_{\alpha \in nP \cap \mathbb{Z}^d,} f(\alpha)$$

is a quasipolynomial of degree $d + D$ with rational coefficients on the variable n. Its leading coefficient equals the integral of f over the polytope P.

Example

Suppose the polytope P is the unit square $[0,1]^2$, and that $f(x,y)$ is of the form $x^k y^k$. Then

$$
\begin{aligned}
i(P,n) =& n^2+2n+1=(n+1)^2 \\
i(P,xy,n) =& 1/4\,n^4+1/2\,n^3+1/4\,n^2 \\
i(P,x^2y^2,n) =& 1/9\,n^6+1/3\,n^5+\frac{13}{36}\,n^4+1/6\,n^3+1/36\,n^2 \\
i(P,x^3y^3,n) =& 1/16\,n^8+1/4\,n^7+3/8\,n^6+1/4\,n^5+1/16\,n^4
\end{aligned}
$$

A Key Structure Theorem.

Theorem. *For a $d \times n$ integral matrix A and a parameter vector $b \in cone(A)$,*

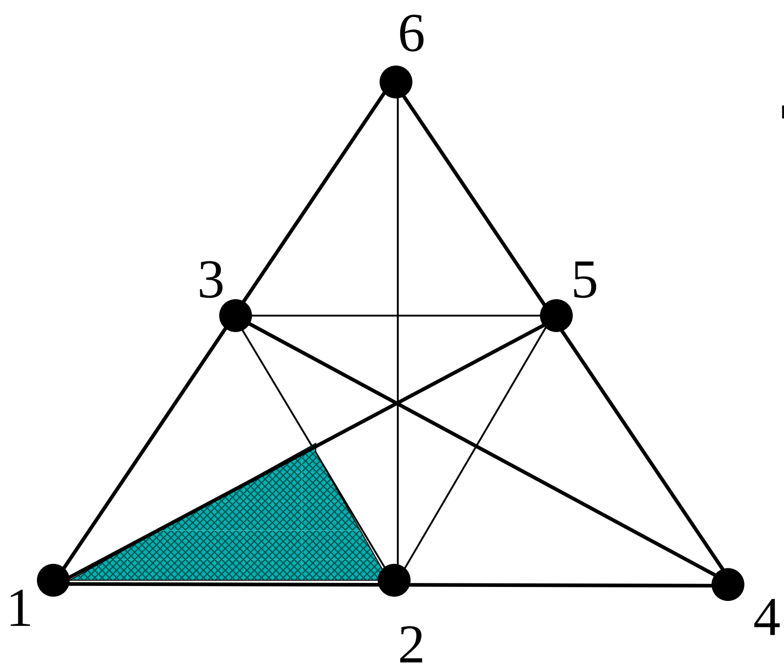
- *there exist a finite decomposition of $\mathbb{Z}^d \cap cone(A)$ such that ϕ_A is a multivariate polynomial of degree $n-d$ in each piece. The number $n-d$ is the dimension of the polytope $\{x | Ax = bx \geq 0\}$.*

- *More precisely, $cone(A)$ can be decomposed into pieces, called* chambers, *such that, for all integral vectors b inside a chamber the function $\phi_A(b)$ can be written as a fixed polynomial function of degree $n-d$ in the variables $b_1, \ldots, b_d$ plus a "correction polynomial" of smaller degree. The correction terms depend periodically on the values of $b_1, b_2, \ldots, b_d$.*

- *The chambers are convex polyhedral subcones of $cone(A)$, that subdivide its interior and their union equals $cone(A)$.*

Example

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 \end{bmatrix}$$



Two dimensional slice of the cone Ax=b x>=0.

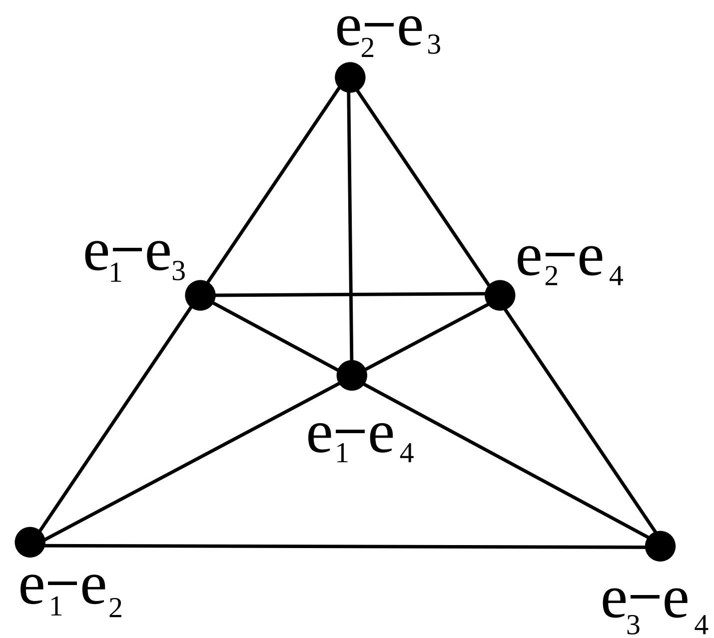
Here is the formula for the chamber marked in the picture.

$$\phi_A(b_1, b_2, b_3) = \frac{b_2 b_3}{2} + \frac{b_2 b_3^2}{8} - \frac{b_3^2}{24} + correction$$

$$correction = \begin{cases} 1 + \frac{b_2}{2} + \frac{2b_3}{3} & \text{if } b_1 = 0 \text{ and } b_2 = 0 \ mod 2 \\ \frac{1}{2} + \frac{b_2}{2} + \frac{5b_3}{12} & \text{if } b_1 = 1 \text{ and } b_2 = 1 \ mod 2 \\ \frac{1}{2} + \frac{3b_2}{8} + \frac{13b_3}{24} & otherwise. \end{cases}$$

Example:

$$A=\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{pmatrix}$$



Two dimensional slice of the cone $Ax=b$ $x>=0$.

1. If $\min\{b_3, -b_2, b_1 + b_2\} \geq 0$ then

$$\phi_{K_4}(b) \quad = \quad (b_1 + b_2 + 3)(b_1 + b_2 + 2)(b_1 + b_2 + 1)/6.$$

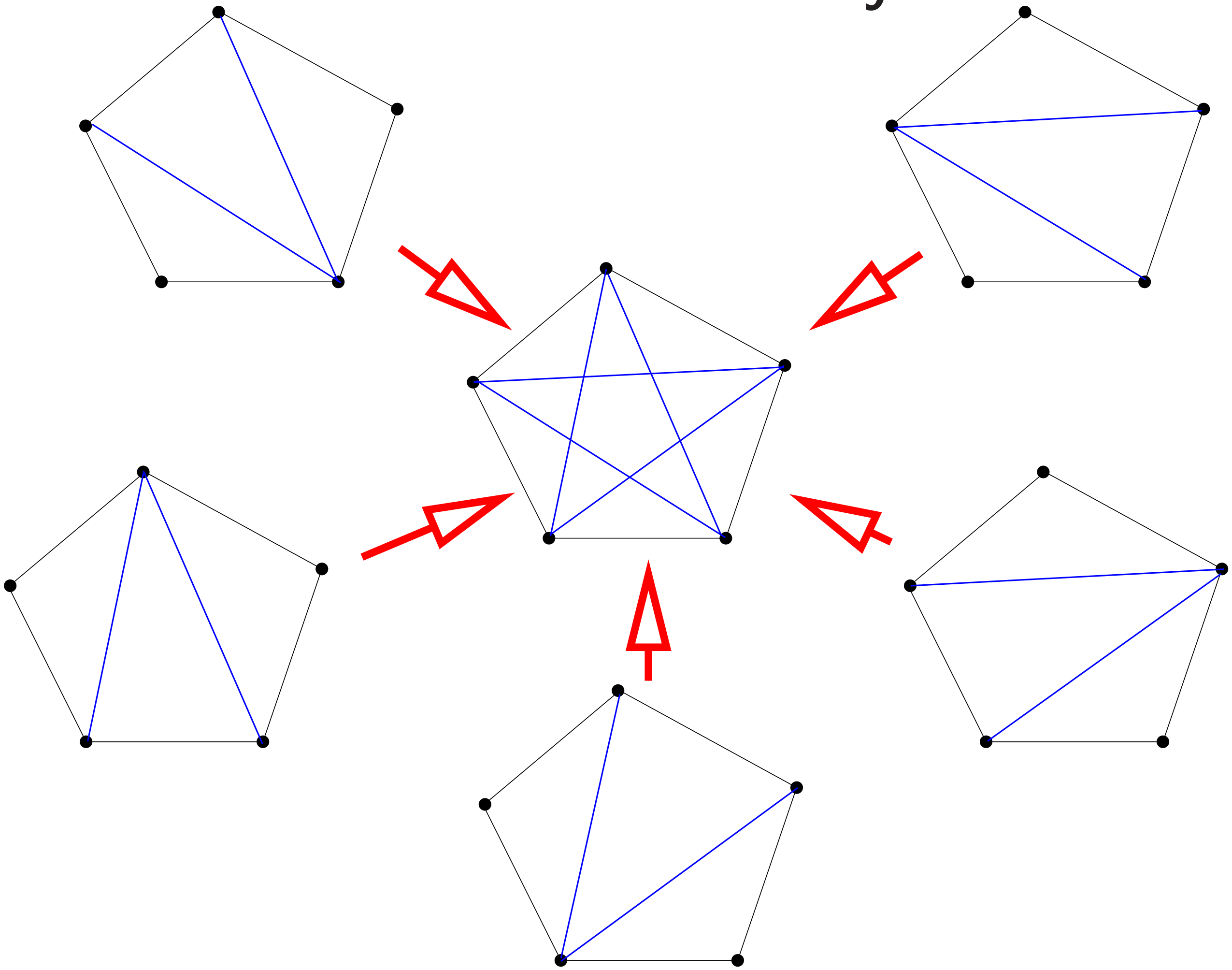
2. If $\min\{b_1, b_2, b_3\} \geq 0$ then

$$\phi_{K_4}(b) \quad = \quad (b_1 + 1)(b_1 + 2)(b_1 + 3b_2 + 3)/6.$$

3. If $\min\{b_1, b_2, b_1 + b_3, b_2 + b_3, -b_3\} \geq 0$ then $\phi_{K_4}(b) \quad = \quad 1 + \frac{11}{6}\, b_1 + 2/3\, b_3 + b_2 + 3/2\, b_1\, b_2 + {b_1}^2 + 1/6\, {b_1}^3 + 1/2\, {b_1}^2 b_2 - 1/6\, {b_3}^3 - 1/2\, b_1\, {b_3}^2 + 1/2\, b_1\, b_3 - 1/2\, {b_3}^2.$

4. If $\min\{b_1, b_2 + b_3, -b_1 - b_3\} \geq 0$ then $\phi_{K_4}(b) \quad = \quad (b_1 + 2)(b_1 + 1)(2b_1 + 3b_2 + 3 + 3b_3).$

Chamber Geometry.



COUNTING LATTICE POINTS INSIDE MORE COMPLICATED REGIONS, CAN WE?

Can one count inside other regions?

When the sets are arbitrary really bad things can happen, even in small fixed dimension!

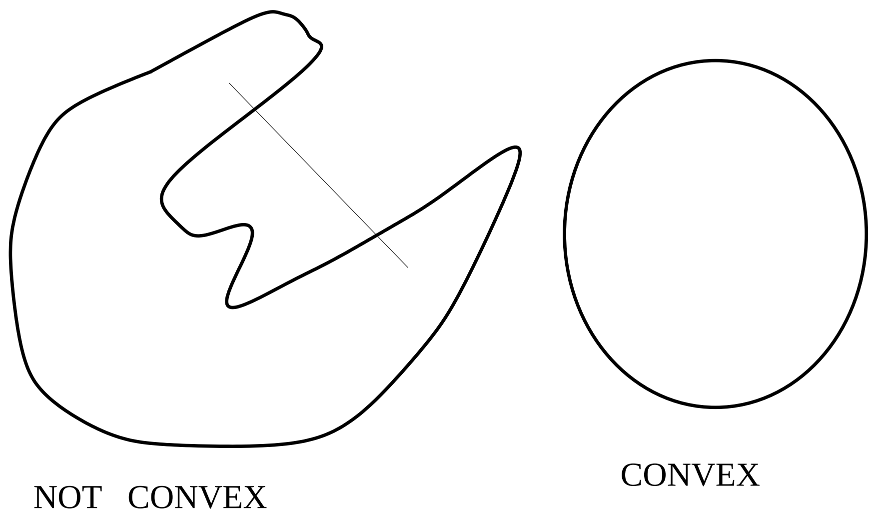
- Given (a, b, c) positive integers, deciding whether there is a lattice point in the set $\{x | ax^2 + bx = c,\ x \geq 0\}$ is an NP-complete problem.

- Deciding whether there is a non-negative integer root for arbitrary polynomials in $\mathbb{Z}[x_1, \dots, x_9]$ is undecidable.

Thus we clearly need to be less ambitious!

But convex sets must be tractable, right?

A convex set C is a set of Euclidean space such that for any pair of points in C the line segment joining x and y is completely inside C. Polyhedra are the simplest case.



CAN ONE EASILY COUNT THE LATTICE POINTS OF CONVEX SETS?

EARLIER WORKERS

CREDIT CARD CYBER-THIEVES CARE

For an integer number n consider the 4-dimensional convex body

$$B(n) = \{x \in R^4 : x_1^2 + x_2^2 + x_3^2 + x_4^2 \leq n\}$$

Jacobi proved that if $|B(n)|$ is the number of lattice points in $B(n)$, for n of the form $pq = n$, where p, q are primes, we have

$$|B(n)| - |B(n-1)| = 8(1 + p + q + n)$$

If we know that $n = pq$, then a factorization of n can be done fast if we know how to compute $|B(n)|$!!

RSA cryptosystems used in Internet transactions can be broken if you know how to count lattice points fast.

HILBERT and GRAVER: INTEGRAL BASES FOR CONES and SUBSPACES

Toward Non-negative Integral Bases for CONES and SUBSPACES

- We know from Weyl-Minkowski's theorem that every cone C can be written using finite linear combination of its **rays**.

- Can we do the same for the lattice points?, Can we find finitely many lattice points $G = \{g_1, g_2, \ldots, g_k\}$ in C that can be used to write any $v \in C \cap \mathbb{Z}^n$ as a **non-negative** linear combination of h_i's? We are looking for Hilbert bases

- How about if we want the same not for a cone but for a subspace? Then we will obtain the so called Graver bases.

Cones of Magic Squares and Cubes

- Recall the magic arrays form a convex polyhedral cones of the form $Ax = 0, x \geq 0$, where A is a matrix with $0, 1, -1$ entries.

- EXAMPLE: The cone C of 3×3 magic matrices is defined by the system of equations (i.e, row sums, column sums, and diagonal sums are equal).

$$x_{11} + x_{12} + x_{13} = x_{21} + x_{22} + x_{23} = x_{31} + x_{32} + x_{33}$$
$$x_{11} + x_{12} + x_{13} = x_{11} + x_{21} + x_{31} = x_{12} + x_{22} + x_{32} = x_{13} + x_{23} + x_{33}$$
$$x_{11} + x_{12} + x_{13} = x_{11} + x_{22} + x_{33} = x_{31} + x_{22} + x_{13},$$

and the inequalities $x_{ij} \geq 0$.

The cone C of magic squares has dimension 3, it is a cone based on a quadrilateral, thus it has 4 rays.

1/3	0	2/3
2/3	1/3	0
0	2/3	1/3

2/3	0	1/3
0	1/3	2/3
1/3	2/3	0

0	2/3	1/3
2/3	1/3	0
1/3	0	2/3

1/3	2/3	0
0	1/3	2/3
2/3	0	1/3

Figure 1: The four RAYS the cone of 3×3 magic squares.

- For a cone C of magic arrays we are interested in $S_C = C \cap \mathbb{Z}^n$, the *semigroup of the cone* C.

- Let $C \subseteq \mathbb{R}^n$ be a polyhedral cone with rational generators and let $\Lambda \subseteq \mathbb{Z}^n$ be a lattice.

  A finite set $H = \{h_1, \ldots, h_t\} \subseteq \Lambda \cap C$ a **generating set** of the $(\Lambda \cap C, +)$ if for every $z \in \Lambda \cap C$ there are non-negative integral multipliers $\lambda_1, \ldots, \lambda_t$ such that $z = \sum_{i=1}^{t} \lambda_i h_i$.

If the generating set of lattice points is **minimal,** then it is called a Hilbert bases.

In general it is hard to compute Hilbert basis. A fast completion and project-and-lift method is implemented in `4ti2`.

1	0	2
2	1	0
0	2	1

2	0	1
0	1	2
1	2	0

0	2	1
2	1	0
1	0	2

1	2	0
0	1	2
2	0	1

1	1	1
1	1	1
1	1	1

Figure 2: The Hilbert bases of the cone of 3×3 magic squares.

Graver Bases

- The lattice $L(A) = \{x \in \mathbb{Z}^n : Ax = 0\}$ has a natural partial order. For $u, v \in \mathbb{Z}^n$ we say that u is *conformal* to v, denoted $u \sqsubseteq v$, if $|u_i| \leq |v_i|$ and $u_i v_i \geq 0$ for $i = 1, \ldots, n$, that is, u and v lie in the same orthant of $\mathbb{R}^n$ and each component of u is bounded by the corresponding component of v in absolute value.

- The Graver basis of an integer matrix A is the set of conformal-minimal nonzero integer dependencies on A.

- **Example:** If $A = [1\ 2\ 1]$ then its Graver basis is

$$\pm\{[2, -1, 0], [0, -1, 2], [1, 0, -1], [1, -1, 1]\}$$

.

Graver Bases (Continued)

- **Theorem:** Graver basis is equal to the union of all Hilbert bases, one for each of the orthants.

- Think of the Graver basis as vectors connecting lattice points All lattice points are in fact connected by these vectors!!

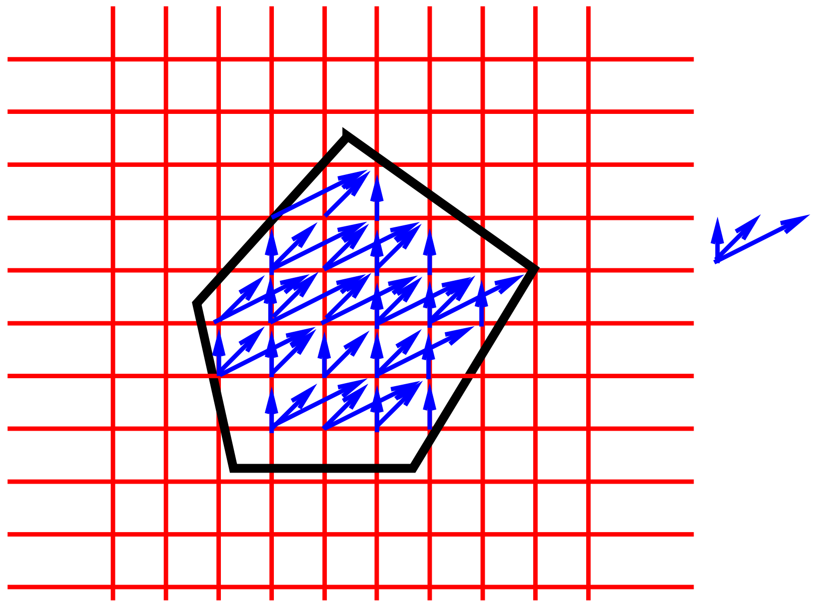
- We have a **connected graph** on the lattice points of

$$P(b) := \{x|\ Ax = b, x \geq 0\}$$

. More precisely:

- Consider $L(b) := \{x|\ Ax = b, x \geq 0,\ x \in \mathbb{Z}^n\}$.

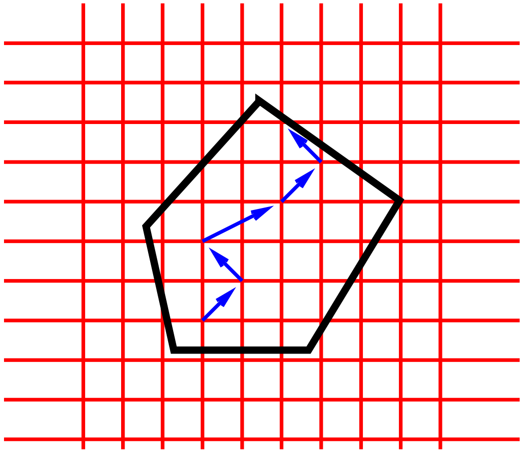
Nodes are lattice points in $L(b)$ and the Graver basis elements give directed edges departing from each lattice point $u \in L(b)$.



- **Theorem** The Graver basis contains all edges for all integer hulls $conv(\{x|\ Ax = b, x \geq 0,\ x \in \mathbb{Z}^n\})$ as b changes.

Graver bases are Test Sets

- A TEST SET is a finite collection of integral vectors with the property that every feasible non-optimal solution of an integer program can be improved by adding a vector in the test set.



- **Theorem** [J. Graver 1975] Graver bases for A can be used to solve the augmentation problem Given $A \in \mathbb{Z}^{m \times n}$, $x \in \mathbb{N}^n$ and $c \in \mathbb{Z}^n$, either find an improving direction $g \in \mathbb{Z}^n$, namely one with $x - g \in \{y \in \mathbb{N}^n : Ay = Ax\}$ and $cg > 0$, or assert that no such g exists.

BAD and GOOD news:

- Graver test sets can be exponentially large even in fixed dimension!

- People typically store a list of the whole test set. Very large indeed. (New ways to store them available, using **Generating Functions**.

- **Theorem:** (Barvinok-Woods) In fixed dimension, the Hilbert bases and the Graver bases of a cone can be computed in polynomial time.

- In arbitrary dimension it is NP-hard to decide whether you have a full Hilbert (Graver) basis.

- NEW RESULTS: Graver bases become very manageable for **concrete families** of matrices. Polynomial time computation!!

Nice Matrices: N-fold Systems

Fix any pair of integer matrices A and B with the same number of columns, of dimensions $r \times q$ and $s \times q$, respectively. The n-fold matrix of the ordered pair A, B is the following $(s + nr) \times nq$ matrix,

$$[A, B]^{(n)} \quad := \quad (\mathbf{1}_n \otimes B) \oplus (I_n \otimes A) \quad = \quad \begin{pmatrix} B & B & B & \cdots & B \\ A & 0 & 0 & \cdots & 0 \\ 0 & A & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A \end{pmatrix} \quad .$$

Theorem Fix any integer matrices A, B of sizes $r \times q$ and $s \times q$, respectively. Then there is a polynomial time algorithm that, given any n, we can compute the Graver bases of $[A, B]^{(n)}$.

Proof by Example

- **Key Lemma** Fix any pair of integer matrices $A \in \mathbb{Z}^{r \times q}$ and $B \in \mathbb{Z}^{s \times q}$. Then there is a polynomial time algorithm that, given n, computes the Graver basis $G([A,B]^{(n)})$ of the n-fold matrix $[A,B]^{(n)}$. In particular, the cardinality and the bit size of $G([A,B]^{(n)})$ are bounded by a polynomial function of n.

- **Key Idea (from Algebraic Geometry)** [Aoki-Takemura, Santos-Sturmfels, Hosten-Sullivant] For every pair of integer matrices $A \in \mathbb{Z}^{r \times q}$ and $B \in \mathbb{Z}^{s \times q}$, there exists a constant $g(A,B)$ such that for all n, the Graver basis of $[A,B]^{(n)}$ consists of vectors with at most $g(A,B)$ the number nonzero components. The smallest constant $g(A,B)$ possible is the Graver complexity of A,B.

Proof by Example

Consider the matrices $A = [1\ 1]$ and $B = I_2$. The Graver complexity of the pair A, B is $g(A, B) = 2$.

$$[A,B]^{(2)} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix},\ G([A,B]^{(2)}) = \pm \begin{pmatrix} 1 & -1 & -1 & 1 \end{pmatrix} \quad .$$

By our theorem, the Graver basis of the 4-fold matrix

$$[A,B]^{(4)} = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix},$$

$$G([A,B]^{(4)}) = \pm \begin{pmatrix} 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \end{pmatrix}.$$