Basic Concepts in Convexity and Computation

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Combinatorial Convexity

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- A subset S of $\mathbb{R}^n$ is called convex if for any two distinct points x_1, x_2 in S the line segment joining x_1, x_2, lies completely in S.

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- Note that a hyperplane divides $\mathbb{R}^d$ into two halfspaces $H_\alpha^+ = \{x \in \mathbb{R}^d : f(x) \geq \alpha\}$ and $H_\alpha^- = \{x \in \mathbb{R}^d : f(x) \leq \alpha\}$. Halfspaces are convex sets.

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- How are POLYTOPES and POLYHEDRA related?

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- This allows us to represent all polytopes in two ways inside a computer!! Either as a list of vertices, or as system of inequalities.

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- **Lemma:** (EXERCISE) For a set of points A in $\mathbb{R}^d$ we have that $conv(A)$ equals all finite convex combinations of A:

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- **Lemma:** A set of $d+2$ or more points in $\mathbb{R}^d$ is affinely dependent.

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Here are three classical theorems about convex sets. We invite you to provide proofs for them (EXERCISE)!!

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- **Theorem:** (Helly's theorem): If C is a collection of closed bounded convex sets in $\mathbb{R}^d$ such that each $d+1$ sets have nonempty intersection then the intersection of all sets in C is non-empty.

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- The dimension of an affine set is the largest number of affinely independent points in the set minus one. The dimension of a set in $\mathbb{R}^d$ is the dimension of its affine hull.
- A face of dimension 0 is called a vertex. A face of dimension 1 is called an edge, and a face of dimension $dim(P) - 1$ is called a facet. The empty set is defined to be a face of P of dimension -1. Faces that are not the empty set or P itself are called proper.

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- **Theorem** (Upper bound theorem) For any d-polytope P with n vertices has no

$$f_i(P) \leq f_i(C(n,d))$$

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- QUESTION: How can we compute the faces of a polyhedron?

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- **idea of proof:** Translate $P \subset \mathbb{R}^d$ to contain the origin as its interior point. For a non-empty face F of P define

$$\hat{F} = \{x \in P^o :< x, y >= 1 for\ all\ y \in F\}$$

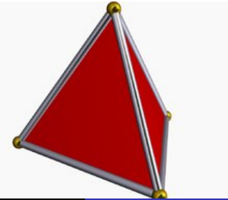
  and for the empty face define $\hat{} = P^0$.
  The hat operation applied to faces of a d-polytope P satisfies

  1. The set $\hat{F}$ is a face of P^o
  2. $dim(F) + dim(\hat{F}) = d - 1$.
  3. The hat operation is involutory: $\hat{\hat{\hat{F}}} = F$.
  4. If $F, G \subset P$ are faces and $F \subset G \subset P$, then $\hat{G}, \hat{F}$ are faces of P^o and $\hat{G} \subset \hat{F}$.

A few key examples to play...

Simplices

- Let $e_1, e_2, \ldots, e_{d+1}$ be the standard unit vectors in $\mathbb{R}^{d+1}$. The *standard d*-dimensional simplex Δ_d is $conv(\{e_1, \ldots, e_{d+1}\})$. Thus $\Delta_d = \{x = (x_1, \ldots, x_{d+1}) : x_i \geq 0 \text{ and } x_1 + x_2 + \cdots + x_{d+1} = 1\}$.
- Note that for polytope $P = conv(\{a_1, \ldots, a_m\})$ we can define a linear map $f : \Delta_{m-1} \to P$ by the formula $f(\lambda_1, \ldots, \lambda_m) = \lambda_1 a_1 + \cdots + \lambda_m a_m$. Thus $f(\Delta_{m-1}) = P$.
- Every polytope is the image of the standard simplex under a linear transformation.



Cubes and Zonotopes

- Let $\{u_i : i \in I\}$ be the set of all 2^d vectors in $\mathbb{R}^d$ whose coordinates are either 1 or -1.
- The polytope $I_d = conv(\{u_i : i \in I\}$ is called the *standard unit d-dimensional cube*. Thus $I^d = \{(x_1, \dots, x_d) : -1 \leq x_i \leq 1\}$.
- The images of a cube under linear transformations receive the name of *zonotopes*.

Birkhoff's polytope

- Let B_n be the convex polytope of real non-negative matrices with all row and column sums equal to one (doubly-stochastic matrices). The dimension of B_n is $(n-1)^2$.
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- Facets: For each pair (i, j) with $1 \leq i, j \leq n$, the doubly stochastic matrices with (i, j) entry equal to 0 is a *facet* (maximal proper face) of B_n and all facets arise in this way.

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- The linear projection of B_n given by multiplying each matrix by a vector $\mathbf{v}$ gives a Permutahedron.

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- **Lemma:** Every hyperplane intersects the moment curve $\gamma(t) = (t, t^2, t^3, \ldots, t^d)$ in no more than d points.
- **Theorem:** The largest possible number of i-dimensional faces of a d-polytope with n vertices is achieved by the cyclic polytope $C(n, d)$.

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- **Examples:**
  - Does the Peterson graph have a clique of size 4?

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- Instances can have more than one representation.
  - A graph $G = (V, E)$ can be represented as a adjacency matrix, incidence matrix, adjacency list, etc.

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  - Worst case difficulty: consider run times of “hard” instances.
  - Express run time as a function of the amount of “memory” needed to represent the instance!

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2. The set $S = \{0, \ldots, n\}$ can also be encoded in around $\log_2 n$ bits.

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- Input size of the normal form representation of $\mathcal{L}$ is the size n.
- Silly algorithm requires around $\binom{n}{2}$ comparisons and re-ordering of the lists.

P vs NP: What you need to know

What do you mean is HARD TO COMPUTE X ??

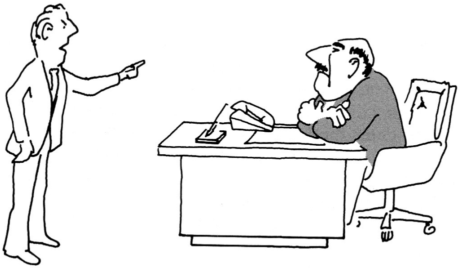

Figure: I tried to compute X, I can't do it, therefore it must be hard!

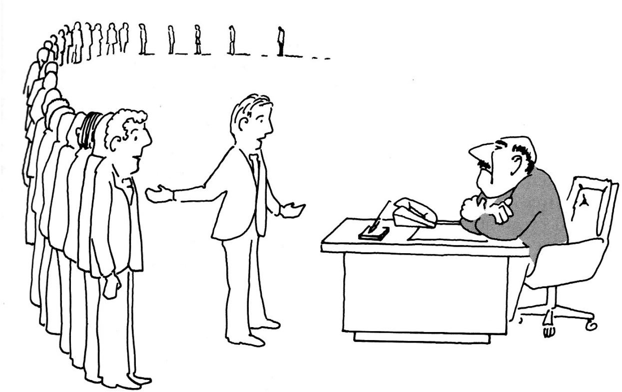

Figure: I can't compute X, but if I could do it, the problems of all these people would be solved too! therefore it must be hard!

$\#P$-complete problems is a family of COUNTING problems, if one finds a fast solution for one, you find it for all the members of the family!

Many of the problems we care about require LISTING all the elements of a set: e.g., list all facets, all vertices. There are no uniformly accepted complexity notions for LISTING algorithms, and the output size can be LARGE.

- An algorithm is output sensitive if it runs in TIME polynomial in both the input size and the output size.
- An algorithm is compact if it runs in SPACE polynomial in the input size ONLY.

  An ideal listing algorithm is a compact output-sensitive algorithm. Hard to find!!!

Thank you