# Polyhedral Algorithms
# Part I: Feasibility and Computation of Facets

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# Fourier-Motzkin Elimination

# Is there any solution of $Ax \geq b$?

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  **Fredholm's Lemma:** $\{x : Ax = b\}$ is non-empty if and only if $\{y : y^T A = 0, y^T b = -1\}$ is empty.
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- **Farkas Lemma:** A polyhedron $\{x : Ax \leq b\}$ is non-empty if and only if there is no solution $\{y : y^T A = 0, y^T b < 0, y \geq 0\}$.
- We will give an (inefficient) algorithmic proof of Farkas lemma using an algorithm that decides whether a polyhedron is feasible: **Fourier-Motzkin' algorithm**.

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$x_1 + (a'_i)^T x' \leq b'_i$, **(if coefficient of $a_{i1}$ is positive) (TYPE I)**

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$(a'_k)^T x' \leq b'_k$, **(if coefficient of $a_{k1}$ is zero) (TYPE III)**

Here $x' = (x_2, x_3, \ldots, x_n)$.

## Fourier-Motzkin continued

Add all possible pairs of inequalities of (TYPE I) and (TYPE II). Create new system $(*)$ with fewer variables:

- $$(a'_j + a'_i)^T x' \leq (b_j + b_i) \text{ for } i \text{ of type I and } j \text{ of type II}$$

  **Keep equations of type III** $(a'_k)^T x' \leq b'_k$
- Original system of inequalities has a solution if and only if the system $(*)$ is feasible **WHY?**

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# Proof of Farkas Lemma

- Indeed, if we reduce until we have no variables. New system becomes

$$\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leq \begin{bmatrix} b'_1 \\ b'_2 \\ \vdots \\ b'_n \end{bmatrix}$$

  Polyhedron $\{x : Ax \leq b\}$ infeasible $\iff$ $b'_i < 0$ for some $i$.
- Rewriting and addition steps correspond to row operations on the original matrix $A$. This is done by matrix multiplication.

  $0 = MAx \geq Mb = b'$, **with matrix** $M$ **with non-negative entries**
- Set $y^T = (e_i)^T M$, with $e_i$ standard $i$-th unit vector then

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Here is another form of Farkas lemma:

- **Corollary:**
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- **proof** $\{x : Ax = b, x \geq 0\} \neq \emptyset \iff \{x : Ax \leq b, \ -Ax \leq -b, \ -Ix \leq 0\} \neq \emptyset$.

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- **proof** $\{x : Ax = b, x \geq 0\} \neq \emptyset \iff \{x : Ax \leq b, \ -Ax \leq -b, \ -Ix \leq 0\} \neq \emptyset.$
  By previous version of Farkas, this happens if and only if no solution exists of $y^T = [y_1 \ y_2 \ y_3]^T$ with

$$[y_1 \ y_2 \ y_3]^T \begin{bmatrix} A \\ -A \\ -I \end{bmatrix} = 0, \ [y_1 \ y_2 \ y_3]^T \begin{bmatrix} b \\ -b \\ 0 \end{bmatrix} < 0, \ y^T \geq 0$$

- The vector $y_1 - y_2$ has the desired property.

# More on Farkas II

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- **proof Necessity:** We know $x \geq 0$, $Ax = b$, if in addition $y^T A \geq 0$ then $y^T b = y^T Ax \geq 0$.

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  **Sufficiency:** Suppose if $y^T A \geq 0$, then $y^T b \geq 0$ but assume $\nexists x \geq 0$ with $Ax = b$. From the previous corollary, $\exists y$ with $y^T A \geq 0, y^T b < 0$. Therefore $0 \leq y^T b < 0$ which is a contradiction.

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Here is another form of Farkas lemma:

- **Corollary:**
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- There are many more consequences and variations of Farkas lemma (ALL theory of Linear Optimization based on it!!!).

# Weyl-Minkowski

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- **Theorem:** [Weyl-Minkowski] Every polytope is a polyhedron. Every bounded polyhedron is a polytope.
- This allows us to represent all polytopes in two ways inside a computer!! Either as a list of vertices, or as system of inequalities.

# Weyl-Minkowski in Steroids

- **Theorem**: (Weyl-Minkowski's Theorem): For a polyhedral subset $P$ of $\mathbb{R}^d$ the following statements are equivalent:
  - $P$ is an **H-polyhedron**, i.e., $P$ is given by a system of linear inequalities $P = \{x : Ax \geq b\}$.
  - $P$ is a **V-polyhedron**, i.e., For finitely many vectors $v_1, \ldots, v_n$ and $r_1, \ldots, r_s$ we can write

$$P = conv(v_1, v_2, \ldots, v_n) + cone(r_1, r_2, \ldots, r_s)$$

  Here $R + S$ denotes the Minkowski sum of two sets, $R + S = \{r + s : r \in R, s \in S\}$.
- We need to design an efficient algorithm for the conversion between the H-polyhedron and V-polyhedron!
- **NOTE:** Cone can be decomposed into a pointed cone plus a linear space.

# Polyhedral Cones

A set $\mathcal{C} \subseteq \mathbb{R}^n$ is a **cone** if it is closed under addition and multiplication by a positive constant.

- A set $\mathcal{C} \subseteq \mathbb{R}^n$ is a **inequality constrained** cone if $\mathcal{C} = \{x \in \mathbb{R}^n : Ax \geq 0\}$ for some matrix $A$.
- A set $\mathcal{C} \subseteq \mathbb{R}^n$ is a **finitely generated** cone if $\mathcal{C} = \{\lambda B : \lambda \in \mathbb{R}^k_+\}$ for some matrix $B$.

**Theorem (Minkowski-Weyl)**

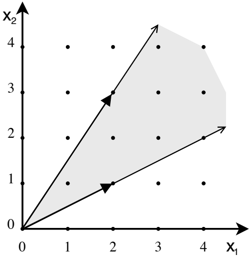
*A cone $\mathcal{C} \subseteq \mathbb{R}^n$ is finitely constrained if and only if it is finitely generated.*

- The set of *extreme rays* of the cone is the *minimal* set of generators of a cone.

FUNDAMENTAL QUESTION: how do we convert between the two repre

# Example 1

Consider the following cone $\mathcal{C}$ and its two representations:




- $\mathcal{C} = \{x \in \mathbb{R}^2 : 3x_1 - 2x_2 \geq 0, -x_1 + 2x_2 \geq 0\}$.
- $\mathcal{C} = \{\lambda_1(2,1) + \lambda_2(2,3) : \lambda_1, \lambda_2 \in \mathbb{R}_+\}$.

## Example2: Magic Squares

A **magic square** is a square grid of non-negative real numbers such that the rows, columns, and diagonals all add up to the same value.

| | | |
|---|---|---|
| 0 | 2 | 1 |
| 2 | 1 | 0 |
| 1 | 0 | 2 |

Magic Squares are closed under non-negative linear combinations

$$3 \times \begin{array}{|c|c|c|}\hline 0 & 2 & 1 \\ \hline 2 & 1 & 0 \\ \hline 1 & 0 & 2 \\ \hline \end{array} = \begin{array}{|c|c|c|}\hline 0 & 6 & 3 \\ \hline 6 & 3 & 0 \\ \hline 3 & 0 & 6 \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|}\hline 0 & 2 & 1 \\ \hline 2 & 1 & 0 \\ \hline 1 & 0 & 2 \\ \hline \end{array} + \begin{array}{|c|c|c|}\hline 1 & 2 & 0 \\ \hline 0 & 1 & 2 \\ \hline 2 & 0 & 1 \\ \hline \end{array} = \begin{array}{|c|c|c|}\hline 1 & 4 & 1 \\ \hline 2 & 2 & 2 \\ \hline 3 & 0 & 3 \\ \hline \end{array}$$

**Question:** Is there a finite set of $n \times n$ magic squares so that we can express every other possible magic square as a linear non-negative combination?

$$\boxed{\textbf{YES!}}$$

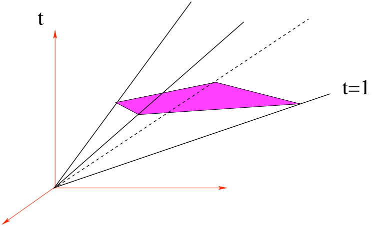
There are four such $3 \times 3$ magic squares:

$$\begin{array}{|c|c|c|}\hline 0 & 2 & 1 \\ \hline 2 & 1 & 0 \\ \hline 1 & 0 & 2 \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\hline 2 & 0 & 1 \\ \hline 0 & 1 & 2 \\ \hline 1 & 2 & 0 \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 0 \\ \hline 0 & 1 & 2 \\ \hline 2 & 0 & 1 \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 0 & 2 \\ \hline 2 & 1 & 0 \\ \hline 0 & 2 & 1 \\ \hline \end{array}$$

IMPORTANT: There is an algorithm for computing a minimal such set of magic squares for $n \times n$ magic squares. These magic squares are the **extreme rays** of the cone of magic squares.

# HOMOGENIZE: Weyl-Minkowski reduces to the case of Cones!!

- We can reduce this problem to problem of transforming between the two representations of a cone. From an H-polyhedron construct a cone from the polytope as follows:




- Observe: If the original polytope was given by inequalities $Ax \geq b$ then the cone is given by inequalities $\bar{A}y \geq 0$, where $\bar{A}$ is the extended matrix $[A, -b]$ and $y = (x, t)$.
- Enough to solve Weyl-Minkowski's Theorem for cones:

# Proof of Weyl-Minkowski

The following are equivalent

- $P$ is an **H-cone**, i.e., $P$ is given by a system of linear inequalities $P = \{x : Ax \geq 0\}$.
- $P$ is a **V-cone**, i.e., For finitely many vectors $r_1, \dots, r_s$ we can write

$$P = cone(r_1, r_2, \dots, r_s)$$

This is equivalent to (Matrix form!!):
The following are equivalent

- $P$ is an **H-cone**, i.e., $\exists$ matrix $A$ such that $P = \{x : Ax \geq 0\}$.
- $P$ is a **V-cone**, i.e., $\exists$ matrix $R$ such that $P = \{x : x = Ry,\ y \geq 0\}$

We say the pair $(A, R)$ is a **double description pair** (DD-pair).

# Polar Cones

- **Definition:** Let $K$ be a convex cone the polar of $K$ is the set

$$K^* = \{x \in \mathbb{R}^n :< x, y > \leq 0 \text{ for all } y \in K\}$$

  pause

- **Lemma:** If $K$ is a cone then $K^*$ is a cone. In fact, $K = cone(\{a_1, a_2, ..., a_m\})$, i.e. $K$ is generated by vectors then $K^*$ is given by inequalities:

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- **Lemma:** $(K^*)^* = K$.
  PUNCH LINE: It is enough to prove Weyl-Minkowski ONE of the implications, the other one follows by polar cone construction!!

# Minkowski-Weyl Algorithmic version

- **LEMMA** For any pair of matrices $A, R$, $(A, R)$ is a DD-pair of cone $C$ if and only if $(R^T, A^T)$ is a double description pair of the polar cone of $C$.

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  **Proof:** (EXERCISE) Use Farkas lemma.
- **An (algorithmic) proof of Minkowski-Weyl's theorem:** Let $R$ be a matrix defining a V-cone, $C$, thus

$$C = \{x : x = Ry,\ y \geq 0\}.$$

  By Fourier-Motzkin we can eliminate all variables $y$ from above system.

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  This proves that every V-cone can be written as an H-cone.
  By previous lemma we are done to prove the converse.
  WARNING: Not an efficient algorithm.

# The Double Description Method (Motzkin-Raiffa-Thompson-Thrall 1953)

# The Double description Method I

- Suppose $A$ is an $m \times d$ matrix, defines cone $C = \{x : Ax \geq 0\}$.
- Let $A_K$ denote the submatrix of $A$ given by rows in index set $K$.
- Suppose we found already a matrix $R$ which is DD pair with $A_K$. From a new row index $i \notin K$ construct new DD pair $(A_{K\cup\{i\}}, R')$ (**but HOW?**):
- Partition the column index set $J$ of $R$ into three parts:
  - $J^+ = \{j \in J : A_i r_j > 0\}$
  - $J^0 = \{j \in J : A_i r_j = 0\}$
  - $J^- = \{j \in J : A_i r_j < 0\}$

  We recover the new $R'$ from the following lemma:

# The Double description Method II

- **Lemma:** The pair $(A_{K\cup\{i\}}, R')$ is a DD pair, when the matrix $R'$ is given by the $d \times J'$ matrix such that
  - the index set is $J' = J^+ \cup J^0 \cup (J^+ \times J^-)$, and
  - the new columns are $r_{jj'} = (A_i r_j) r_{j'} - (A_i r_{j'}) r_j$ for each $(j, j') \in J^+ \times J^-$.
- **Proof:** Let $C(A_{K\cup\{i\}}) = \{x : A_{K\cup\{i\}} x \geq 0\}$ and $C(R') = \{x : x = R'y,\ y \geq 0\}$. We wish $C(A_{K\cup\{i\}}) = C(R')$.

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- Clearly $C(R') \subset C(A_{K\cup\{i\}})$ because $r_{jj'} \in C(A_{K\cup\{i\}})$.

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- Clearly $C(R') \subset C(A_{K\cup\{i\}})$ because $r_{jj'} \in C(A_{K\cup\{i\}})$.
- Take $x \in C(A_{K\cup\{i\}})$. Then

$$x = \sum_{j\in J} \lambda_j r_j, \quad \text{with } \lambda_j \geq 0$$

# The Double description Method II

- **Lemma:** The pair $(A_{K\cup\{i\}}, R')$ is a DD pair, when the matrix $R'$ is given by the $d \times J'$ matrix such that
  - the index set is $J' = J^+ \cup J^0 \cup (J^+ \times J^-)$, and
  - the new columns are $r_{jj'} = (A_i r_j) r_{j'} - (A_i r_{j'}) r_j$ for each $(j, j') \in J^+ \times J^-$.
- **Proof:** Let $C(A_{K\cup\{i\}}) = \{x : A_{K\cup\{i\}} x \geq 0\}$ and $C(R') = \{x : x = R'y,\ y \geq 0\}$. We wish $C(A_{K\cup\{i\}}) = C(R')$.
- Clearly $C(R') \subset C(A_{K\cup\{i\}})$ because $r_{jj'} \in C(A_{K\cup\{i\}})$.
- Take $x \in C(A_{K\cup\{i\}})$. Then

$$x = \sum_{j\in J} \lambda_j r_j, \quad \text{with } \lambda_j \geq 0$$

- If there is no $\lambda_k > 0$ for $k \in J^-$ then $x \in C(R')$ already. Thus assume such $\lambda_k$ exists.

## The Double description Method III

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- Therefore since $A_i x \geq 0$ there must also be $\lambda_h > 0$ with $h \in J^+$.
- Substract a suitable multiple of $r_{kh} = (A_i r_h) r_k - (A_i r_k) r_h$ from $x = \sum_{j \in J} \lambda_j r_j$
- We are left with a new expression of $x$ with smaller non-zero coefficients. This process can be repeated as long as $\lambda_k > 0$ with $k \in J^-$ exists.
- So in finitely many steps we must get rid of all such $\lambda$ at which point we have $x \in C(R')$.

# The Double description Method IV

- We can refine the above construction, finding a matrix $R'$ which has no redundant columns!!
- We say $r_j$ is a **extreme ray** if it cannot be written as a non-negative combination of two other rays.
  Thus all we need to do is throw away columns of the matrix which are not extreme rays. How to tell???
- **Lemma:** Let $Z(x)$ be the set of indices of inequalities such that $A_i x = 0$. A ray $r$ is an extreme ray of the cone $\{x : x \in \mathbb{R}^d,\ Ax \geq 0\} \iff$ the rank of the submatrix $A_{Z(r)} = d - 1$.
- How to do the initial DD pair?? Select a maximal submatrix $A_K$ with linearly independent rows of $A$.
- Initial matrix $R$ is the solution to $A_K R = I$. WHY? $rank(A) = d$ then $A_K$ must be square then $R = A_K^{-1}$. Then $(A_K, R)$ is DD pair since $A_K x \geq 0 \iff A_K^{-1} y, y \geq 0$.

# The Double description Method V

- The double description method has a dual version called the **Beneath-Beyond method**.
- DD is practical for low dimensions (see CDD).
- The size of intermediate polytopes can be very very sensitive to the order in which the subspaces are introduced.
- D. Bremner (1999) showed a family of polytopes for which the double description method is exponential.

# The Ellipsoid Method

Given: A set $S$ is a polyhedron (bounded and convex) with $\text{vol}(S) > 0$, an ellipsoid $E_{M,z}$ such that $S \subseteq E_{M,z}$.
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3. Find a nonzero vector $a$ such that $a^T x \leq a^T z^k, \ \forall x \in S$; (separating hyperplane)
4. Construct the smaller volume ellipsoid that contains
$$E_{M,z} \cap \{x \in \mathbb{R}^n \mid a^T(x - z^k) \leq 0\}.$$
Let this ellipsoid have matrix $M^{k+1}$ and center $z^{k+1}$.
5. $k = k + 1$;
6. Go back to Step 2.

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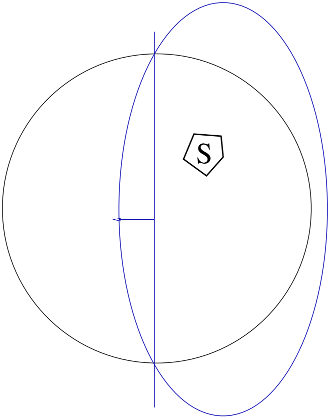

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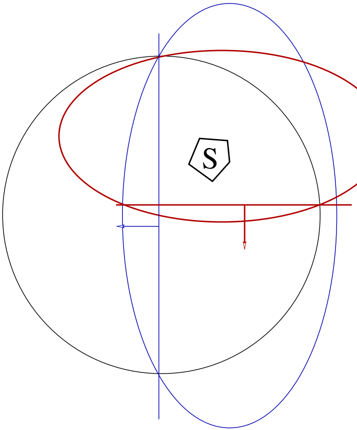

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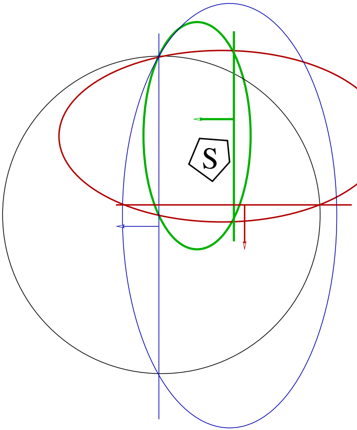

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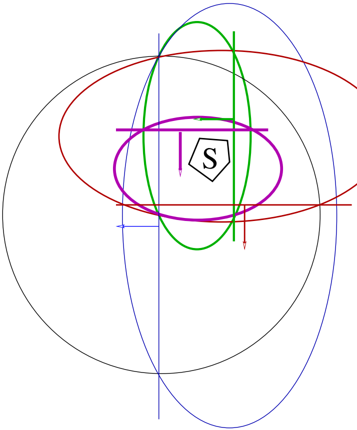

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$$E_k = \{x : (x - x_k)^T A_k^{-1} (x - x_k) \leq 1\}$$

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- There are many many polytopes that encode combinatorial information on their vertices: Matching polytopes, Traveling salesman polytopes, matroid polytopes, etc.
- **Theorem** (Gröstchel, Lovász, Schrijver) If given a polyhedron $P$ and a point $x \notin P$ you can SEPARATE them in polynomial time, then you can OPTIMIZE any linear functional over $P$ in

# Thank you