Polyhedral Algorithms
Part I: Feasibility and Computation of Facets

Jesús A. De Loera, UC Davis

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Fourier-Motzkin Elimination
Is there any solution of $Ax \geq b$?

We say that the system of inequalities $Ax \geq b$ is **feasible** if there is at least one $x$ that satisfies all the inequalities. We wish to know when and certify the feasibility/infeasibility of polyhedra.
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- Analogously, in linear algebra, **Fredholm’s Lemma:** $\{x : Ax = b\}$ is non-empty if and only if $\{y : y^T A = 0, y^T b = -1\}$ is empty. Such a vector $y$ is a **mathematical proof** that $Ax = b$ has no solution.
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- **Farkas Lemma:** A polyhedron $\{x : Ax \leq b\}$ is non-empty if and only if there is no solution $\{y : y^T A = 0, y^T b < 0, y \geq 0\}$.
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**Fredholm’s Lemma:** $\{x : Ax = b\}$ is non-empty if and only if $\{y : y^TA = 0, y^Tb = -1\}$ is empty.

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**Farkas Lemma:** A polyhedron $\{x : Ax \leq b\}$ is non-empty if and only if there is no solution $\{y : y^TA = 0, y^Tb < 0, y \geq 0\}$.

We will give an (inefficient) algorithmic proof of Farkas lemma using an algorithm that decides whether a polyhedron is feasible: **Fourier-Motzkin’ algorithm**.
Fourier-Motzkin Algorithm

**INPUT:** Polyhedron $P = \{ x : Ax \leq b \}$

**OUTPUT:** Yes/No depending whether $P$ is empty or not.
Fourier-Motzkin Algorithm

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- Else we eliminate leading variable \((x_1)\). Re-write the inequalities to be regrouped in 3 groups:
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  $x_1 + (a'_i)^T x' \leq b'_i$,  \hspace{0.5cm} (if coefficient of $a_{i1}$ is positive) \hspace{0.5cm} (TYPE I)
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Fourier-Motzkin Algorithm

**INPUT:** Polyhedron \( P = \{x : Ax \leq b\} \)

**OUTPUT:** Yes/No depending whether \( P \) is empty or not.

- Else we eliminate leading variable \((x_1)\). Re-write the inequalities to be regrouped in 3 groups:

  \[ x_1 + (a_i')^T x' \leq b_i', \quad \text{(if coefficient of } a_{i1} \text{ is positive)} \quad \text{(TYPE I)} \]

  \[ -x_1 + (a_j')^T x' \leq b_j', \quad \text{(if coefficient of } a_{j1} \text{ is negative)} \quad \text{(TYPE II)} \]

  \[ (a_k')^T x' \leq b_k', \quad \text{(if coefficient of } a_{k1} \text{ is zero)} \quad \text{(TYPE III)} \]
Fourier-Motzkin Algorithm

**INPUT:** Polyhedron $P = \{ x : Ax \leq b \}$

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- Else we eliminate leading variable ($x_1$). Re-write the inequalities to be regrouped in 3 groups:

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  $$-x_1 + (a'_j)^T x' \leq b'_j, \quad \text{(if coefficient of } a_{j1} \text{ is negative)} \quad \text{(TYPE II)}$$

  $$(a'_k)^T x' \leq b'_k, \quad \text{(if coefficient of } a_{k1} \text{ is zero)} \quad \text{(TYPE III)}$$

Here $x' = (x_2, x_3, \ldots, x_n)$. 
Fourier-Motzkin continued

Add all possible pairs of inequalities of (TYPE I) and (TYPE II). Create new system (*) with fewer variables:

\[(a'_j + a'_i)^T x' \leq (b_j + b_i) \text{ for } i \text{ of type I and } j \text{ of type II}\]

Keep equations of type III \[(a'_k)^T x' \leq b'_k\]

Original system of inequalities has a solution if and only if the system (*) is feasible WHY?
Fourier-Motzkin continued

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Original system of inequalities has a solution if and only if the system (*) is feasible WHY?

Clearly new system

\((*) \text{ is equivalent to } (a'_j)^T x - b'_j \leq b'_i - (a'_i)^T x', \text{ and } (a'_k)^T x' \leq b'_k\)
Fourier-Motzkin continued

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Thus, if we manage to find \(x_2, x_3, \ldots, x_n\) satisfying (*), then we find \(x_1\) (squeezed in between).
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Thus, if we manage to find \(x_2, x_3, \ldots, x_n\) satisfying (*), then we find \(x_1\) (squeezed in between).

\[\max((a'_i)^T x - b'_i) \leq x_1 \leq \min(b'_i - (a'_i)^T x').\]
Fourier-Motzkin continued

Add all possible pairs of inequalities of (TYPE I) and (TYPE II). Create new system (*) with fewer variables:

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- Original system of inequalities has a solution if and only if the system (*) is feasible \textbf{WHY}?
- Clearly new system

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Thus, if we manage to find \(x_2, x_3, \ldots, x_n\) satisfying (*), then we find \(x_1\) (squeezed in between).

\[
\max((a'_i)^T x - b'_i) \leq x_1 \leq \min(b'_i - (a'_i)^T x').
\]
Proof of Farkas Lemma

- Indeed, if we reduce until we have no variables. New system becomes

\[
\begin{bmatrix}
0 \\
0 \\
\vdots \\
0
\end{bmatrix} \leq 
\begin{bmatrix}
b'_1 \\
\vdots \\
b'_n
\end{bmatrix}
\]

Polyhedron \( \{ x : A x \leq b \} \) infeasible \( \iff \) \( b'_i < 0 \) for some \( i \).

- Rewriting and addition steps correspond to row operations on the original matrix \( A \). This is done by matrix multiplication.

\[
0 = MAx \geq Mb = b', \text{ with matrix } M \text{ with non-negative entries}
\]

- Set \( y^T = (e_i)^T M \), with \( e_i \) standard \( i \)-th unit vector then
Here is another form of Farkas lemma:

- **Corollary:**
  \[ \{ x : Ax = b, x \geq 0 \} = \emptyset \iff \{ y : y^T A \geq 0, y^T b < 0 \} \neq \emptyset \]
More on Farkas I

Here is another form of Farkas lemma:

- **Corollary:**
  \[
  \{ x : Ax = b, x \geq 0 \} = \emptyset \iff \{ y : y^T A \geq 0, y^T b < 0 \} \neq \emptyset.
  \]

- **Proof:**
  \[
  \{ x : Ax = b, x \geq 0 \} \neq \emptyset \iff \{ x : Ax \leq b, -Ax \leq -b, -Ix \leq 0 \} \neq \emptyset.
  \]
Here is another form of Farkas lemma:

**Corollary:**

\[ \{ x : Ax = b, x \geq 0 \} = \emptyset \iff \{ y : y^T A \geq 0, y^T b < 0 \} \neq \emptyset. \]

**proof** \{ x : Ax = b, x \geq 0 \} \neq \emptyset \iff \{ x : Ax \leq b, -Ax \leq -b, -l x \leq 0 \} \neq \emptyset.

By previous version of Farkas, this happens if and only if no solution exists of \( y^T = [y_1 \ y_2 \ y_3]^T \) with

\[
[y_1 \ y_2 \ y_3]^T \begin{bmatrix} A & -A & -l \end{bmatrix} = 0, \quad [y_1 \ y_2 \ y_3]^T \begin{bmatrix} b & -b & 0 \end{bmatrix} < 0, \quad y^T \geq 0
\]

- The vector \( y_1 - y_2 \) has the desired property.
More on Farkas II

Here is another form of Farkas lemma:

**Corollary:**

\[ \{ x : Ax \leq b, x \geq 0 \} \neq \emptyset \iff \text{When } y^T A \geq 0, \text{ then } y^T b \geq 0 \]
More on Farkas II

Here is another form of Farkas lemma:

- **Corollary:**
  \[ \{ x : Ax \leq b, x \geq 0 \} \neq \emptyset \iff \text{When } y^T A \geq 0, \text{ then } y^T b \geq 0 \]

- **Proof Necessity:** We know \( x \geq 0, \ Ax = b \), if in addition \( y^T A \geq 0 \) then \( y^T b = y^T Ax \geq 0 \).
Here is another form of Farkas lemma:

- **Corollary:**
  \[ \{ x : A x \leq b, x \geq 0 \} \neq \emptyset \iff \text{When } y^T A \geq 0, \text{ then } y^T b \geq 0 \]

- **proof Necessity:** We know \( x \geq 0, A x = b \), if in addition \( y^T A \geq 0 \) then \( y^T b = y^T A x \geq 0 \).
  
**Sufficiency:** Suppose if \( y^T A \geq 0 \), then \( y^T b \geq 0 \) but assume \( \exists x \geq 0 \text{ with } A x = b \). From the previous corollary, \( \exists y \text{ with } y^T A \geq 0, y^T b < 0 \). Therefore \( 0 \leq y^T b < 0 \) which is a contradiction.
More on Farkas II

Here is another form of Farkas lemma:

- **Corollary:**
  \[ \{x : Ax \leq b, x \geq 0\} \neq \emptyset \iff \text{When } y^TA \geq 0, \text{ then } y^Tb \geq 0 \]

- **proof Necessity:** We know \( x \geq 0, \ Ax = b \), if in addition \( y^TA \geq 0 \) then \( y^Tb = y^TAx \geq 0 \).

- **Sufficiency:** Suppose if \( y^TA \geq 0 \), then \( y^Tb \geq 0 \) but assume \( \nexists x \geq 0 \) with \( Ax = b \). From the previous corollary, \( \exists y \) with \( y^TA \geq 0, y^Tb < 0 \). Therefore \( 0 \leq y^Tb < 0 \) which is a contradiction.

- There are many more consequences and variations of Farkas lemma (ALL theory of Linear Optimization based on it!!!).
Weyl-Minkowski

Theorem: [Weyl-Minkowski] Every polytope is a polyhedron. Every bounded polyhedron is a polytope.
Weyl-Minkowski

**Theorem:** [Weyl-Minkowski] Every polytope is a polyhedron. Every bounded polyhedron is a polytope.

This allows us to represent all polytopes in two ways inside a computer!! Either as a list of vertices, or as system of inequalities.
Theorem: (Weyl-Minkowski’s Theorem): For a polyhedral subset $P$ of $\mathbb{R}^d$ the following statements are equivalent:

- $P$ is an **H-polyhedron**, i.e., $P$ is given by a system of linear inequalities $P = \{x : Ax \geq b\}$.
- $P$ is a **V-polyhedron**, i.e., For finitely many vectors $v_1, \ldots, v_n$ and $r_1, \ldots, r_s$ we can write

$$P = \text{conv}(v_1, v_2, \ldots, v_n) + \text{cone}(r_1, r_2, \ldots, r_s)$$

Here $R + S$ denotes the **Minkowski sum** of two sets, $R + S = \{r + s : r \in R, s \in S\}$.

We need to design an efficient algorithm for the conversion between the H-polyhedron and V-polyhedron!

**NOTE:** Cone can be decomposed into a pointed cone plus a linear space.
Polyhedral Cones

A set $C \subseteq \mathbb{R}^n$ is a **cone** if it is closed under addition and multiplication by a positive constant.

- A set $C \subseteq \mathbb{R}^n$ is an **inequality constrained** cone if $C = \{x \in \mathbb{R}^n : Ax \geq 0\}$ for some matrix $A$.
- A set $C \subseteq \mathbb{R}^n$ is a **finitely generated** cone if $C = \{\lambda B : \lambda \in \mathbb{R}^k_+\}$ for some matrix $B$.

**Theorem (Minkowski-Weyl)**

*A cone $C \subseteq \mathbb{R}^n$ is finitely constrained if and only if it is finitely generated.*

- The set of *extreme rays* of the cone is the *minimal* set of generators of a cone.

**FUNDAMENTAL QUESTION:** how do we convert between the two representations?
Example 1

Consider the following cone $C$ and its two representations:

- $C = \{x \in \mathbb{R}^2 : 3x_1 - 2x_2 \geq 0, -x_1 + 2x_2 \geq 0\}$.
- $C = \{\lambda_1(2, 1) + \lambda_2(2, 3) : \lambda_1, \lambda_2 \in \mathbb{R}_+\}$. 
Example 2: Magic Squares

A **magic square** is a square grid of non-negative real numbers such that the rows, columns, and diagonals all add up to the same value.

\[
\begin{array}{ccc}
0 & 2 & 1 \\
2 & 1 & 0 \\
1 & 0 & 2 \\
\end{array}
\]

Magic Squares are closed under non-negative linear combinations

\[
\begin{array}{ccc}
0 & 2 & 1 \\
2 & 1 & 0 \\
1 & 0 & 2 \\
\end{array} \times 3 = \begin{array}{ccc}
0 & 6 & 3 \\
6 & 3 & 0 \\
3 & 0 & 6 \\
\end{array}
\]

\[
\begin{array}{ccc}
0 & 2 & 1 \\
2 & 1 & 0 \\
1 & 0 & 2 \\
\end{array} + \begin{array}{ccc}
1 & 2 & 0 \\
0 & 1 & 2 \\
2 & 0 & 1 \\
\end{array} = \begin{array}{ccc}
1 & 4 & 1 \\
2 & 2 & 2 \\
3 & 0 & 3 \\
\end{array}
\]
**Question:** Is there a finite set of $n \times n$ magic squares so that we can express every other possible magic square as a linear non-negative combination?

**YES!**

There are four such $3 \times 3$ magic squares:

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**IMPORTANT:** There is an algorithm for computing a minimal such set of magic squares for $n \times n$ magic squares. These magic squares are the **extreme rays** of the cone of magic squares.
HOMOGENIZE: Weyl-Minkowski reduces to the case of Cones!!

- We can reduce this problem to problem of transforming between the two representations of a cone. From an H-polyhedron construct a cone from the polytope as follows:

  $$ t = 1 $$

  - Observe: If the original polytope was given by inequalities $Ax \geq b$ then the cone is given by inequalities $\bar{A}y \geq 0$, where $\bar{A}$ is the extended matrix $[A, -b]$ and $y = (x, t)$.
  - Enough to solve Weyl-Minkowski’s Theorem for cones:
Proof of Weyl-Minkowski

The following are equivalent

- **$P$ is an $H$-cone**, i.e., $P$ is given by a system of linear inequalities $P = \{x : Ax \geq 0\}$.
- **$P$ is a $V$-cone**, i.e., For finitely many vectors $r_1, \ldots, r_s$ we can write
  \[
P = \text{cone}(r_1, r_2, \ldots, r_s)\]

This is equivalent to (Matrix form!!):
The following are equivalent

- **$P$ is an $H$-cone**, i.e., $\exists$ matrix $A$ such that $P = \{x : Ax \geq 0\}$.
- **$P$ is a $V$-cone**, i.e., $\exists$ matrix $R$ such that
  \[
P = \{x : x = Ry, \ y \geq 0\}\]

We say the pair $(A, R)$ is a **double description pair** (DD-pair).
Polar Cones

- **Definition:** Let $K$ be a convex cone the **polar** of $K$ is the set

$$K^* = \{ x \in \mathbb{R}^n : \langle x, y \rangle \leq 0 \text{ for all } y \in K \}$$

- **Lemma:** If $K$ is a cone then $K^*$ is a cone. In fact, $K = \text{cone}( \{a_1, a_2, \ldots, a_m\} )$, i.e. $K$ is generated by vectors then $K^*$ is given by inequalities:

$$K^* = \{ x : \langle x, a_i \rangle \leq 0, \ i = 1, \ldots, m \}.$$
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**Lemma:** $(K^*)^* = K$.

**PUNCH LINE:** It is enough to prove Weyl-Minkowski ONE of the implications, the other one follows by polar cone construction!!
Minkowski-Weyl Algorithmic version

**LEMMA** For any pair of matrices $A, R$, $(A, R)$ is a DD-pair of cone $C$ if and only if $(R^T, A^T)$ is a double description pair of the polar cone of $C$. 

**Proof:** (EXERCISE) Use Farkas lemma.

An (algorithmic) proof of Minkowski-Weyl's theorem:

Let $R$ be a matrix defining a V-cone, $C$, thus $C = \{ x : x = Ry, y \geq 0 \}$. By Fourier-Motzkin we can eliminate all variables $y$ from above system. The resulting system of inequalities is written as $Ax \geq 0$ (since Fourier-Motzkin respects the direction of inequalities). This proves that every V-cone can be written as an H-cone. By previous lemma we are done to prove the converse. WARNING: Not an efficient algorithm.
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- **LEMMA** For any pair of matrices $A, R$, $(A, R)$ is a DD-pair of cone $C$ if and only if $(R^T, A^T)$ is a double description pair of the polar cone of $C$.
  
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**WARNING:** Not an efficient algorithm.
The Double Description Method (Motzkin-Raiffa-Thompson-Thrall 1953)
The Double description Method I

- Suppose $A$ is an $m \times d$ matrix, defines cone $C = \{x : Ax \geq 0\}$.
- Let $A_K$ denote the submatrix of $A$ given by rows in index set $K$.
- Suppose we found already a matrix $R$ which is DD pair with $A_K$. From a new row index $i / \notin K$ construct new DD pair $(A_{K \cup \{i\}}, R')$ (but HOW?):
- Partition the column index set $J$ of $R$ into three parts:
  - $J^+ = \{j \in J : A_ir_j > 0\}
  - J^0 = \{j \in J : A_ir_j = 0\}
  - J^- = \{j \in J : A_ir_j < 0\}

  We recover the new $R'$ from the following lemma:
The Double description Method II

- **Lemma**: The pair \((A_{K \cup \{i\}}, R')\) is a DD pair, when the matrix \(R'\) is given by the \(d \times J'\) matrix such that
  - the index set is \(J' = J^+ \cup J^0 \cup (J^+ \times J^-)\), and
  - the new columns are \(r_{jj'} = (A_i r_j) r_{j'} - (A_i r_{j'}) r_j\) for each \((j, j') \in J^+ \times J^-\).

- **Proof**: Let \(C(A_{K \cup \{i\}}) = \{x : A_{K \cup \{i\}} x \geq 0\}\) and \(C(R') = \{x : x = R'y, y \geq 0\}\). We wish \(C(A_{K \cup \{i\}}) = C(R')\).
The Double description Method II

**Lemma:** The pair \((A_{K \cup \{i\}}, R')\) is a DD pair, when the matrix \(R'\) is given by the \(d \times J'\) matrix such that

- the index set is \(J' = J^+ \cup J^0 \cup (J^+ \times J^-)\), and
- the new columns are \(r_{jj'} = (A_ir_j)r_j' - (A_i'r_{j'})r_j\) for each \((j, j') \in J^+ \times J^-\).

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Clearly \(C(R') \subset C(A_{K \cup \{i\}})\) because \(r_{jj'} \in C(A_{K \cup \{i\}})\).
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x = \sum_{j \in J} \lambda_j r_j, \quad \text{with } \lambda_j \ge 0
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x = \sum_{j \in J} \lambda_j r_j, \quad \text{with } \lambda_j \geq 0
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If there is no \(\lambda_k > 0\) for \(k \in J^-\) then \(x \in C(R')\) already. Thus assume such \(\lambda_k\) exists.
Therefore since $A_ix \geq 0$ there must also be $\lambda_h > 0$ with $h \in J^+$. 
Therefore since $A_i x \geq 0$ there must also be $\lambda_h > 0$ with $h \in J^+$. 

Subtract a suitable multiple of $r_{kh} = (A_i c_h) r_k - (A_i c_k) r_h$ from $x = \sum_{j \in J} \lambda_j r_j$.
Therefore since $A_i x \geq 0$ there must also be $\lambda_h > 0$ with $h \in J^+$.

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We are left with a new expression of $x$ with smaller non-zero coefficients. This process can be repeated as long as $\lambda_k > 0$ with $k \in J^-$ exists.

So in finitely many steps we must get rid of all such $\lambda$ at which point we have $x \in C(R')$. 
The Double description Method IV

- We can refine the above construction, finding a matrix $R'$ which has no redundant columns!!
- We say $r_j$ is a extreme ray if it cannot be written as a non-negative combination of two other rays. Thus all we need to do is throw away columns of the matrix which are not extreme rays. How to tell???
- **Lemma:** Let $Z(x)$ be the set of indices of inequalities such that $A_i x = 0$. A ray $r$ is an extreme ray of the cone \( \{x : x \in \mathbb{R}^d, Ax \geq 0\} \iff \text{the rank of the submatrix } A_{Z(r)} = d - 1. \)
- **How to do the initial DD pair??** Select a maximal submatrix $A_K$ with linearly independent rows of $A$.
- Initial matrix $R$ is the solution to $A_K R = I$. WHY? $\text{rank}(A) = d$ then $A_K$ must be square then $R = A_K^{-1}$. Then $(A_K, R)$ is DD pair since $A_K x \geq 0 \iff A_K^{-1} y, y \geq 0$. 
The double description method has a dual version called the **Beneath-Beyond method**.

DD is practical for low dimensions (see CDD).

The size of intermediate polytopes can be very very sensitive to the order in which the subspaces are introduced.

D. Bremner (1999) showed a family of polytopes for which the double description method is exponential.
The Ellipsoid Method
The Ellipsoid Method

The Algorithm

Given: A set $S$ is a polyhedron (bounded and convex) with $\text{vol}(S) > 0$, an ellipsoid $E_{M,z}$ such that $S \subseteq E_{M,z}$.

Want: $s \in S$. 

1. $k = 0$; $M_k = M_0$, $z_k = z_0$.
2. If $z_k \in S$: STOP; otherwise
3. Find a nonzero vector $a$ such that $a^T x \leq a^T z_k$, $\forall x \in S$; (separating hyperplane)
4. Construct the smaller volume ellipsoid that contains $E_{M,z} \cap \{x \in \mathbb{R}^n | a^T (x - z_k) \leq 0\}$.
   Let this ellipsoid have matrix $M_{k+1}$ and center $z_{k+1}$.
5. $k = k + 1$;
6. Go back to Step 2.
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Technical point: How to compute successive ellipsoids. Each ellipsoid is given by a positive definite matrix $A_k$ and a center $x_k$ namely

$$E_k = \{x : (x - x_k)^T A_k^{-1} (x - x_k) \leq 1\}$$

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Theorem (Grötschel, Lovász, Schrijver) If given a polyhedron $P$ and a point $x \not\in P$ you can SEPARATE them in polynomial time, then you can OPTIMIZE any linear functional over $P$ in polynomial time.
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Thank you