Polyhedral Algorithms

Part II: Optimization and Pivoting Algorithms

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- **Theorem** If $P = \{x \mid Ax \leq b\} \neq \emptyset$ and $P \subset \mathbb{R}^n$ then $\dim(P) = n - \text{rank}(A^\top, b^\top)$ where $A^\top x \leq b^\top$ is the set of implicit equalities.
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**Theorem** If $P = \{x \mid Ax \leq b\} \neq \emptyset$ and $P \subset \mathbb{R}^n$ then \(\dim(P) = n - \text{rank}(A^=, b^=)\) where $A^=x \leq b^=$ is the set of implicit equalities.
What is the dimension of the following polytope?

(1) \( x_1 + x_2 + x_3 \geq 2 \)
(2) \( x_1 + x_2 \leq 1 \)
(3) \( x_3 \leq 1 \)
(4) \( x_1 \leq \frac{1}{2} \)
(5) \( x_1, x_2, x_3 \geq 0 \)
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We have \( x_1 + x_2 + x_3 = 2 \) (one implied equality)
\[ \Rightarrow \dim(P) = n - \text{rank}(A^=, b^=). \]
Given a polyhedron \( P = \{ x \mid Ax \leq b \} \) the inequality \( a'x \leq b' \) is valid for \( P \) if it is satisfied by all points in \( P \).
How to compute FACES? Eliminate REDUNDANCY, FACETS

- Given a polyhedron $P = \{ x \mid Ax \leq b \}$ the inequality $a'x \leq b'$ is valid for $P$ if it is satisfied by all points in $P$.
- Let $v'x \leq g$ be a valid inequality for $P$ and let $F = \{ x \in P \mid v'x = g \}$. Then $F$ is a face of $P$. A face is proper is $F \neq \emptyset, P$. 
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- A face of $P$ represented by $v'x \geq g$ is a facet if $\dim(F) = \dim(P) - 1$. This is a facet defining inequality.
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- A face of $P$ represented by $v'x \geq g$ is a facet if $\dim(F) = \dim(P) - 1$. This is a facet defining inequality.
- **Theorem** For each facet $F$ of $P$, at least one inequality representing $F$ is necessary in any description of $P$. If an inequality represents a face of smaller dimension that $\dim(P) - 1$, then it can be dropped (IRREDUNDANT SYSTEM).
How to compute FACES? LOW DIMENSION

- **Theorem** Let $P = \{ x : Ax \leq b \}$. Then a nonempty subset $F$ of $P$ is a face of $P$ if and only if $F$ is represented as the set of solutions to an inequality system obtained from $Ax \leq b$ by setting some of the inequalities to equalities in an irredundant system of $P$. 
Theorem  Let $P = \{x : Ax \leq b\}$. Then a nonempty subset $F$ of $P$ is a face of $P$ if and only if $F$ is represented as the set of solutions to an inequality system obtained from $Ax \leq b$ by setting some of the inequalities to equalities in an irredundant system of $P$.

Corollary Every minimal nonempty face of $P$ is an affine subspace of form $\{x : A_1x = b_1\}$ where $A_1x = b_1$ is a subsystem of $Ax = b$. 

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**Theorem** Let $P = \{x : Ax \leq b\}$. Then a nonempty subset $F$ of $P$ is a face of $P$ if and only if $F$ is represented as the set of solutions to an inequality system obtained from $Ax \leq b$ by setting some of the inequalities to equalities in an irredundant system of $P$.

**Corollary** Every minimal nonempty face of $P$ is an affine subspace of form $\{x : A_1x = b_1\}$ where $A_1x = b_1$ is a subsystem of $Ax = b$.

**Corollary** There are finitely many faces!
We have been looking at the problem: Is there a point $x$ such that $Ax \leq b$? This is the Feasibility problem.

There is another problem, the Optimization problem:

Maximize/Minimize linear functional $c^T x$ subject to $Ax \leq b$.

Surprise: From the point of view theory if you know how to solve one problem, you know how to solve the other?

Of course, in practice they may perform differently, but I do not have time to make the distinction!!

Recall Farkas:

\[
\{ x \mid Ax \leq b \} \text{ is non-empty } \iff \text{there is no solution } \{ y \mid y \geq 0, y^T A = 0, y b < 0 \}.
\]

It has OPTIMIZATION VERSION TOO!!

Theorem: If a finite optimum for max \[
\{ c x \mid Ax \leq b \}
\]

then min \[
\{ y b \mid y \geq 0, y^T A = c \}
\]

has a finite optimum too!!

Optimum is at a vector $y \geq 0$ whose positive components correspond to the linearly independent rows of $A$!!!
Optimization = Feasibility

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**Theorem:** If a finite optimum for \( \max \{ cx \mid Ax \leq b \} \) exists then \( \min \{ yb \mid y \geq 0, yA = c \} \) has a finite optimum too!!
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**Theorem:** If a finite optimum for $\max \{cx \mid Ax \leq b\}$ exists then $\min \{yb \mid y \geq 0, yA = c\}$ has a finite optimum too!! Optimum is at a vector $y \geq 0$ whose positive components correspond to the linearly independent rows of $A$!!
**Basic Idea:** Search or traverse the graph of a polytope OR a hyperplane arrangement by *pivoting* operations that move us from one vertex to the next. That way we can generate them all.
The Simplex Method
Is there any solution of $Ax \geq b$?

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- **Lemma** Given any system of inequalities $Ax \leq b$, $Cx \geq d$, then it can be transformed into a new system of the form

  $$D\bar{x} = f, \bar{x} \geq 0$$

  with the property that one system has a solution $\iff$ the other system has a solution.
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- **Proof** The inequality $\sum_{j=1}^{n} a_{ij}x_j \leq b_i$ can be turned into an equation:

  Add the variable $s_i \rightarrow \sum_{j=1}^{n} a_{ij}x_j + s_i = b_i$ with $s_i \geq 0$

  Similarly, $\sum_{j=1}^{n} c_{ij}x_j \geq d_i \rightarrow \sum_{j=1}^{n} c_{ij}x_j - t_i = d_i$

  Finally, note that a variable $x_j$ unrestricted can be replaced by
Example: From Inequalities to Equations

Solve the system of inequalities:
7x + 3y - 20z \leq -2
4x - 3y + 9z \leq 3
-x + 2y - z \geq 4
11x - 2y + 2z \geq 11

Using the previous lemma, we can now modify the system:

\[ 7x^+ - 7x^- + 3y^+ - 3y^- + 20z^+ - 20z^- + s_1 = -2 \]
\[ 4x^+ - 4x^- - 3y^+ + 3y^- + 9z^+ - 9z^- + s_2 = 3 \]
\[ -x^+ + x^- + 2y^+ - 2y^- - z^+ + z^- - t_1 = 4 \]
\[ 11x^+ - 11x^- - 2y^+ + 2y^- - 2z^+ + 2z^- - t_2 = 11 \]

where \( x^\pm, y^\pm, z^\pm, t_1, t_2, S_1, s_2 \geq 0 \)

but how can solve it???
The Simplex Method “Expresso” version

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**Algorithm:** B-Rule Algorithm (aka Simplex method)
The Simplex Method “Expresso” version

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- **Algorithm:** B-Rule Algorithm (aka Simplex method)

**input:** $A \in \mathbb{Q}^{m \times n}$ of full row rank and $b \in \mathbb{Q}^m$.

**output:** Either a nonnegative vector $x$ with $Ax = b$ or a vector $y$ certifying infeasibility.

**Step 1:** Find an invertible $m \times m$ submatrix $B$ of $A$. Rewrite the system $Ax = b$ leaving the variables associated to $B$ in the left.

**Step 2:** Set all the non-basic variables to zero. Find the smallest index of a basic variable with negative solution.
Else, select the equation corresponding to that basic variable continue to Step 3.

- **Step 3:** Find the non-basic variable in the equation chosen in Step 2, that has smallest index and a positive coefficient.

- If there is none, then the problem is infeasible, stop! Else, solve this equation for the non-basic variable and substitute the result in all other equations. This variable becomes now basic, the former basic variable becomes non-basic. Go to Step 2.

**NOTE:** This last switch of variables is called a PIVOT.

**NOTE:** The simplex algorithm in general will have different PIVOT RULE to choose which variable leaves which variable enters the set of basic variables.
Example 1

Solve the next system for $x_i \geq 0$, $i = 1, 2, \ldots, 7$.

\begin{align*}
2x_1 + x_2 + 3x_3 + x_4 + x_5 &= 8 \\
2x_1 + 3x_2 + 4x_4 + x_6 &= 12 \\
3x_1 + 2x_2 + 2x_3 + x_7 &= 18.
\end{align*}

- **Step 1 of the $B$-Rule Algorithm**: find a basis in the matrix $A$.
  We choose the easiest basis, which is given by the 5th, 6th and 7th columns of $A$.
  Denote the basis by $B = \{5, 6, 7\}$ and the set of the remaining vectors by $NB = \{1, 2, 3, 4\}$.
- Next we solve the equation $Ax = b$ for the basic variables $X_B = \{x_5, x_6, x_7\}$. 
\[ X_B = B^{-1}b - B^{-1}CX_{NB} = \begin{bmatrix} 8 \\ 12 \\ 18 \end{bmatrix} - \begin{bmatrix} 2 & 1 & 3 & 1 \\ 2 & 3 & 0 & 4 \\ 3 & 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \]
Example 1 Continued

- **Step 2 in the B-Rule Algorithm:** Set all non-basic variables equal to zero. We obtain the non-negative values $x_5 = 8$, $x_6 = 12$, $x_7 = 18$.

- Therefore, this problem is feasible, and its solution is given by $x_1 = x_2 = x_3 = x_4 = 0$, $x_5 = 8$, $x_6 = 12$ and $x_7 = 18$.

- Suppose we choose a different basis from the above, say $B' = (1, 4, 7)$, and solve the problem keeping this election. It is not difficult to obtain solution $x_1 = 10/3$, $x_2 = x_3 = 0$, $x_4 = 10/3$, $x_5 = x_6 = 0$, $x_7 = 8$.

- We observe that the two solutions are completely different. In general, the solution always depends on the election of the basis.
Example 2

Next we solve the system $Ax = b$ for $x_i \geq 0$, $i = 1, 2, ..., 6$, where $A$ and $b$ are given by

\[
A = \begin{bmatrix}
-1 & -2 & 1 & 1 & 0 & 0 \\
1 & -3 & -1 & 0 & 1 & 0 \\
-1 & -2 & 2 & 0 & 0 & 1
\end{bmatrix} \quad b = \begin{bmatrix}
-1 \\
2 \\
-2
\end{bmatrix}
\]

- **Step 1** Choose a basis from $A$, say $B = \{4, 5, 6\}$. Next we solve the system for the basic variables $X_B = \{x_4, x_5, x_6\}$.

\[
x_4 = -1 + x_1 + 2x_2 - x_3 \\
x_5 = 2 - x_1 + 3x_2 + x_3 \\
x_6 = -2 + x_1 + 2x_2 - 2x_3
\]

- **Step 2**: Setting all non-basic variables equal to zero, we get $x_4 = -1$, $x_5 = 2$ and $x_6 = -2$. Note that $x_4$ and $x_6$ are basic variables with negative solution. Choose that one with smallest index!!
Example 2, Continued

- Choose the equation that corresponds to $x_4$ in the equation above.
- Next, we must find the non-basic variable in the equation that has smallest index and a positive coefficient, in this case $x_1$ is such a variable.
- **Step 3** Solve the first equation for that non-basic variable, taking from now $x_1$ as basic variable and go back to Step 2 of the algorithm.

\[
\begin{align*}
  x_1 &= 1 - 2x_2 + x_3 + x_4 = 1 - 2x_2 + x_3 + x_4 \\
  x_5 &= 2 - (1 - 2x_2 + x_3 + x_4) + 3x_2 + x_3 = 1 + 5x_2 - x_4 \\
  x_6 &= -2 + (1 - 2x_2 + x_3 + x_4) + 2x_2 - 2x_3 = -1 - x_3 + x_4
\end{align*}
\]

- Set non-basic variables equal to zero, we obtain $x_1 = 1$, $x_5 = 1$ and $x_6 = -1$. We see $x_6$ has negative solution so we
Example 2, Continued

- The non-basic variable selected is $x_4$. Solve that equation for $x_4$ and rewrite the system as follow.

\[
\begin{align*}
    x_1 &= 1 - 2x_2 + x_3 + (1 + x_3 + x_6) = 2 - 2x_2 + 2x_3 + x_6 \\
    x_4 &= 1 + x_3 + x_6 = 1 + x_3 + x_6 \\
    x_5 &= 1 + 5x_2 - (1 + x_3 + x_6) = 0 + 5x_2 - x_3 - x_6
\end{align*}
\]

- Back again to Step 2: now with $x_1$, $x_4$ and $x_5$ as basic variables, we set all non-basic variables equal to zero obtaining non-negative solutions for the basic variables.

- So we have found that one solution to the problem is $x_1 = 2$, $x_2 = x_3 = 0$ $x_4 = 1$ and $x_5 = x_6 = 0$. 
Example 3

Solve the system $Ax = b$ for $x_i \geq 0$, $i = 1, \ldots, 6$, where $A$ and $b$ are given as follow.

$$A = \begin{bmatrix} -1 & 2 & 1 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 & 1 & 0 \\ -1 & -6 & 23 & 0 & 0 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 3 \\ -17 \\ 19 \end{bmatrix}$$

First choose a basis from $A$, say $B = \{4, 5, 6\}$, and solve the system for the basic variables $x_4$, $x_5$ and $x_6$.
We obtain...
Example 3, Continued

\[ x_4 = 3 + x_1 - 2x_2 - x_3 \]
\[ x_5 = -17 - 3x_1 + 2x_2 - x_3 \]
\[ x_6 = 19 + x_1 + 6x_2 + 23x_3 \]

- Set all non-basic variables equal to zero. We obtain \( x_4 = 3 \), \( x_5 = -17 \) and \( x_6 = 19 \). Since \( x_5 \) has negative solution we have to find the non-basic variable in the second equation that has smallest index and a positive coefficient.
- The variable selected for the pivot is \( x_2 \).
Example 3, Continued

Solve that equation for $x_2$ and after substitute the variable $x_5$ by the variable $x_2$ in the basis.

\[
x_2 = \frac{17}{2} + \frac{3}{2}x_1 + \frac{1}{2}x_3 + \frac{1}{2}x_5 \\
x_4 = 3 + x_1 - 2\left(\frac{17}{2} + \frac{3}{2}x_1 + \frac{1}{2}x_3 + \frac{1}{2}x_5\right) - x_3 \\
x_5 = 19 + x_1 + 6\left(\frac{17}{2} + \frac{3}{2}x_1 + \frac{1}{2}x_3 + \frac{1}{2}x_5\right) + 23x_3
\]

then

\[
x_2 = \frac{17}{2} + \frac{3}{2}x_1 + \frac{1}{2}x_3 + \frac{1}{2}x_5 \\
x_4 = -14 - 2x_1 - 2x_3 - x_5 \\
x_5 = 70 + 10x_1 + 26x_3 + 3x_5
\]

- We are in step two again. Set all non-basic variables equal to zero. The only solution negative is $x_4 = -14$, so we must choose the corresponding equation to $x_4$. 
The algorithm works!!

Note that by construction of the algorithm gives the desired answer IF the algorithm ever terminates!!

**Lemma** If $x_n$ is the last variable, during the $B$-rule iterations, $x_n$ cannot enter the basic variables and then leave OR leave and then enter.
The algorithm works!!

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**Lemma** If $x_n$ is the last variable, during the $B$-rule iterations, $x_n$ cannot enter the basic variables and then leave OR leave and then enter.

**(proof of lemma)**

- When $x_n$ is chosen to enter the basic variables among the equations of the dictionary one finds

\[ x_i = b_i + \sum_{j \notin B} a_{ij}x_j + a_{in}x_n \]

where $a_{ij} \leq 0$, $b_i < 0$ and $a_{in} > 0$. 
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  where $a_{ij} \leq 0$, $b_i < 0$ and $a_{in} > 0$.

- Therefore, any solution of the whole system with $x_l \geq 0$ for $l \neq n$ must necessarily have $x_n > 0$, otherwise a contradiction occurs!
When $x_n$ is chosen to leave the basic variables we have

$$x_i = b'_i + \sum_{j \in N} a'_{ij}x_j \quad (i \in B \setminus \{n\})$$

$$x_n = b'_n + \sum_{j \in N} a'_{nj}x_j$$

This shows $b'_n < 0$ while $b'_i \geq 0$ for all others indices. Setting the non-basic variables to zero shows there is a solution with $x_1, \ldots, x_{n-1} \geq 0$ but $x_n < 0$. Thus leaving and entering or entering and leaving are incompatible situations.
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Thus leaving and entering or entering and leaving are incompatible situations.
**Theorem** The $B$-Rule Algorithm terminates

(proof of Theorem): By contradiction.

- Suppose there is a matrix $A$ and a vector $b$ for which the algorithm does not terminate. Let us assume that $A$ is an example with smallest number of rows plus columns.
Theorem  The $B$-Rule Algorithm terminates  
(proof of Theorem): By contradiction. 
- Suppose there is a matrix $A$ and a vector $b$ for which the algorithm does not terminate. Let us assume that $A$ is an example with smallest number of rows plus columns. 
- Since there is a finite number of bases, in fact no more than $\binom{n}{m}$, then if the algorithm does not terminate one can find a cycle of bases in the iterations. One starts at one basis $B_1$, then moves to $B_2, B_3, \ldots$, and after say $k$ iterations one returns to $B_1$. 
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- By the lemma, during this cycle of bases, the last variable $x_n$ is either in all $B_i$ or in none of them.
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- By the lemma, during this cycle of bases, the last variable $x_n$ is either in all $B_i$ or in none of them.
- If $x_n$ is a basic variable discard the associated equation, it did not affect the choice of variables entering or leaving the basis. We have a smaller counterexample (fewer rows!)
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- By the lemma, during this cycle of bases, the last variable $x_n$ is either in all $B_i$ or in none of them.
- If $x_n$ is a basic variable discard the associated equation, it did not affect the choice of variables entering or leaving the basis. We have a smaller counterexample (fewer rows!)
- If $x_n$ is always non-basic then delete $x_n$. A counterexample (fewer columns).
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- It is called the reverse-search method because it reverses the simplex method: It goes from the optimal vertex in unique paths toward all possible other vertices, recovering a spanning tree of the polytope.
- For simple polytopes reverse-search is an output-sensitive algorithm.
Application 1: Hyperplane arrangements
Central arrangements—Zonotopes

- Given a central arrangement of hyperplanes represented by a $d \times m$ matrix $A$, i.e., $h_i = x : A_i x = 0$ Look at the cut section of the arrangement with the unit $(d - 1)$-sphere $S^{d-1}$
- Each hyperplane cuts a $(d - 2)$-sphere. Each is an arrangement of spheres (or great-circles), giving a spherical polytope!!!
**Theorem** The face lattice of this spherical polytope is the dual to the zonotope generated as the Minkowski sum of the line segments \([-A_i, A_i]\).
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Theorem: For a hyperplane arrangement with in \(\mathbb{R}^d\) and \(m\) hyperplanes, there is an efficient implementation of Reverse Search that lists all the regions of an arrangement (equivalently the vertices of a zonotope) with time complexity \(O(md(\#\text{regions}))\) and space complexity \(O(md)\). Similarly the vertices of the arrangement can be listed efficiently.
Application 2: Voronoi Diagrams
Question: How to properly assign regions of vigilance to AAA repair shops or Fire stations??
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Given a finite set of point in space $\mathbb{R}^d$. The Voronoi diagram of $S$ is a decomposition of space into regions associated to each of the points $p \in S$:

$$\text{near}(p) = \{x \in \mathbb{R}^d : \text{dist}(x, p) \leq \text{dist}(x, q) \text{ for all } q \in S - p\}$$

Each region is a polyhedral cell, a Voronoi cell. Finding those cells has many applications.
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HOW? Lift the points of $S$ to the paraboloid $x_{d+1} = x_1^2 + x_2^2 + \cdots + x_d^2$. The consider the polyhedron whose inequalities are precisely the tangent planes at the lifted points.
Replace each equation with inequality $\geq$ for each $p \in S$ to obtain a polyhedron $P_S$ given by inequalities
\[ \sum_{j=1}^{d} p_j^2 - 2p_j x_j + x_{d+1} \geq 0 \]
Replace each equation with inequality $\geq$ for each $p \in S$ to obtain a polyhedron $P_S$ given by inequalities
\[
\sum_{j=1}^{d} p_j^2 - 2p_jx_j + x_{d+1} \geq 0
\]
- **Theorem** The Voronoi diagram of $S$ is the orthogonal projection of each facet of $P_S$ back into the original space.
Thank you