Polyhedral Algorithms
Part II: Optimization and Pivoting Algorithms

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- **Theorem** If $P = \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\} \neq \emptyset$ and $P \subset \mathbb{R}^n$ then $\dim(P) = n - \text{rank}(A^=, \mathbf{b}^=)$ where $A^=\mathbf{x} \leq \mathbf{b}^=$ is the set of implicit equalities.

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- **Theorem** If $P = \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\} \neq \emptyset$ and $P \subset \mathbb{R}^n$ then $\dim(P) = n - \text{rank}(A^=, \mathbf{b}^=)$ where $A^=\mathbf{x} \leq \mathbf{b}^=$ is the set of implicit equalities.

Example

What is the dimension of the following polytope?

(1) $x_1 + x_2 + x_3 \geq 2$

(2) $x_1 + x_2 \leq 1$

(3) $x_3 \leq 1$

(4) $x_1 \leq \frac{1}{2}$

(5) $x_1, x_2, x_3 \geq 0$

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We have $x_1 + x_2 + x_3 = 2$ (one implied equality)
$\Rightarrow \dim(P) = n - \text{rank}(A^=, b^=)$.

How to compute FACES? Eliminate REDUNDACY, FACETS

- Given a polyhedron $P = \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\}$ the inequality $\mathbf{a}'\mathbf{x} \leq \mathbf{b}'$ is valid for P if it is satisfied by all points in P.

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- Let $\mathbf{v}'\mathbf{x} \leq \mathbf{g}$ be a valid inequality for P and let $F = \{\mathbf{x} \in P \mid \mathbf{v}'\mathbf{x} = \mathbf{g}\}$. Then F is a face of P. A face is proper is $F \neq \emptyset, P$.

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- A face of P represented by $\mathbf{v}'\mathbf{x} \geq \mathbf{g}$ is a facet if $\dim(F) = \dim(P) - 1$. This is a facet defining inequality.

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- A face of P represented by $\mathbf{v}'\mathbf{x} \geq \mathbf{g}$ is a facet if $\dim(F) = \dim(P) - 1$. This is a facet defining inequality.
- **Theorem** For each facet F of P, at least one inequality representing F is necessary in any description of P. If an inequality represents a face of smaller dimension that $\dim(P) - 1$, then it can be dropped (IRREDUNDANT SYSTEM).

How to compute FACES? LOW DIMENSION

- **Theorem** Let $P = \{x : Ax \leq b\}$. Then a nonempty subset F of P is a face of P if and only if F is represented as the set of solutions to an inequality system obtained from $Ax \leq b$ by setting some of the inequalities to equalities in an irredundant system of P.

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- **Corollary** There are finitely many faces!

Optimization=Feasibility

- We have been looking at the problem: Is there a point x such that $Ax \leq b$?. This is the Feasibility problem.
- There is another problem, the Optimization problem: Maximize/Minimize linear functional $c^T x$ subject to $Ax \leq b$.

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- Recall Farkas:$\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\}$ is non-empty $\Leftrightarrow$ there is no solution $\{\mathbf{y} \mid \mathbf{y} \geq 0,\ \mathbf{y}^T A = 0,\ \mathbf{y}\mathbf{b} < 0\}$. It has **OPTIMIZATION VERSION TOO!!**

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- **Theorem**: If a finite optimum for $\max\{\mathbf{cx} \mid A\mathbf{x} \leq \mathbf{b}\}$ exists then $\min\{\mathbf{yb} \mid \mathbf{y} \geq 0,\ \mathbf{y}A = \mathbf{c}\}$ has a finite optimum too!!

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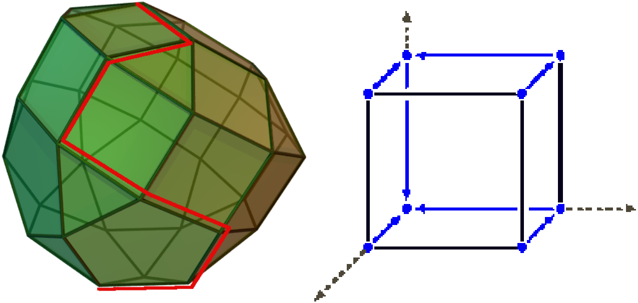
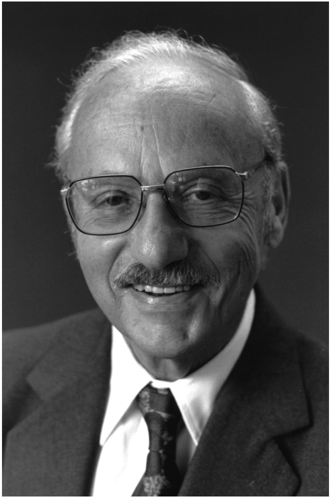
Basic Idea: Search or traverse the graph of a polytope OR a hyperplane arrangement by pivoting operations that move us from one vertex to the next. That way we can generate them all.



The Simplex Method



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- **lemma** Given any system of inequalities Ax $\leq$ b, Cx $\geq$ d, then it can be transformed into a new system of the form

  $$D\bar{x} = f, \bar{x} \geq 0$$

  with the property that one system has a solution $\Leftrightarrow$ the other system has a solution.

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proof The inequality $\sum_{j=1}^{n} a_{ij}x_j \leq b_i$ can be turned into an equation:
Add the variable $s_i \rightarrow \sum_{j=1}^{n} a_{ij}x_j + s_i = b_i$ with $s_i \geq 0$

Similarly, $\sum_{j=1}^{n} c_{ij}x_j \geq d_i \rightarrow \sum_{j=1}^{n} c_{ij}x_j - t_i = d_i$
Finally, note that a variable x_i unrestricted can be replaced by

Example: From Inequalities to Equations

Solve the system of inequalities:

7x + 3y - 20z $\leq$ -2

4x - 3y + 9z $\leq$ 3

-x + 2y - z $\geq$ 4

11x - 2y + 2z $\geq$ 11

Using the previous lemma, we can now modify the system:

$$7x^+ - 7x^- + 3y^+ - 3y^- + 20z^+ - 20z^- + s_1 = -2$$
$$4x^+ - 4x^- - 3y^+ + 3y^- + 9z^+ - 9z^- + s_2 = 3$$
$$-x^+ + x^- + 2y^+ - 2y^- - z^+ + z^- - t_1 = 4$$
$$11x^+ - 11x^- - 2y^+ + 2y^- - 2z^+ + 2z^- - t_2 = 11$$

where $x^{\pm}, y^{\pm}, z^{\pm}, t_1, t_2, S_1, s_2 \geq 0$

but how can solve it???

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- The key idea was introduced by Robert Bland (1970's) and developed in this form by Avis and Kaluzny.
- **Algorithm:** B-Rule Algorithm (aka Simplex method)
- **input:** $A \in Q^{m \times n}$ of full row rank and $b \in Q^m$.
- **output:** Either a nonnegative vector x with $Ax = b$ or a vector y certifying infeasibility.
- **Step 1**: Find an invertible $m \times m$ submatrix B of A. Rewrite the system $Ax = b$ leaving the variables associated to B in the left
- **Step 2:** Set all the non-basic variables to zero. Find the *smallest index* of a basic variable with negative solution

Else, select the equation corresponding to that basic variable continue to Step 3.

- **Step 3:** Find the non-basic variable in the equation chosen in Step 2, that has smallest index and a positive coefficient.
- If there is none, then the problem is infeasible, stop!
  Else, solve this equation for the non-basic variable and substitute the result in all other equations.
  This variable becomes now basic, the former basic variable becomes non-basic. Go to Step 2.

NOTE: This last switch of variables is called a PIVOT.
NOTE: The simplex algorithm in general will have different PIVOT RULE to choose which variable leaves which variable enters the set of basic variables.

Example 1

Solve the next system for $x_i \geq 0$, $i = 1, 2, ..., 7$.

$$\begin{aligned}
2x_1 + x_2 + 3x_3 + x_4 + x_5 &= 8 \\
2x_1 + 3x_2 + 4x_4 + x_6 &= 12 \\
3x_1 + 2x_2 + 2x_3 + x_7 &= 18.
\end{aligned}$$

- **Step 1 of the *B*-Rule Algorithm:** find a basis in the matrix A,.
  We choose the easiest basis, which is given by the 5th, 6th and 7th columns of A.
  Denote the basis by $B = \{5, 6, 7\}$ and the set of the remaining vectors by $NB = \{1, 2, 3, 4\}$.
- Next we solve the equation $Ax = b$ for the basic variables $X_B = \{x_5, x_6, x_7\}$.

$$X_B = B^{-1}b - B^{-1}CX_{NB} = \begin{bmatrix} 8 \\ 12 \\ 18 \end{bmatrix} - \begin{bmatrix} 2 & 1 & 3 & 1 \\ 2 & 3 & 0 & 4 \\ 3 & 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Example1 Continued

- **Step 2 in the B-Rule Algorithm:** Set all non-basic variables equal to zero. We obtain the non-negative values $x_5 = 8$, $x_6 = 12$, $x_7 = 18$.
- Therefore, this problem is feasible, and its solution is given by $x_1 = x_2 = x_3 = x_4 = 0$, $x_5 = 8$, $x_6 = 12$ and $x_7 = 18$.
- Suppose we choose a different basis from the above, say $B' = (1, 4, 7)$, and solve the problem keeping this election. It is not difficult to obtain solution $x_1 = 10/3$, $x_2 = x_3 = 0$, $x_4 = 10/3$, $x_5 = x_6 = 0$, $x_7 = 8$.
- We observe that the two solutions are completely different. In general, the solution always depends on the election of the basis.

Example 2

Next we solve the system $Ax = b$ for $x_i \geq 0$, $i = 1, 2, \ldots, 6.$, where A and b are given by

$$A = \begin{bmatrix} -1 & -2 & 1 & 1 & 0 & 0 \\ 1 & -3 & -1 & 0 & 1 & 0 \\ -1 & -2 & 2 & 0 & 0 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}$$

- **Step 1** Choose a basis from A, say $B = \{4, 5, 6\}$. Next we solve the system for the basic variables $X_B = \{x_4, x_5, x_6\}$.

$$\begin{aligned} x_4 &= -1 + x_1 + 2x_2 - x_3 \\ x_5 &= 2 - x_1 + 3x_2 + x_3 \\ x_6 &= -2 + x_1 + 2x_2 - 2x_3 \end{aligned}$$

- **Step 2:** Setting all non-basic variables equal to zero, we get $x_4 = -1$, $x_5 = 2$ and $x_6 = -2$. Note that x_4 and x_6 are basic variables with negative solution. Choose that one with

Example 2, Continued

- Choose the equation that corresponds to x_4 in the equation above.
- Next, we must find the non-basic variable in the equation that has smallest index and a positive coefficient, in this case x_1 is such a variable.
- **Step 3** Solve the first equation for that non-basic variable, taking from now x_1 as basic variable and go back to Step 2 of the algorithm.

$$\begin{aligned}
x_1 &= 1 - 2x_2 + x_3 + x_4 && = 1 - 2x_2 + x_3 + x_4 \\
x_5 &= 2 - (1 - 2x_2 + x_3 + x_4) + 3x_2 + x_3 && = 1 + 5x_2 - x_4 \\
x_6 &= -2 + (1 - 2x_2 + x_3 + x_4) + 2x_2 - 2x_3 && = -1 - x_3 + x_4
\end{aligned}$$

- Set non-basic variables equal to zero, we obtain $x_1 = 1$, $x_5 = 1$ and $x_6 = -1$. We see x_6 has negative solution so we

Example 2, Continued

- The non-basic variable selected is x_4. Solve that equation for x_4 and rewrite the system as follow.

$$
\begin{aligned}
x_1 &= 1 - 2x_2 + x_3 + (1 + x_3 + x_6) &&= 2 - 2x_2 + 2x_3 + x_6 \\
x_4 &= 1 + x_3 + x_6 &&= 1 + x_3 + x_6 \\
x_5 &= 1 + 5x_2 - (1 + x_3 + x_6) &&= 0 + 5x_2 - x_3 - x_6
\end{aligned}
$$

- Back again to Step 2: now with x_1, x_4 and x_5 as basic variables, we set all non-basic variables equal to zero obtaining non-negative solutions for the basic variables.
- So we have found that one solution to the problem is $x1 = 2$, $x_2 = x_3 = 0$ $x_4 = 1$ and $x_5 = x_6 = 0$.

Example 3

Solve the system $Ax = b$ for $x_i \geq 0$, $i = 1, \ldots 6$, where A and b are given as follow.

$$A = \begin{bmatrix} -1 & 2 & 1 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 & 1 & 0 \\ -1 & -6 & 23 & 0 & 0 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 3 \\ -17 \\ 19 \end{bmatrix}$$

First choose a basis from A, say $B = \{4, 5, 6\}$, and solve the system for the basic variables x_4, x_5 and x_6.
We obtain...

Example 3, Continued

$$\begin{aligned} x_4 &= 3 + x_1 - 2x_2 - x_3 \\ x_5 &= -17 - 3x_1 + 2x_2 - x_3 \\ x_6 &= 19 + x_1 + 6x_2 + 23x_3 \end{aligned}$$

- Set all non-basic variables equal to zero. We obtain $x_4 = 3$, $x_5 = -17$ and $x_6 = 19$. Since x_5 has negative solution we have to find the non-basic variable in the second equation that has smallest index and a positive coefficient.
- The variable selected for the pivot is x_2.

Example 3, Continued

Solve that equation for x_2 and after substitute the variable x_5 by the variable x_2 in the basis.

$$\begin{aligned}
x_2 &= 17/2 + 3/2x_1 + 1/2x_3 + 1/2x_5 \\
x_4 &= 3 + x_1 - 2(17/2 + 3/2x_1 + 1/2x_3 + 1/2x_5) - x_3 \\
x_5 &= 19 + x_1 + 6(17/2 + 3/2x_1 + 1/2x_3 + 1/2x_5) + 23x_3
\end{aligned}$$

then

$$\begin{aligned}
x_2 &= 17/2 + 3/2x_1 + 1/2x_3 + 1/2x_5 \\
x_4 &= -14 - 2x_1 - 2x_3 - x_5 \\
x_5 &= 70 + 10x_1 + 26x_3 + 3x_5
\end{aligned}$$

- We are in step two again. Set all non-basic variables equal to zero. The only solution negative is $x_4 = -14$, so we must choose the corresponding equation to x_4.

The algorithm works!!

Note that by construction of the algorithm gives the desired answer IF the algorithm ever terminates!!!

Lemma If x_n is the last variable, during the B-rule iterations, x_n cannot enter the basic variables and then leave OR leave and then enter.

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(proof of lemma)

- When x_n is chosen to enter the basic variables among the equations of the dictionary one finds

$$x_i = b_i + \sum_{j \notin B} a_{ij} x_j + a_{in} x_n$$

  where $a_{ij} \leq 0$, $b_i < 0$ and $a_{in} > 0$.

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  where $a_{ij} \leq 0$, $b_i < 0$ and $a_{in} > 0$.
- Therefore, any solution of the whole system with $x_l \geq 0$ for $l \neq n$ must necessarily have $x_n > 0$, otherwise a contradiction occurs!

- When x_n is chosen to leave the basic variables we have

$$x_i = b'_i + \sum_{j \in N} a'_{ij} x_j \quad (i \in B \setminus \{n\})$$

$$x_n = b'_n + \sum_{j \in N} a'_{nj} x_j$$

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- This shows $b'_n < 0$ while $b'_i \geq 0$ for all others indices. Setting the non-basic variables to zero shows there is a solution with $x_1, \ldots, x_{n-1} \geq 0$ but $x_n < 0$.

- When x_n is chosen to leave the basic variables we have

$$x_i = b_i' + \sum_{j \in N} a_{ij}' x_j \quad (i \in B \setminus \{n\})$$

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- This shows $b_n' < 0$ while $b_i' \geq 0$ for all others indices. Setting the non-basic variables to zero shows there is a solution with $x_1, \ldots, x_{n-1} \geq 0$ but $x_n < 0$.
- Thus leaving and entering or entering and leaving are incompatible situations.

Theorem The B-Rule Algorithm terminates
(proof of Theorem): By contradiction.

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- By the lemma, during this cycle of bases, the last variable x_n is either in all B_i or in none of them.
- If x_n is a basic variable discard the associated equation, it did not affect the choice of variables entering or leaving the basis. We have a smaller counterexample (fewer rows!)

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- Since there is a finite number of bases, in fact no more than $\binom{n}{m}$, then if the algorithm does not terminate one can find a cycle of bases in the iterations. One starts at one basis B_1, then moves to $B_2, B_3, \ldots$, and after say k iterations one returns to B_1.
- By the lemma, during this cycle of bases, the last variable x_n is either in all B_i or in none of them.
- If x_n is a basic variable discard the associated equation, it did not affect the choice of variables entering or leaving the basis. We have a smaller counterexample (fewer rows!)
- If x_n is always non-basic then delete x_n. A counterexample

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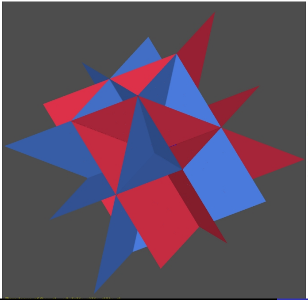
- It is called the reverse-search method because it reverses the simplex method: It goes from the optimal vertex in unique paths toward all possible other vertices, recovering a spanning tree of the polytope.
- For simple polytopes reverse-search is an output-sensitive algorithm.

Application 1: Hyperplane arrangements

Central arrangements=Zonotopes

- Given a central arrangement of hyperplanes represented by a $d \times m$ matrix A, i.e, $h_i = x : A_i x = 0$ Look at the cut section of the arrangement with the unit $(d-1)$-sphere S^{d-1}
- Each hyperplane cuts a $(d-2)$-sphere. Each is an arrangement of spheres (or great-circles), giving a spherical polytope!!!



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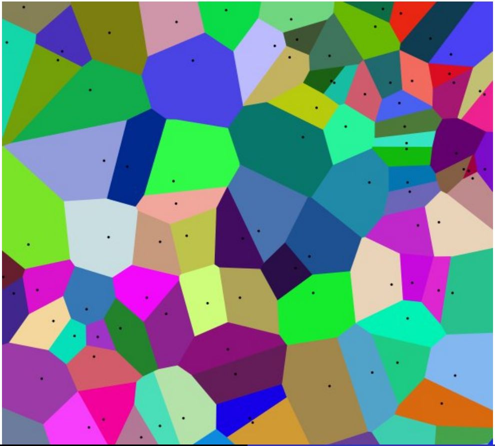
- **Theorem:** For a hyperplane arrangement with in $\mathbb{R}^d$ and m hyperplanes, there is an efficient implementation of Reverse Search that lists all the regions of an arrangement (equivalently the vertices of a zonotope) with time complexity $O(md(\#\textbf{regions}))$ and space complexity $O(md)$. Similarly the vertices of the arrangement can be listed efficiently

Application 2: Voronoi Diagrams



- Question: How to properly assign regions of vigilance to AAA repair shops or Fire stations??

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- Given a finite set of point in space $\mathbb{R}^d$. The Voronoi diagram of S is a decomposition of space into regions associated to each of the points $p \in S$:

$$\mathbf{near}(p) = \{x \in \mathbb{R}^d : dist(x,p) \leq dist(x,q) \text{ for all } q \in S - p\}$$

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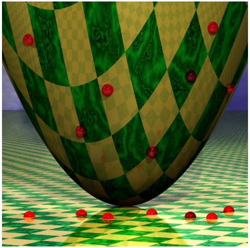
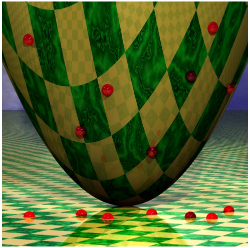
- IDEA: Voronoi cells can be computed are the projections of the facets of certain nice polytope!!!
- **HOW?** Lift the points of S to the paraboloid $x_{d+1} = x_1^2 + x_2^2 + \cdots + x_d^2$. The consider the polyhedron whose inequalities are precisely the tangent planes at the lifted points.



- Replace each equation with inequality $\geq$ for each $p \in S$ to obtain a polyhedron P_S given by inequalities $\sum_{j=1}^{d} p_j^2 - 2p_jx_j + x_{d+1} \geq 0$



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- **Theorem** The Voronoi diagram of S is the orthogonal projection of each facet of P_S back into the original space.

Thank you