POLYTOPES ARISING IN REPRESENTATION THEORY

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MENU

(1) The background and the questions.

(2) The polyhedral language.

(3) Some answers and conjectures.

BACKGROUND & QUESTIONS

Finite-dimensional Complex Lie Algebras

For us a **complex Lie Algebra** is a finite-dimensional vector space $\mathfrak{g}$ over $\mathbb{C}$ together with a bilinear form $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, the so called *Lie bracket*, that satisfies:

- $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0,$ for all $x, y, z \in \mathfrak{g}$
- $[x,x] = 0$ for all $x \in \mathfrak{g}$.

Example: $\mathfrak{gl}_n(\mathbb{C})$, the *general linear Lie-Algebra* The elements of $\mathfrak{gl}_n(V)$ are all linear maps from $\mathbb{C}^n$ into itself. Define $[a,b] := a \cdot b - b \cdot a$.

Fact: Every **Lie group** (i.e. a group with a compatible complex manifold structure) has always an associated Lie Algebra.

Example: the *special linear group* $\mathbf{SL}_n(\mathbb{C})$ of $n \times n$ matrices with determinant 1 has an associated Lie Algebra $\mathfrak{sl}_n(\mathbb{C})$ the vector space of traceless $n \times n$ matrices.

Fact: The **Simple Lie Algebras** are building blocks of all other Lie Algebras. They behave a bit like finite group. **Semisimple Lie Algebras** are direct sums of simple Lie Algebras.

Fact: All complex Simple finite-dimensional Lie Algebras have been classified using **discrete geometry**! In terms of the **root systems**.

Root Systems and Classification

There are four infinite series of indecomposable root systems :

- $A_r = \{e_i - e_j, \delta + e_i\}_{i \neq j}$, where $\delta = \sum_{k=1}^{r} e_k$. This system corresponds to $\mathfrak{sl}_{r+1}$.

- $B_r = \{\pm e_i \pm e_j\}_{i<j} \cup \{e_i\}$. This system corresponds to $\mathfrak{so}_{2r+1}$.

- $C_r = \{\pm e_i \pm e_j\}_{i<j} \cup \{2e_i\}$. This system corresponds to $\mathfrak{sp}_{2r}$.

- $D_r = \{\pm e_i \pm e_j\}_{i<j}$. This system corresponds to $\mathfrak{so}_{2r}$.

There are also five exceptional root systems G_2, F_4, E_6, E_7, E_8. All these appear, for example, in the classification of finite groups of reflections!

Representation Theory

Definition A representation of a Lie Algebra $\mathfrak{g}$ on a vector space V is a linear transformation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$, such that $\rho([x, y]) = [\rho(x), \rho(y)]$.

Fact: The representations of semisimple Lie Algebras decompose into direct sums of **irreducible representations**. The irreducible representations are indexed by **dominant weight vectors** or **highest weight vectors**.They belong to the *weight lattice*.

MAIN OBJECT: Given highest weights λ, μ, and ν dominant weights for irreducible representations of a finite dimensional complex semisimple Lie algebra,

$$V_\lambda \otimes V_\mu = \bigoplus_\nu C^\nu_{\lambda\mu} V_\nu. \tag{1}$$

The non-negative integers $C^\nu_{\lambda\mu}$ are the Clebsch–Gordan coefficients.

Why should one care about these numbers?

- Representation theorists, Combinatorialists need these numbers! For Lie algebras of type A_r these are, the Littlewood-Richardson coefficients.

- Physicists use these number in quantum mechanics and high-energy physics (e.g. Angular moment problems).

- Some computer scientists are interested because of the connection to the conjecture $P \neq NP$. Representation theory of Lie algebras plays a role on **geometric Complexity theory**. Conversely, complexity theory suggests several conjectures about Clebsch-Gordan coefficients!!

PROBLEM 1: Clebsch Gordan Coefficients Computation

From their approach to the $P \neq NP$ conjecture, Mulmuley and Sohoni (2001) have stated:

CONJECTURE: For all semisimple Lie Algebras of rank r and types A_r, B_r, C_r, D_r, given dominant weight vectors λ, μ, ν, one can decide, in polynomial time, whether the Clebsch-Gordan coefficient $C^{\nu}_{\lambda,\mu}$ is equal to zero.

Prior work: Narayanan 2005 has proved that it is $\#P$-complete to compute Littlewood-Richardson coefficients explicitly (type A_r case).

QUESTION What are fastest practical algorithms to compute the Clebsch–Gordan coefficients $C^{\nu}_{\lambda\mu}$?

In practice, TODAY, it would be impossible to compute with these numbers for A_{12} when the weights have entries bigger than 100.

PROBLEM 2: The Stretching Effect

Definition: *stretching function of Clebsch-Gordan coefficients*, is the function $e\colon \mathbb{Z}_{>0} \to Z_{\geq 0}$ given by $e(n) = C^{n\nu}_{n\lambda,n\mu}$.

Example: For B_3 with $\lambda = (0, 15, 5)$, $\mu = (12, 15, 3)$, and $\nu = (6, 15, 6)$ we have

$$C^{\mu}_{\lambda,\nu} = 2644, \quad C^{4\mu}_{4\lambda,4\nu} = 1393547, \quad C^{100\mu}_{100\lambda,100\nu} = 10784511824151$$

Prior work: Derksen-Weyman 2002 for the root system A_r, $e(n)$ is a **polynomial**.

QUESTION (King, Tollu and Toumazet 2003) Which properties are satisfied by the Clebsch-Gordan coefficients for semisimple Lie Algebras of types B_r, C_r, and D_r under the operation of *stretching*?

Theorem (Saturation condition) (Knutson Tao) For the roots system A_n, $C^{\mu}_{\lambda,\nu} \neq 0$ if and only if $C^{n\mu}_{n\lambda,n\nu} \neq 0$ for all values of $n > 1$.

POLYHEDRAL

LANGUAGE

The Polyhedral Approach

- IDEA: Represent Clebsch-Gordan Coefficients as the number of lattice points of some special convex polytopes!

- First people to think using polytopes Gelf'and-Tsetlin (1950's)! This approach has generated alot of work from the Representation theory community in the last 6 years. Knutson-Tao(1999), Berenstein-Zelevinsky (2001), Pak-Vallejo (2002), Cochet (2003), Baldoni-Silva-Vergne (2004).

Hive Polytopes

A hive pattern of rank r is a triangular array of real numbers

$$\begin{array}{ccccccccc} & & & & h_{0,0,r} & & & & \\ & & & h_{1,0,r-1} & & h_{0,1,r-1} & & & \\ & & h_{2,0,r-2} & & h_{1,1,r-2} & & h_{0,2,r-2} & & \\ & \iddots & & & \vdots & & & \ddots & \\ h_{r,0,0} & & h_{r-1,1,0} & & \cdots & & h_{1,r-1,0} & & h_{0,r,0} \end{array}$$

such that, in every little “little rhombus” of entries

$$
\begin{array}{ccc}
c & & b \\
& & \\
& a & & d
\end{array}
\qquad
\begin{array}{ccc}
& c & \\
a & & b \\
& d &
\end{array}
\qquad
\begin{array}{cccc}
& a & & c \\
& & & \\
d & & b &
\end{array}
$$

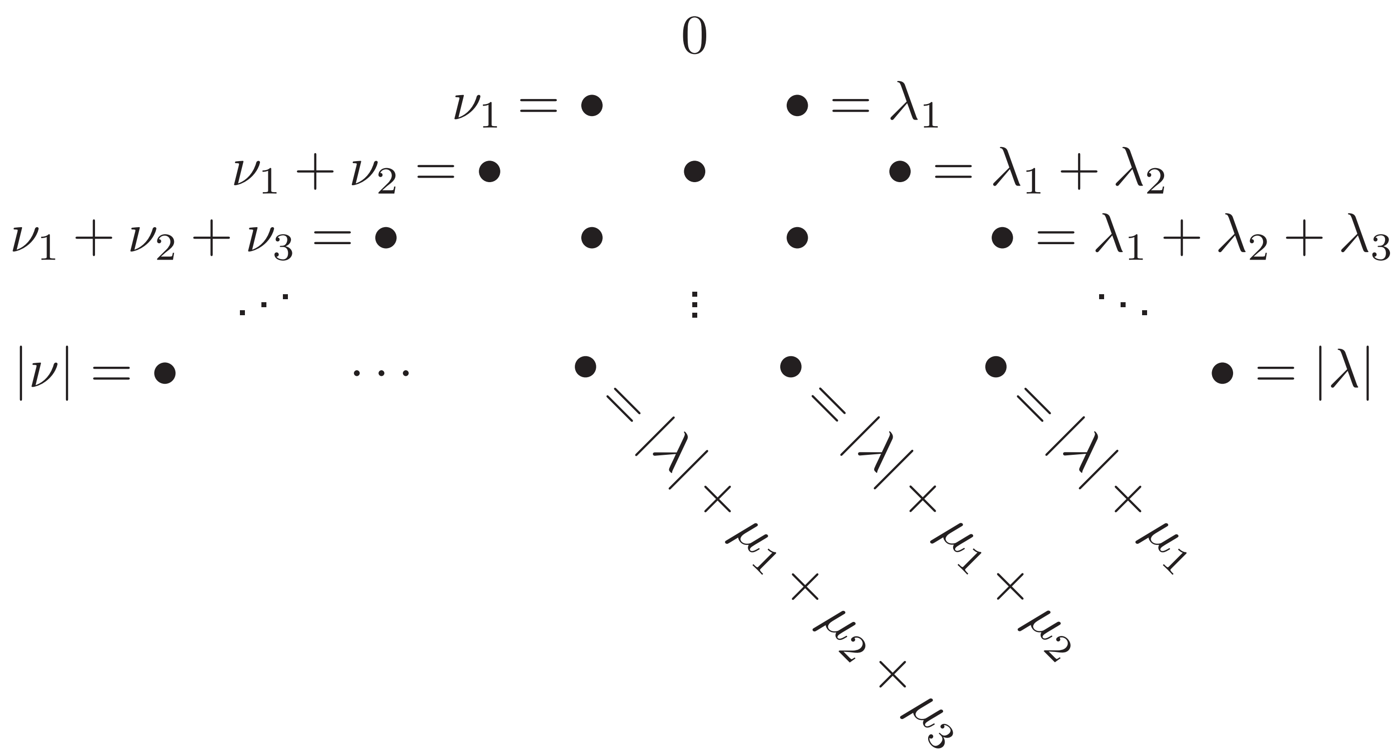
we have $a + b \geq c + d$.

Here is example of a Hive Pattern with $r = 4$:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & 5 & & 8 & & & \\
 & & 8 & & 12 & & 13 & & \\
 & 11 & & 15 & & 17 & & 18 & \\
12 & & 16 & & 18 & & 20 & & 20
\end{array}
$$

FACT: When $\mathfrak{g}$ is of type A_r, i.e. that $\mathfrak{g} \cong \mathfrak{sl}_{r+1}(\mathbb{C})$, the highest weights are (with respect to the standard basis) partitions of length $r+1$, *i.e.*, sequences λ of integers $\lambda_1 \geq \cdots \geq \lambda_{r+1} \geq 0$.

Definition Given partitions $\lambda, \mu, \nu \in \mathbb{Z}^n_{\geq 0}$, put $|\lambda| = \sum_i \lambda_i$, $|\mu| = \sum_i \mu_i$, and $|\nu| = \sum_i \nu_i$. The *hive polytope* $H^{\nu}_{\lambda\mu}$ is the set of those hive patterns with boundary entries fixed as below.



KEY LEMMAS

- **Lemma**(Knutson-Tao 1999) For type A_r, the Clebsch-Gordan coefficient (also called the Littlewood–Richardson coefficient) $C^{\nu}_{\lambda\mu}$ equals the number of integer lattice points in the **hive polytope** $H^{\nu}_{\lambda\mu}$.

- **Lemma**(Berenstein-Zelevinsky 2001) Fix a finite dimensional complex semisimple Lie algebra $\mathfrak{g}$ and a triple of highest weights (λ, μ, ν) for $\mathfrak{g}$. Then the Clebsch–Gordan coefficient $C^{\nu}_{\lambda\mu}$ equals the number of integer lattice points in certain **BZ-polytope**.

ANSWERS &

AND CONJECTURES

THEOREM

- For complex semisimple Lie algebras of type A_r, one can decide in polynomial time whether $C^{\nu}_{\lambda\mu} \neq 0$, even for non-fixed rank.

- For fixed rank r, if $\mathfrak{g}$ is a complex semisimple Lie algebra of rank r, then one can compute Clebsch–Gordan coefficients $C^{\nu}_{\lambda\mu}$ of irreducible representations of $\mathfrak{g}$ in time polynomial in the input size of the defining weights.

Ehrhart Quasipolynomials & Stretching

Lemma: Given highest weights λ, μ, ν of a semisimple Lie algebra, the stretching function $e(n) = C^{n\nu}_{n\lambda,n\mu}$ is a **Ehrhart quasi-polynomial** of a BZ-polytope or a hive polytope.

Recall, given a convex *rational* polytope P, let

$$nP = \{x : (1/n)x \in P\}, \qquad n = 1, 2, \ldots .$$

If P is a d-dimensional rational polytope, the counting function $i_P(n) = \#(nP \cap \mathbb{Z}^d)$ is a quasi-polynomial function of degree d; that is, there are polynomials $f_0(n), \ldots, f_{M-1}(n)$ of degree d s.t.

$$i_P(n) = \begin{cases} f_0(n) & \text{if } n \equiv 0 \mod M, \\ \vdots & \\ f_{M-1}(n) & \text{if } n \equiv M-1 \mod M. \end{cases}$$

λ, μ, ν	$C^{n\mu}_{n\lambda,n\nu}$
(0, 15, 5) (12, 15, 3) (6, 15, 6)	$\begin{cases} \frac{68339}{64}n^5 + \frac{407513}{384}n^4 + \frac{13405}{32}n^3 + \frac{9499}{96}n^2 + \frac{107}{8}n + 1, & n \text{ even} \\ \frac{68339}{64}n^5 + \frac{407513}{384}n^4 + \frac{13405}{32}n^3 + \frac{16355}{192}n^2 + \frac{659}{64}n + \frac{75}{128}, & n \text{ odd} \end{cases}$
(4, 8, 11) (3, 15, 10) (10, 1, 3)	$\begin{cases} \frac{13}{4}n^2 + 3n + 1, & n \text{ even} \\ \frac{13}{4}n^2 + 3n + 3/4, & n \text{ odd} \end{cases}$
(8, 1, 3) (11, 13, 3) (8, 6, 14)	$\begin{cases} \frac{121}{576}n^6 + \frac{1129}{640}n^5 + \frac{6809}{1152}n^4 + \frac{163}{16}n^3 + \frac{2771}{288}n^2 + \frac{191}{40}n + 1, & n \text{ even} \\ \frac{121}{576}n^6 + \frac{1129}{640}n^5 + \frac{6809}{1152}n^4 + \frac{1933}{192}n^3 + \frac{659}{72}n^2 + \frac{8003}{1920}n + \frac{93}{128}, & n \text{ odd} \end{cases}$
(8, 9, 14) (8, 4, 5) (1, 5, 15)	$\begin{cases} \frac{4117}{192}n^6 + \frac{50369}{640}n^5 + \frac{14829}{128}n^4 + \frac{703}{8}n^3 + \frac{3541}{96}n^2 + \frac{341}{40}n + 1, & n \text{ even} \\ \frac{4117}{192}n^6 + \frac{50369}{640}n^5 + \frac{14829}{128}n^4 + \frac{5599}{64}n^3 + \frac{3451}{96}n^2 + \frac{5001}{640}n + \frac{97}{128}, & n \text{ odd} \end{cases}$
(10, 5, 6) (5, 4, 10) (0, 7, 12)	$\begin{cases} \frac{669989}{960}n^5 + \frac{286355}{384}n^4 + \frac{10803}{32}n^3 + \frac{7993}{96}n^2 + \frac{1427}{120}n + 1, & n \text{ even} \\ \frac{669989}{960}n^5 + \frac{286355}{384}n^4 + \frac{10803}{32}n^3 + \frac{15509}{192}n^2 + \frac{10081}{960}n + \frac{65}{128}, & n \text{ odd} \end{cases}$

Figure 1: Stretched Clebsch–Gordan coefficients for B_3.

λ, μ, ν	$C_{n\lambda,n\nu}^{n\mu}$
(1,13,6) (14,15,5) (5,11,7)	$\begin{cases} \frac{5937739}{5760}\,n^6 + \frac{87023}{40}\,n^5 + \frac{936097}{576}\,n^4 + \frac{27961}{48}\,n^3 + \frac{85397}{720}\,n^2 + \frac{883}{60}\,n + 1, \ n \text{ eve} \\ \frac{5937739}{5760}\,n^6 + \frac{87023}{40}\,n^5 + \frac{936097}{576}\,n^4 + \frac{27961}{48}\,n^3 + \frac{657931}{5760}\,n^2 + \frac{3097}{240}\,n + 3/4, \ n \end{cases}$
(4,15,14) (12,12,10) (4,9,8)	$\begin{cases} \frac{22199219}{2880}\,n^6 + \frac{8154617}{960}\,n^5 + \frac{4500665}{1152}\,n^4 + \frac{31297}{32}\,n^3 + \frac{226903}{1440}\,n^2 + \frac{2021}{120}\,n + 1, \\ \frac{22199219}{2880}\,n^6 + \frac{8154617}{960}\,n^5 + \frac{4500665}{1152}\,n^4 + \frac{31297}{32}\,n^3 + \frac{217363}{1440}\,n^2 + \frac{13513}{960}\,n + \frac{85}{128} \end{cases}$
(9,0,8) (8,12,9) (7,7,3)	$1/30\,n^5 + 3/8\,n^4 + \frac{19}{12}\,n^3 + \frac{25}{8}\,n^2 + \frac{173}{60}\,n + 1$
(10,2,7) (8,10,1) (7,5,5)	$\begin{cases} \frac{596153}{1152}\,n^6 + \frac{53425}{48}\,n^5 + \frac{502621}{576}\,n^4 + \frac{5577}{16}\,n^3 + \frac{11941}{144}\,n^2 + \frac{149}{12}\,n + 1, \ n \text{ even} \\ \frac{596153}{1152}\,n^6 + \frac{53425}{48}\,n^5 + \frac{502621}{576}\,n^4 + \frac{5577}{16}\,n^3 + \frac{94097}{1152}\,n^2 + \frac{131}{12}\,n + \frac{23}{32}, \ n \text{ odd} \end{cases}$
(10,10,15) (11,3,15) (10,7,15)	$\begin{cases} \frac{6084163}{320}\,n^6 + \frac{507527}{30}\,n^5 + \frac{1185853}{192}\,n^4 + \frac{59995}{48}\,n^3 + \frac{43039}{240}\,n^2 + \frac{357}{20}\,n + 1, \ n \text{ e} \\ \frac{6084163}{320}\,n^6 + \frac{507527}{30}\,n^5 + \frac{1185853}{192}\,n^4 + \frac{59995}{48}\,n^3 + \frac{144751}{960}\,n^2 + \frac{883}{80}\,n + \frac{25}{64}, \ n \end{cases}$

Figure 2: Stretched Clebsch–Gordan coefficients for C_3.

λ, μ, ν	$C_{n\lambda,n\nu}^{n\mu}$
(0, 2, 10, 5) (4, 11, 9, 11) (5, 8, 6, 9)	$\begin{cases} \frac{625007}{10080}\,n^7 + \frac{729157}{2880}\,n^6 + \frac{77197}{180}\,n^5 + \frac{449539}{1152}\,n^4 + \frac{298979}{1440}\,n^3 + \frac{95189}{1440}\,n^2 + \\ \frac{625007}{10080}\,n^7 + \frac{729157}{2880}\,n^6 + \frac{77197}{180}\,n^5 + \frac{449539}{1152}\,n^4 + \frac{298979}{1440}\,n^3 + \frac{95189}{1440}\,n^2 + \end{cases}$
(2, 7, 12, 2) (11, 10, 5, 9) (13, 11, 1, 1)	$\begin{cases} \frac{34675903}{80640}\,n^8 + \frac{3037051}{1680}\,n^7 + \frac{9121453}{2880}\,n^6 + \frac{241181}{80}\,n^5 + \frac{615083}{360}\,n^4 + \frac{8947}{15}\,n^3 \\ \frac{34675903}{80640}\,n^8 + \frac{3037051}{1680}\,n^7 + \frac{9121453}{2880}\,n^6 + \frac{241181}{80}\,n^5 + \frac{615083}{360}\,n^4 + \frac{8947}{15}\,n^3 \end{cases}$
(3, 11, 0, 10) (2, 15, 10, 15) (10, 12, 11, 0)	$\begin{cases} \frac{53609}{60}\,n^6 + \frac{25631}{15}\,n^5 + \frac{63779}{48}\,n^4 + \frac{1627}{3}\,n^3 + \frac{2497}{20}\,n^2 + \frac{239}{15}\,n + 1, & n \text{ eve} \\ \frac{53609}{60}\,n^6 + \frac{25631}{15}\,n^5 + \frac{63779}{48}\,n^4 + \frac{1627}{3}\,n^3 + \frac{2497}{20}\,n^2 + \frac{239}{15}\,n + \frac{15}{16}, & n \text{ od} \end{cases}$
(10, 1, 12, 4) (1, 12, 0, 3) (0, 5, 3, 4)	$5\,n^2 + 4\,n + 1$
(12, 2, 5, 13) (15, 6, 10, 11) (2, 0, 12, 13)	$\begin{cases} \frac{455263}{2016}\,n^7 + \frac{447281}{576}\,n^6 + \frac{198433}{180}\,n^5 + \frac{971011}{1152}\,n^4 + \frac{108787}{288}\,n^3 + \frac{28969}{288}\,n^2 + \\ \frac{455263}{2016}\,n^7 + \frac{447281}{576}\,n^6 + \frac{198433}{180}\,n^5 + \frac{971011}{1152}\,n^4 + \frac{108787}{288}\,n^3 + \frac{28969}{288}\,n^2 + \end{cases}$

Figure 3: Stretched Clebsch–Gordan coefficients for D_4.

THEOREM

For any complex semisimple Lie algebra associated to root systems A_r,B_r,C_r,D_r, the stretching function of Clebsch-Gordan coefficients

$$e(n) = C_{n\lambda,n\mu}^{n\nu}.$$

is a quasipolynomial of period at most TWO. In other words,

$$C_{n\lambda,n\mu}^{n\nu} = \begin{cases} f_0(n) & \text{if } n \equiv 0 \mod 2, \\ \\ f_1(n) & \text{if } n \equiv 1 \mod 2; \end{cases}$$

The Stretching Conjecture

Conjecture (Stretching Conjecture) Given highest weights λ, μ, ν of a Lie algebra of type A_r, B_r, C_r, or D_r, the stretching function

$$C_{n\lambda,n\mu}^{n\nu} = \begin{cases} f_0(n) & \text{if } n \equiv 0 \mod 2, \\ \\ f_1(n) & \text{if } n \equiv 1 \mod 2; \end{cases}$$

has only POSITIVE coefficients in each polynomial f_i.

Experiments with LattE

In the table below, we express highest weights in terms of the canonical basis $e_1, \ldots, e_r$, so that the highest weights are partitions with r parts.

λ, μ, ν	$c^{\nu}_{\lambda\mu}$	LattE runtime	LiE runtime
(9,7,3,0,0), (9,9,3,2,0), (10,9,9,8,6)	2	0m00.74s	0m00.01s
(18,11,9,4,2), (20,17,9,4,0), (26,25,19,16,8)	453	0m03.86s	0m00.12s
(30,24,17,10,2), (27,23,13,8,2), (47,36,33,29,11)	5231	0m05.21s	0m02.71s
(38,27,14,4,2), (35,26,16,11,2), (58,49,29,26,13)	16784	0m06.33s	0m25.31s
(47,44,25,12,10),(40,34,25,15,8),(77,68,55,31,29)	5449	0m04.35s	1m55.83s
(60,35,19,12,10),(60,54,27,25,3),(96,83,61,42,23)	13637	0m04.32s	23m32.10s
(64,30,27,17,9), (55,48,32,12,4), (84,75,66,49,24)	49307	0m04.63s	45m52.61s
(73,58,41,21,4), (77,61,46,27,1), (124,117,71,52,45)	557744	0m07.02s	$>$ 24 hours

Figure 4: A sample comparison of running times between LattE and LiE case of A_r

	λ, μ, ν	$C^{\mu}_{\lambda\nu}$	LattE runtime	LiE runtime
B_3	(46,42,38), (38,36,42), (41,36,44)	354440672	0m09.58s	1m45.27s
	(46,42,41), (14,58,17), (50,54,38)	88429965	0m06.38s	3m16.01s
	(15,60,67), (58,70,52), (57,38,63)	626863031	0m07.14s	6m01.43s
C_3	(25,42,22), (36,38,50), (31,33,48)	87348857	0m07.48s	0m17.21s
	(34,56,36), (44,51,49), (37,51,54)	606746767	0m08.42s	2m57.27s
	(39,64,58), (65,15,72), (70,41,44)	519379044	0m07.63s	8m00.35s
D_4	(13,20,10,14), (10,20,13,20), (5,11,15,18)	41336415	2m46.88s	0m12.29s
	(12,22,9,30),(28,14,15,26),(10,24,10,26)	322610723	3m04.31s	7m03.44s
	(37,16,31,29),(40,18,35,41),(36,27,19,37)	18538329184	4m29.63s	$>$60m

Figure 5: A sample comparison of running times between LattE and LiE