The role of Triangulations Of Polytopes in Mathematics

Jesús A. De Loera
University of California, Davis

Outline of the talk

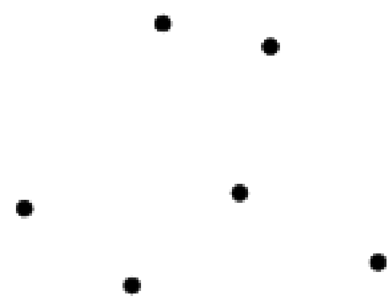
1. **Triangulations of polytopes**. (A brief overview).
2. **Triangulations and algebraic geometry:** Bernstein's Theorem; Viro's Theorem;
3. **Triangulations and optimization:** parametric linear programming; Sperner's lemma and fixed points.

1. Triangulations

Polytopes

A **polytope** is the convex hull of finitely many points

$$\operatorname{conv}(p_1, \ldots, p_n) := \{\sum \alpha_i p_i : \alpha_i \geq 0 \, \forall i = 1, \ldots, n, \sum \alpha_i =$$



A finite point set

Polytopes

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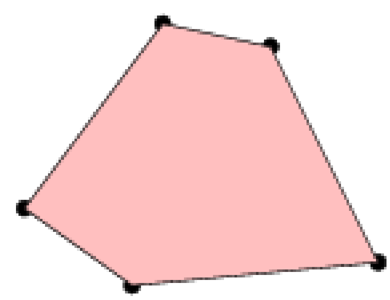


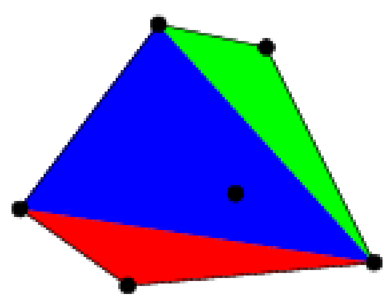
Its convex hull

Triangulations

A **triangulation** is a partition of the convex hull into **simplices** such that

The union of all these simplices equals $\mathrm{conv}(A)$. (Union Property.)

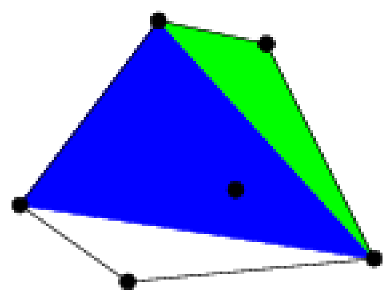
Any pair of them intersects in a (possibly empty) common face. (Intersec. Prop.)



A triangulation of P

Triangulations

The following are **not** triangulations:



The union is not the whole convex hull

Triangulations

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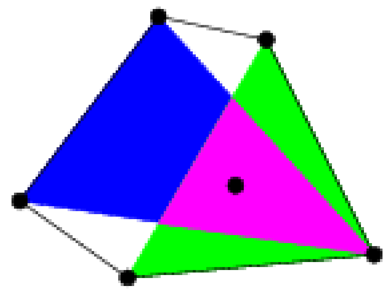
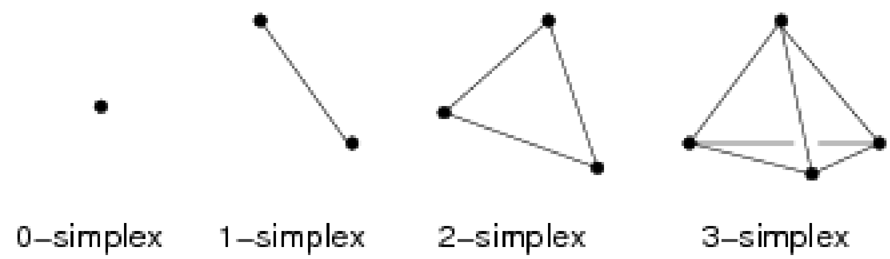


The intersection is not okay

Triangulations

Recall that a **simplex** is the convex hull of any set of **affinely independent points**. Equivalently, it is any polytope of dimension d with $d+1$ vertices.

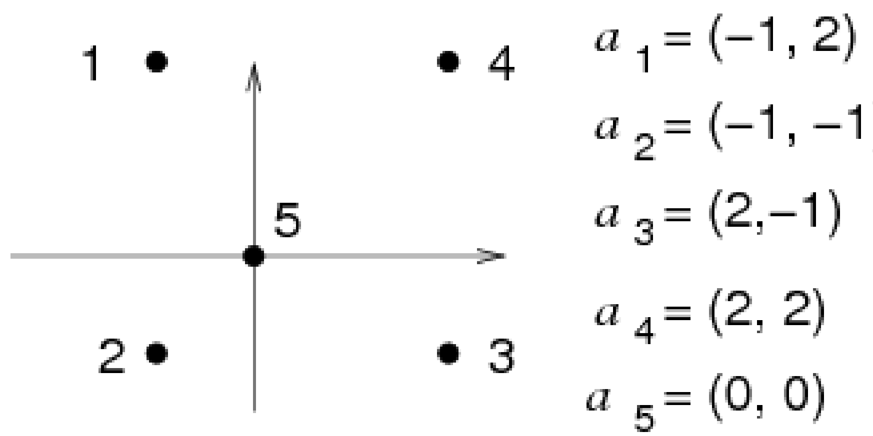


The simplest simplices

A d-simplex has exactly $\binom{d+1}{i+1}$ faces of dimension i, $(i = -1, 0, \ldots, d)$, which are themselves i-simplices.

Triangulations of a point configuration

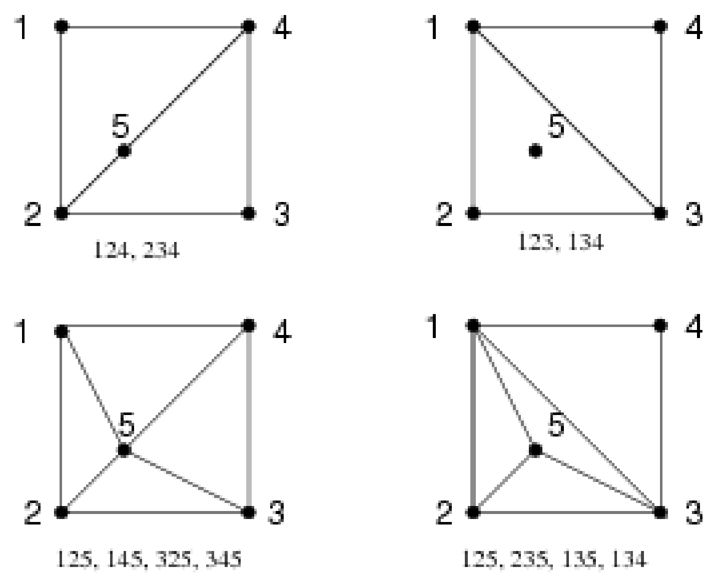
A **point configuration** is a finite set of points in $\mathbb{R}^d$, possibly with repetitions.



A point set with 5 elements

Triangulations of a point configuration

A triangulation of a point set $\mathcal{A}$ is a triangulation of $\operatorname{conv} \mathcal{A}$ with **vertex set contained in** $\mathcal{A}$. **Remark:** Don't need to use all points



Example: Triangulations of a convex n-gon

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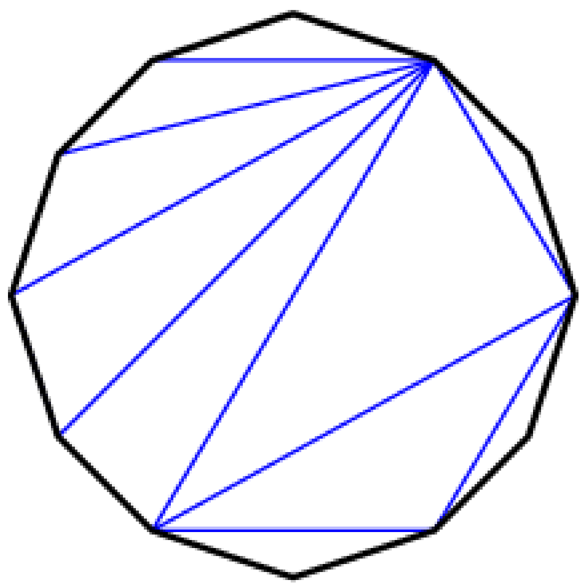
To triangulate the n-gon, you just need to insert $n-3$ non-crossing diagonals:



A triangulation of the 12-gon

Example: Triangulations of a convex n-gon

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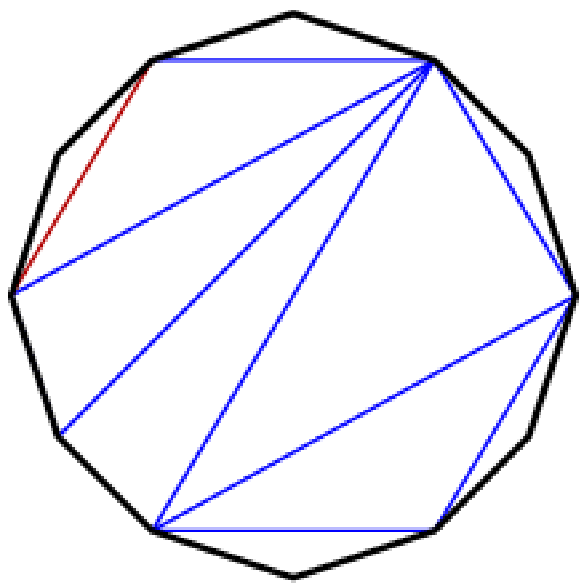
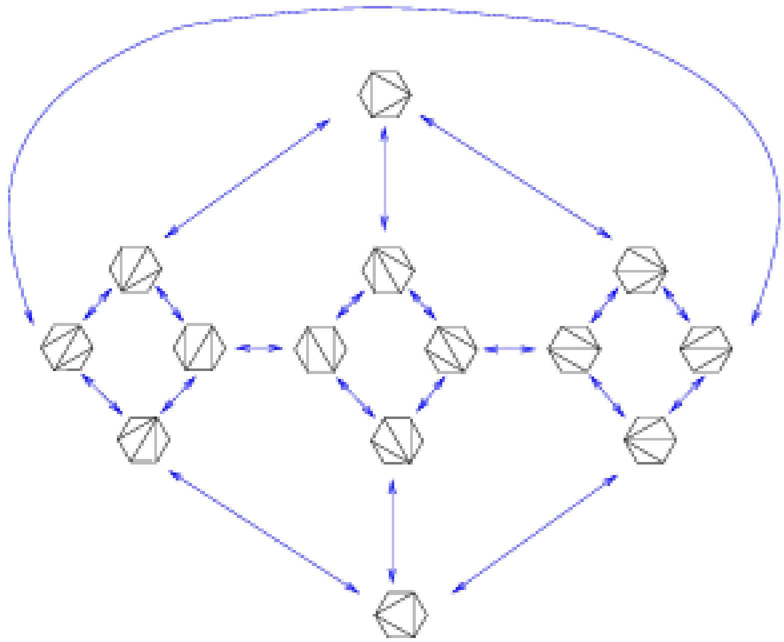


Another triangulation of the 12-gon, obtained by **flipping** an edge



The Graph of flips for a hexagon

Some obvious properties of triangulations and flips of an n-gon

- The graph is regular of degree $n-3$.
- The graph has dihedral symmetry.

Some non-obvious properties of triangulations and flips of an n-gon

- It is the graph of a polytope of dimension $n-3$, called **the associahedron** [Stasheff 1963, Haiman 1984, Lee 1989].

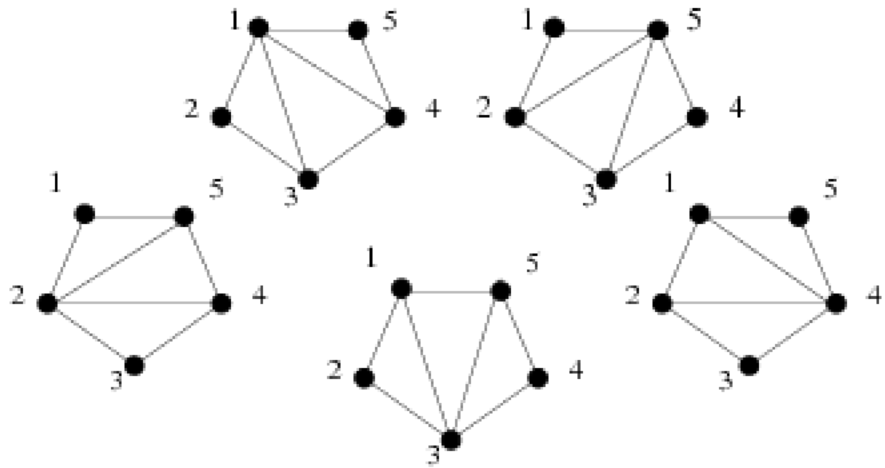
- The graph has diameter equal to $2n-10$ for all $n \geq 12$ [Sleator-Tarjan-Thurston, 1988].

- There are exactly $\frac{1}{n-1}\binom{2n-4}{n-2}$ triangulations.
  That is to say, the **Catalan number** C_{n-2}:

$C_n := \frac{1}{n+1}\binom{2n}{n}$,

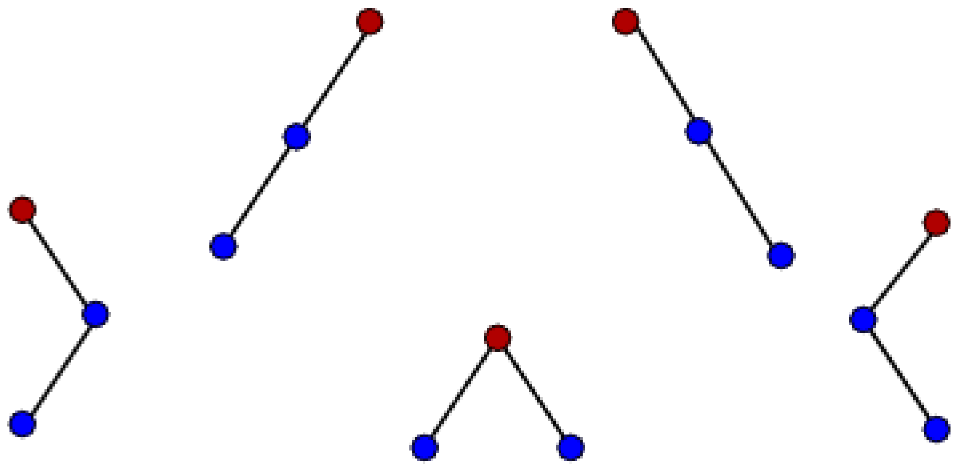
n	0	1	2	3	4	5	6
C_n	1	1	2	5	14	42	132

The Catalan number C_n not only counts the triangulations of a $n+2$-gon:



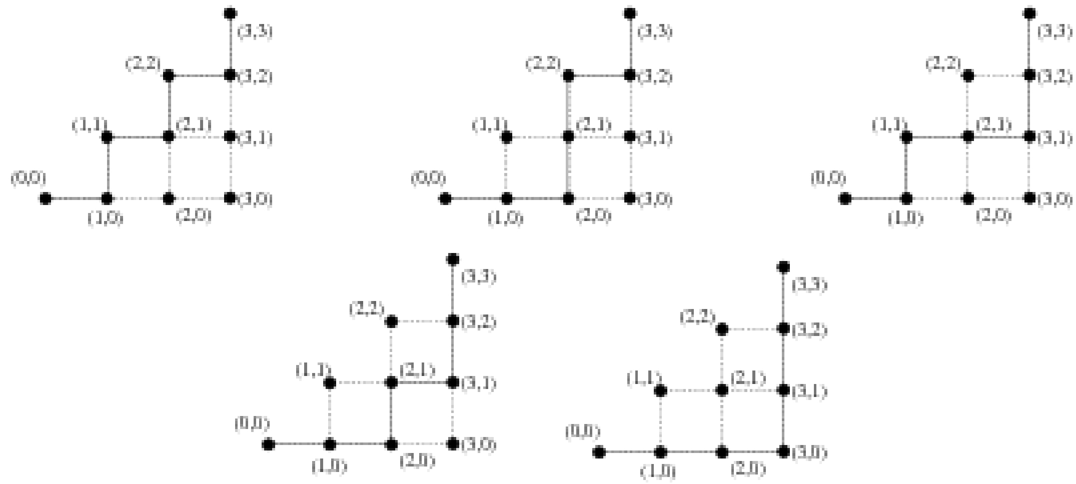
It also counts. . .

1. Binary trees on n-nodes.



1. Binary trees on n-nodes.

2. Monotone lower-diagonal lattice (integer) paths from $(0, 0)$ to (n, n).



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2. Monotone lower-diagonal lattice (integer) paths from $(0, 0)$ to (n, n).

3. Sequences of $2n$ signs with exactly n of each and with more $+$'s than $-$'s in every initial segment.

$$+-+-+- \qquad ++--+- \qquad +-++-$$

$$+-+-+- \qquad +++---$$

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2. Monotone lower-diagonal lattice (integer) paths from $(0,0)$ to (n,n).

3. Sequences of $2n$ signs with exactly n of each and with more $+$'s than $-$'s in every initial segment.

. . . and **some other 60 combinatorial structures**,

according to Exercise 6.19 in

R. Stanley, *Enumerative combinatorics*, Cambridge University Press, 1999.

Regular triangulations

Let $A = \{a_, \dots, a_n\} \subset \mathbb{R}^d$ be a vector configuration. Let $h = (h_1, \dots, h_n) \in \mathbb{R}^n$ be a vector.

Consider the lifted vector configuration $\tilde{A} = \begin{Bmatrix} a_1 & \cdots & a_n \\ h_1 & \cdots & h_n \end{Bmatrix} \subset \mathbb{R}^{d+1}$. The lower envelope of $\mathrm{cone}(\tilde{A})$ (projected down to $\mathbb{R}^d$ forms a polyhedral subdivision of A. If h is "sufficiently generic" then it forms a triangulation.

Remark: Different h's may provide different triangulations. But, for some A's, **not all triangulations can be obtained in this way.**

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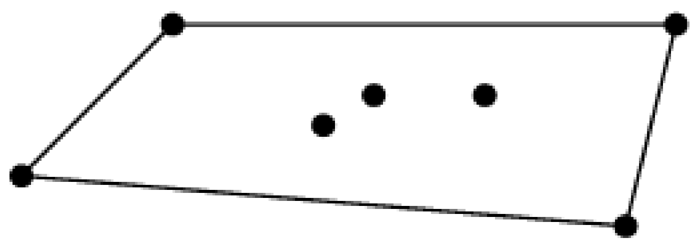
The triangulations that **can** be obtained like this are called regular.

Regular triangulations



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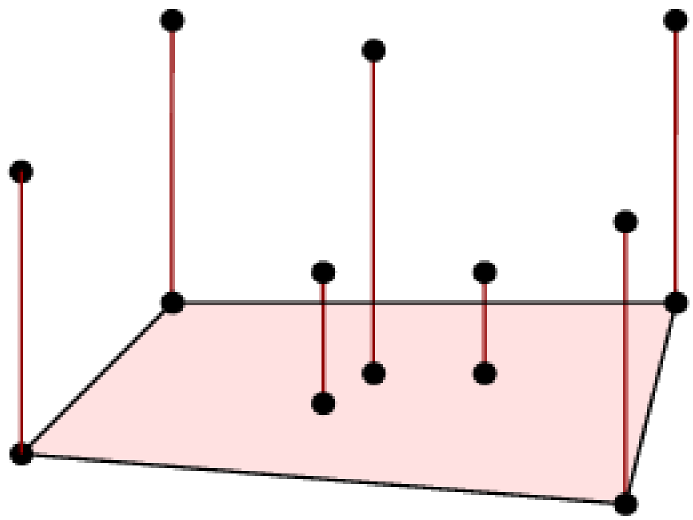

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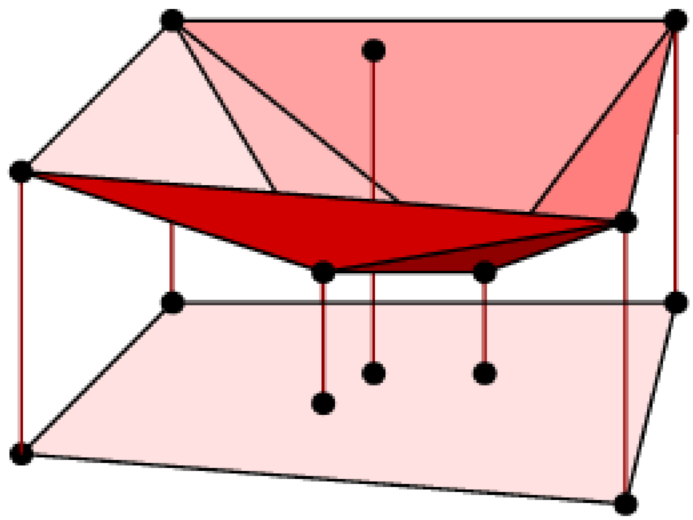

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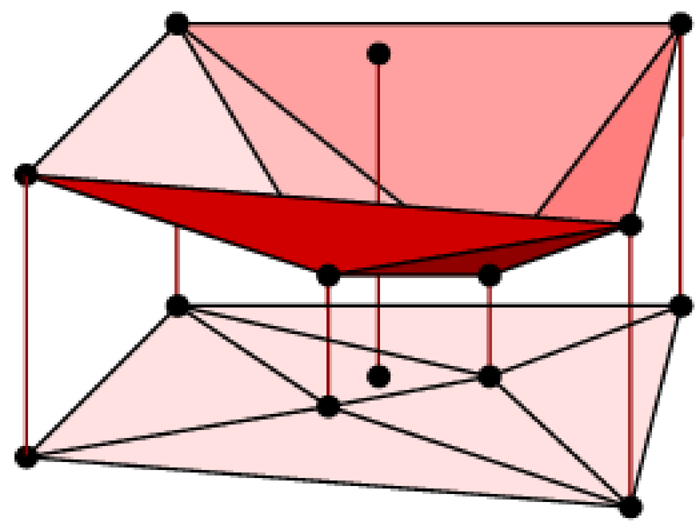
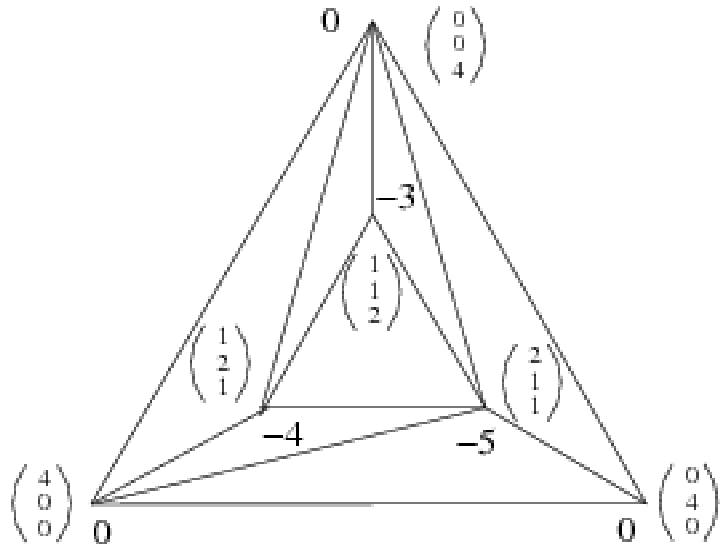


Example:

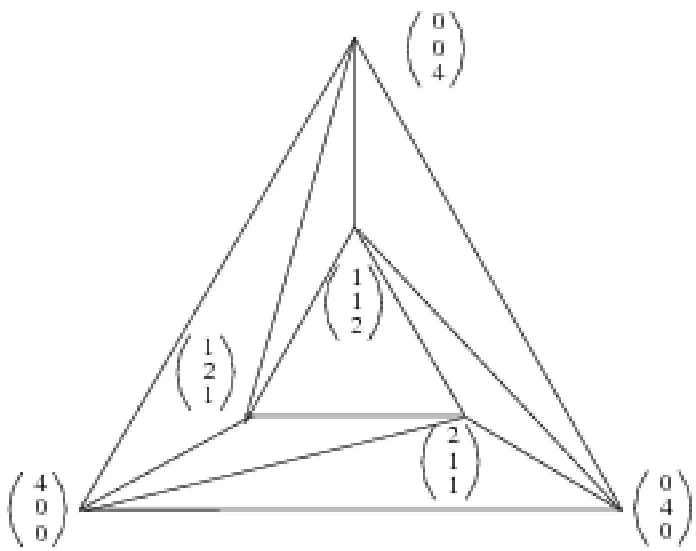
$h = (0, 0, 0, -5, -4, -3)$, $A = \begin{bmatrix} 4 & 0 & 0 & 2 & 1 & 1 \\ 0 & 4 & 0 & 1 & 2 & 1 \\ 0 & 0 & 4 & 1 & 1 & 2 \end{bmatrix}$,



The triangulation associated with the lifting vector h.
This shows a two-dimensional slice of the 3d-cone.

Another example:

$$h = (\ ?,\ ?,\ ?,\ ?,\ ?,\ ?), \quad A = \begin{bmatrix} 4 & 0 & 0 & 2 & 1 & 1 \\ 0 & 4 & 0 & 1 & 2 & 1 \\ 0 & 0 & 4 & 1 & 1 & 2 \end{bmatrix},$$

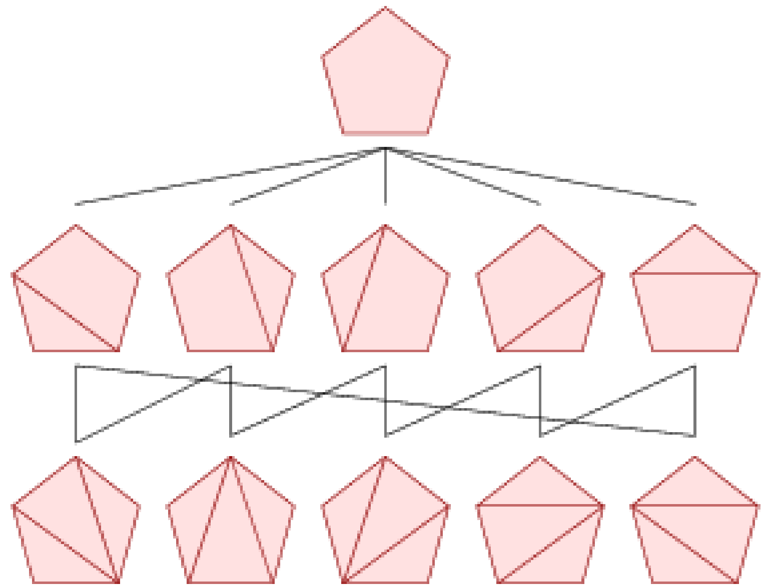


A triangulation not associated with any lifting vector h. That is to say, a non-regular triangulation.

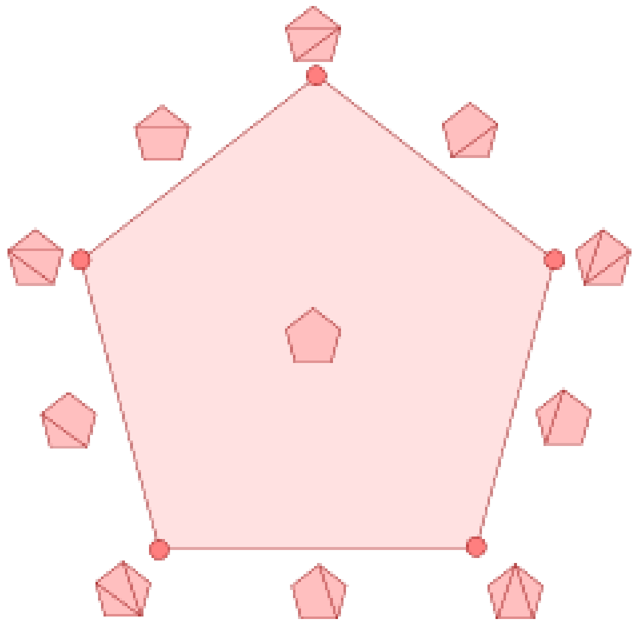
The secondary polytope

Theorem [Gelfand-Kapranov-Zelevinskii, 1990]
The poset (partially ordered set) of **regular** polyhedral subdivisions of a point set (or acyclic vector configuration) A equals the face poset of a certain polytope of dimension $n-k$ ($n =$ number of points, $k =$ rank $=$ dimension $+1$).

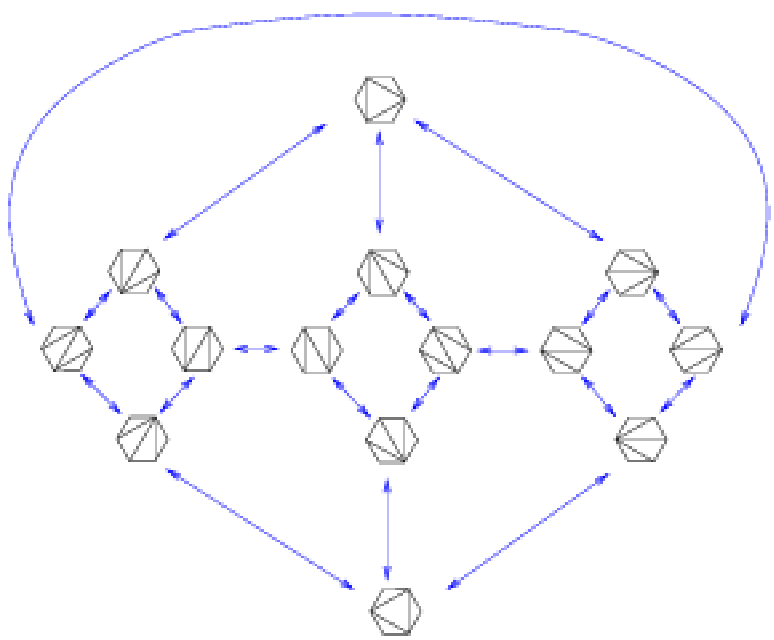
This is called the secondary polytope of A.



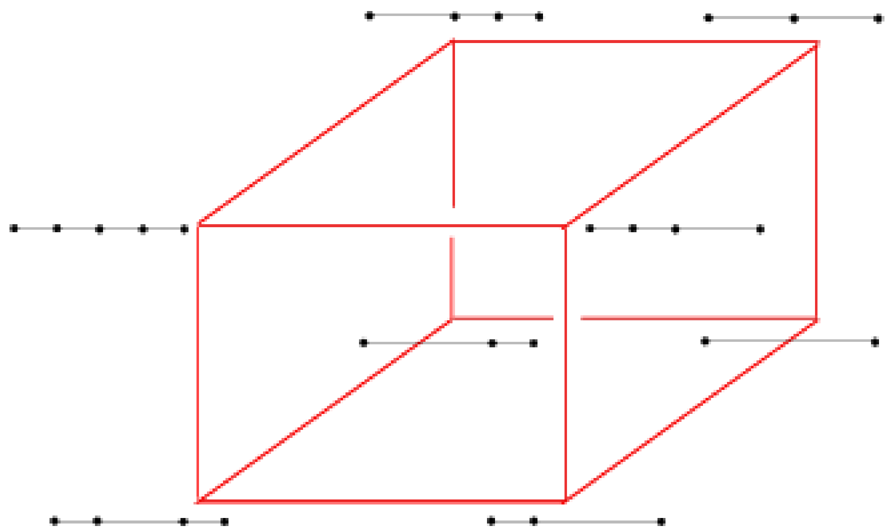
The poset of subdivisions of a pentagon



Secondary polytope of a pentagon (again a pentagon)



Secondary polytope of a polygon (associahedron)



Secondary polytope of a 1-dimensional configuration
(a cube)

Bistellar flips

They are the "minimal possible changes" among triangulations. They correspond to edges in the secondary polytope.

Definition 1: In the poset of polyhedral subdivisions of a point set A, the minimal elements are the triangulations. It is a fact that if a subdivision is only refined by triangulations then it is refined by exactly two of them. We say these two triangulations differ by a flip.

That is to say, flips correspond to next to minimal elements in the poset of polyhedral subdivisions of A

Bistellar flips

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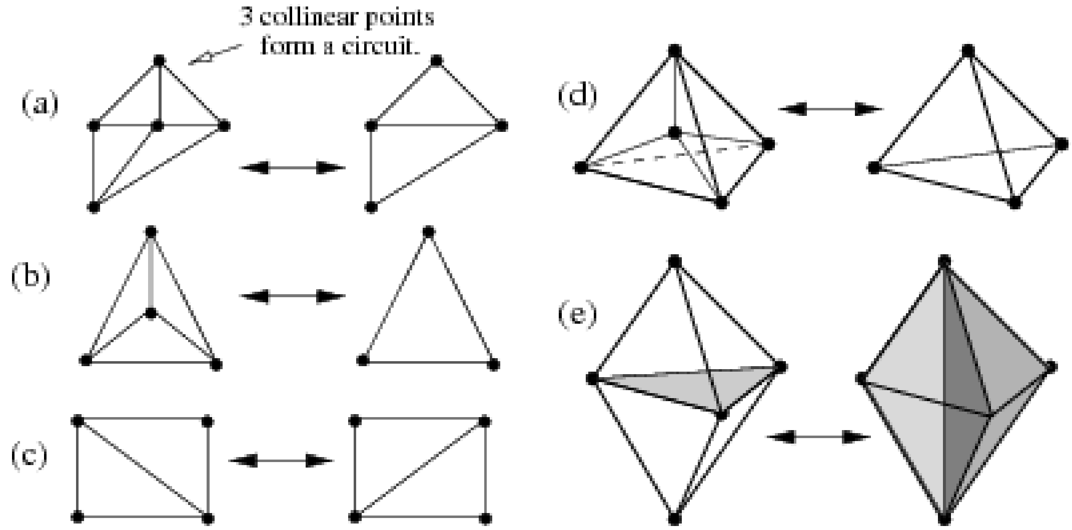
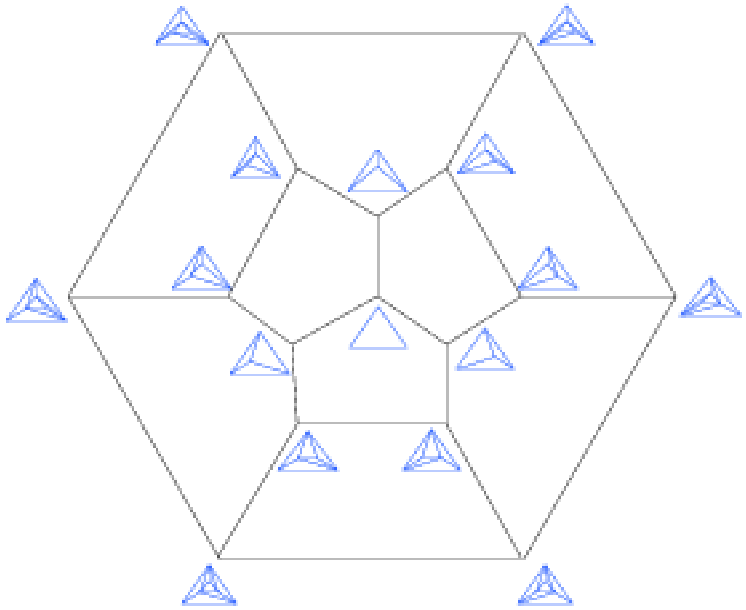
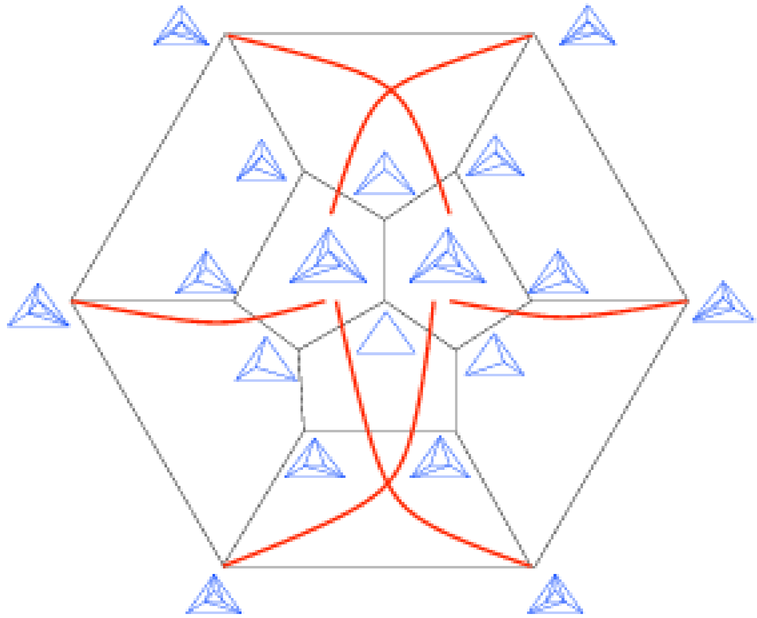
Definition 2: A circuit is a minimally (affinely/linearly) dependent set of (points/vectors). It is a fact that a circuit has exactly two triangulations. If a triangulation T of A contains one of the two triangulations of a circuit C, a flip on C consists on changing that part of T to become the other triangulation of C.



Triangulated circuits and their flips, in dimensions 2 and 3



Flips between regular triangulations of a point set
(i.e., secondary polytope



... and flips between all the triangulations of the same point set

The number of flips of a triangulation

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Theorem: For every point set A, the graph of flips between regular triangulations is $n-d-1$-connected (in particular, every triangulation has at least $n-d-1$ flips).

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The same happens for non-regular triangulations if d or n are "small":

If $d = 2$, then the graph is connceted [Lawson 1977] and every triangulation has at least $n-3$ flips [de Loera-Santos-Urrutia, 1997] (but it is not known if the graph is $n-3$-connected).

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The same happens for non-regular triangulations if d or n are "small":

If $d=3$ **and** the points are in convex position, then every triangulation has at least $n-4$ flips [de Loera-Santos-Urrutia, 1997] (but it is not known if the graph is even connected).

The number of flips of a triangulation

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The same happens for non-regular triangulations if d or n are "small":

If $n \leq d+4$, then every triangulation has at least 3 flips and the graph is 3-connected [Azaola-Santos, 2001].

The number of flips of a triangulation

But:

1. In **dimension 3**, there are triangulations with arbitrarily large n and only O$(\sqrt{n})$ flips [Santos, 1999].

2. In **dimension 4**, there are triangulations with arbitrarily large n and only O(1) flips [Santos, 1999].

3. In **dimension 5**, there are point sets with a disconnected graph of triangulations [Santos, 2004].

4. In **dimension 6**, there are triangulations with arbitrarily large n and no flips [Santos, 2000].

3. Triangulations and algebraic geometry

1. Bernstein's Theorem

2. Viro's Theorem

Bernstein's Theorem

Counting solutions of sparse polynomial systems

Let f and g be two polynomials in two unknowns, of degrees d and d'. Bezout's Theorem says that if the number of (complex, projective) solutions of $f(x,y) = g(x,y) = 0$ is finite, then it is bounded above by dd'.

Question: Can we get better bounds if we know that most of the possible monomials in f and g have zero coefficient? Observe that in one dimension:

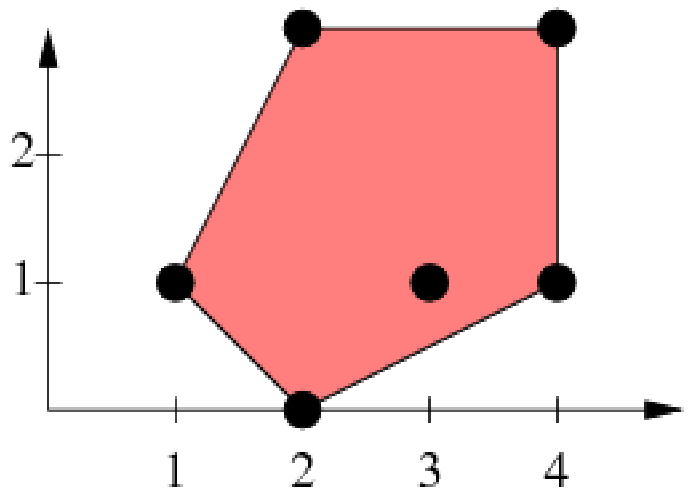
• The number of distinct roots of a polynomial is at most equal to twice the number of monomials minus one (Descartes rule of signs)

• The number of non-zero roots, counted with multiplicity, cannot exceed the difference between the highest and lowest degree monomials in the polynomial (as follows from "Bezout in one dimension").

What we want is a generalization of the second statement. For this:

• To every possible monomial $x^i y^j$ we associate its corresponding integer point (i, j) (as in Viro's Theorem).

• To a polynomial $f(x, y) = \sum c_{i,j} x^i y^j$ we associate the corresponding integer point set. Its convex hull is the Newton polytope of f, $N(f)$.



The Newton polytope for the polynomial $x^2 + xy + x^3y + x^4y + x^2y^3 + x^4y^3$

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Can we bound the number of common zeroes of f and g, counted with multiplicities, in terms of $N(f)$ and $N(g)$? (For example, in terms of their areas...)

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Remark: sparse polynomial $\sim$ polynomial with a fixed set of allowed monomials.

YES !

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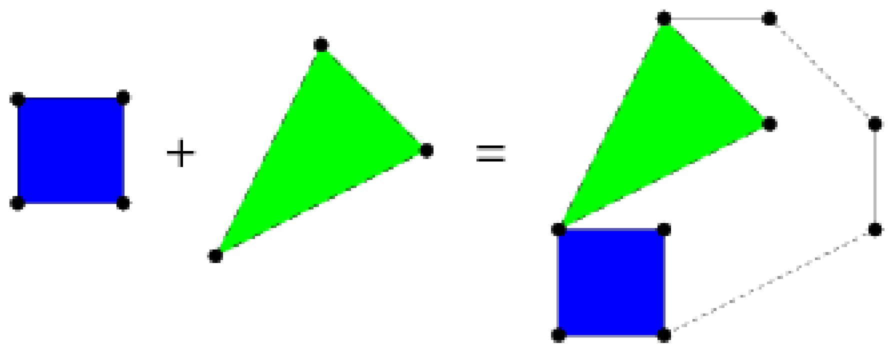
Theorem (Bernstein, 1975) The number of common zeroes of f and g in $(\mathbb{R}^*)^2$ (that is, out of the coordinate axes) is bounded above by the mixed area of the two polygons $N(f)$ and $N(g)$.



Mixed area of a triangle and a rectangle.

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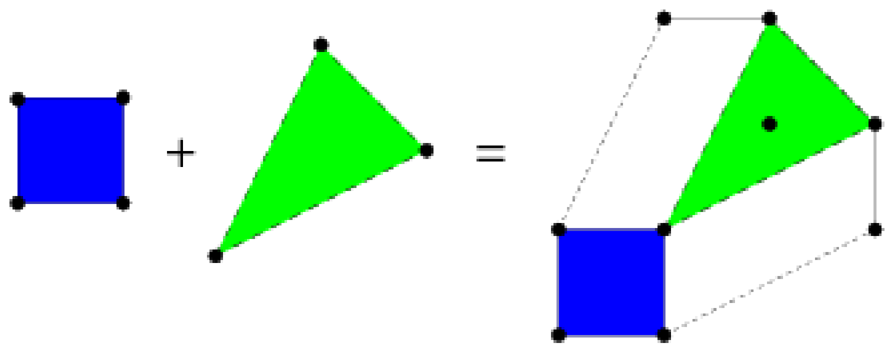

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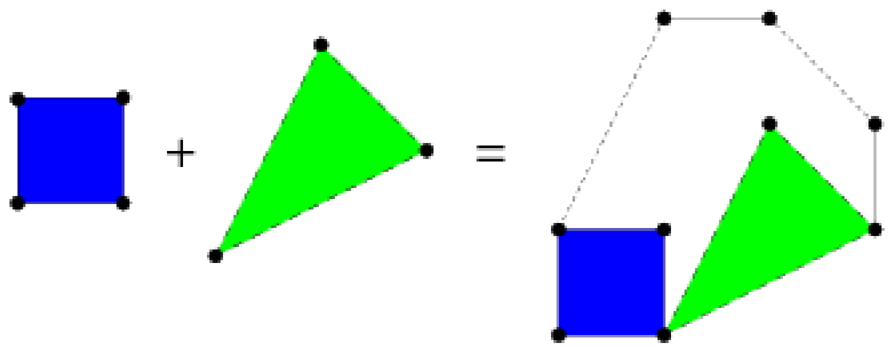

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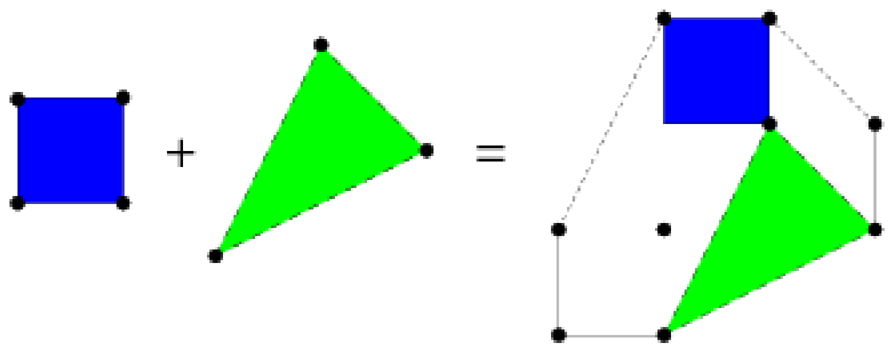

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The theorem is valid for n polynomials $f_1, \ldots, f_n$ in n variables, except we have to define their mixed volume.

Mixed volume

Definition 1: Let $Q_1, Q_2, \ldots, Q_n$ be n polytopes in $\mathbb{R}^n$. Their mixed volume $\mu(Q_1, \ldots, Q_n)$ equals

$$\sum_{I \subset \{1,2,\ldots,n\}} (-1)^{|I|} \operatorname{vol}\left(\sum_{j \in I} Q_i\right).$$

Mixed volume

Definition 2: Let $Q_1, Q_2, \ldots, Q_n$ be n polytopes in $\mathbb{R}^n$. Their mixed volume $\mu(Q_1, \ldots, Q_n)$ equals

the coefficient of $x_1 x_2 \cdots x_n$ in the homogeneous polynomial $\mathrm{vol}(x_1 Q_1 + \cdots + x_n Q_n)$.

Mixed volume

Definition 3: Let $Q_1, Q_2, \ldots, Q_n$ be n polytopes in $\mathbb{R}^n$. Their mixed volume $\mu(Q_1, \ldots, Q_n)$ equals

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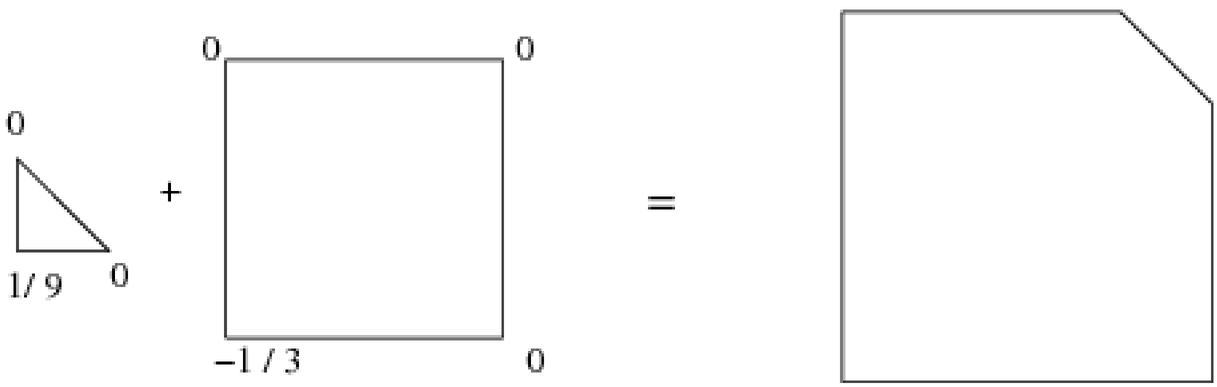
In particular, to compute the number of zeroes of a sparse system of polynomials $f_1, \ldots, f_n$ one only needs to compute a "(fine) mixed subdivision" of $N(f_1) + \cdots + N(f_n)$.

A cooking recipe for fine mixed subdivisions:



Choose sufficiently generic (e.g. random) numbers $w_a \in \mathbb{R}$, one for each a in each of the Q_i's

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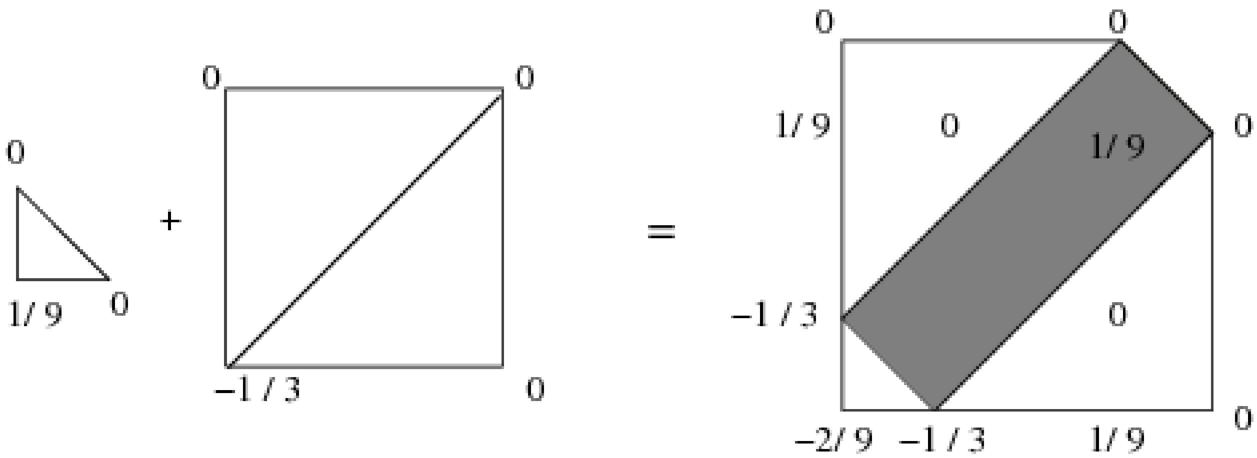

Use the numbers to lift the points of $Q_1 + \cdots + Q_n$ and compute the lower envelope of the lifted point configuration.

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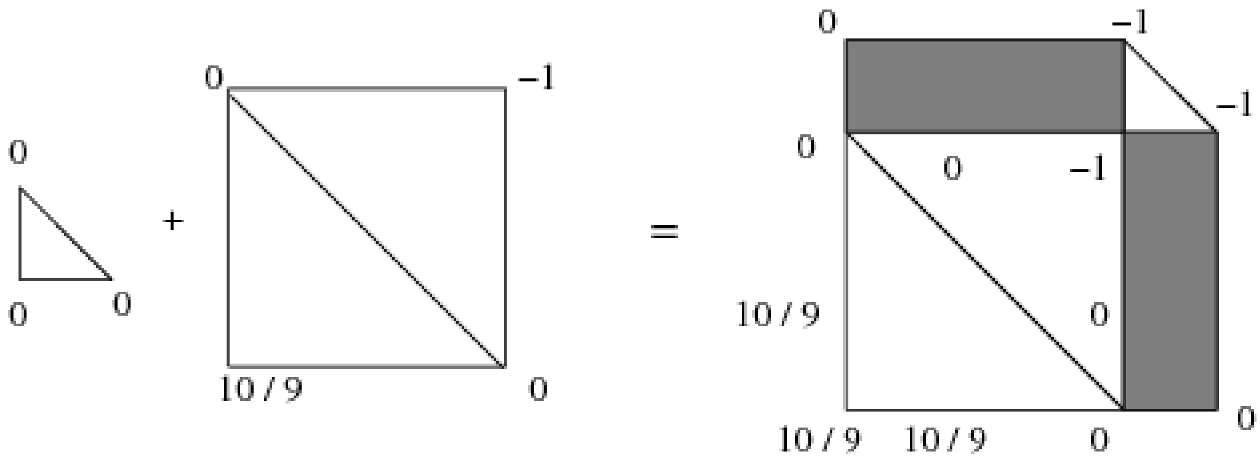

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- But, moreover, for every family of polytopes $Q_1, \ldots, Q_n$ in $\mathbb{R}^d$ there is another polytope $\mathcal{C}(Q_1, \ldots, Q_n)$ in $\mathbb{R}^{n+d-1}$ such that

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(This is the polyhedral Cayley Trick).

That is to say, the number of roots of a sparse system of polynomials can be computed via triangulations.

How to compute the roots

From (the proof of) Bernstein's theorem one not only gets the number of roots, but also a germ at $t = 0$ of an algebraic curve $(x(t), y(t))$ such that $(x(1), y(1))$ is a root (roots are in bijection to the mixed cells in the mixed subdivision, counted with their volume; the germs are given by the slopes of mixed cells in the lifting that was used to construct the mixed subdivision).

Using the germ, one can follow the curve numerically until reaching the solution.

These are the so-called homotopy methods or numerical continuation methods.

Recently, [Verschelde et al.] have extended this method to positive-dimensional sparse systems of polynomials.

Viro's Theorem

Hilbert's sixteenth problem (1900)

"What are the possible (topological) types of non-singular real algebraic curves of a given degree d?"

Observation: Each connected component is either a pseudo-line or an oval. A curve contains one or zero pseudo-lines depending in its parity.

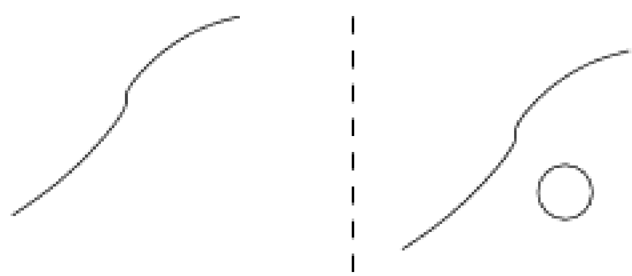
A pseudoline. Its complement has one component, homeomorphic to an open circle. The picture only shows the "affine part"; think the two ends as meeting at infinity.

An oval. It
is a (topologica
and its ext
Möbius

Partial answers:

Bezout's Theorem: A curve of degree d cuts every line in at most d points. In particular, there cannot be nestings of depth greater than $\lfloor d/2 \rfloor$

Harnack's Theorem: A curve of degree d cannot have more than $\binom{d-1}{2}+1$ connected components (recall that $\binom{d-1}{2}=$ genus)



Two configurations are possible in degree 3

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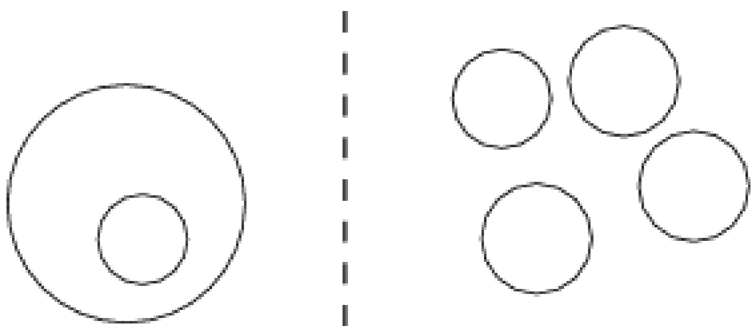
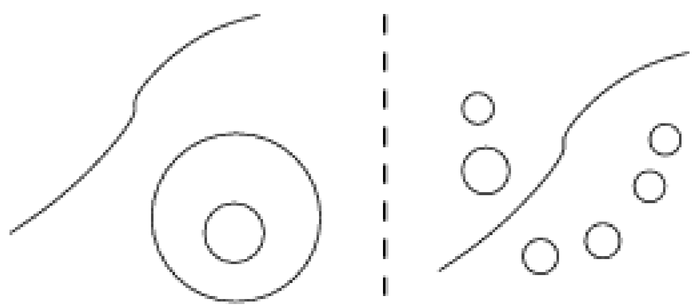


Six configurations are possible in degree 4. Only the maximal ones are shown.

Partial answers:

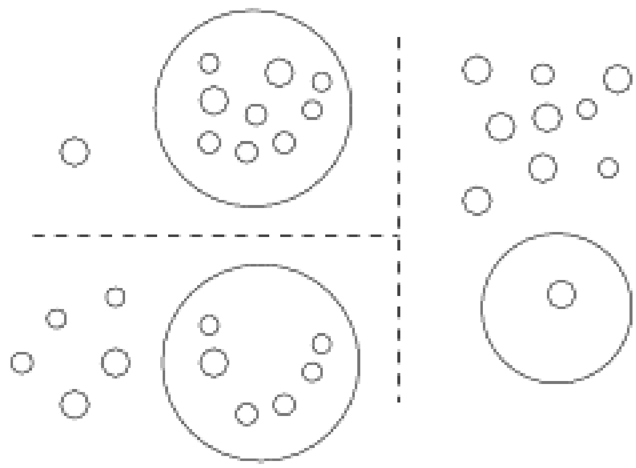
Bezout's Theorem: A curve of degree d cuts every line in at most d points. In particular, there cannot be nestings of depth greater than $\lfloor d/2 \rfloor$

Harnack's Theorem: A curve of degree d cannot have more than $\binom{d-1}{2}+1$ connected components (recall that $\binom{d-1}{2} =$ genus)

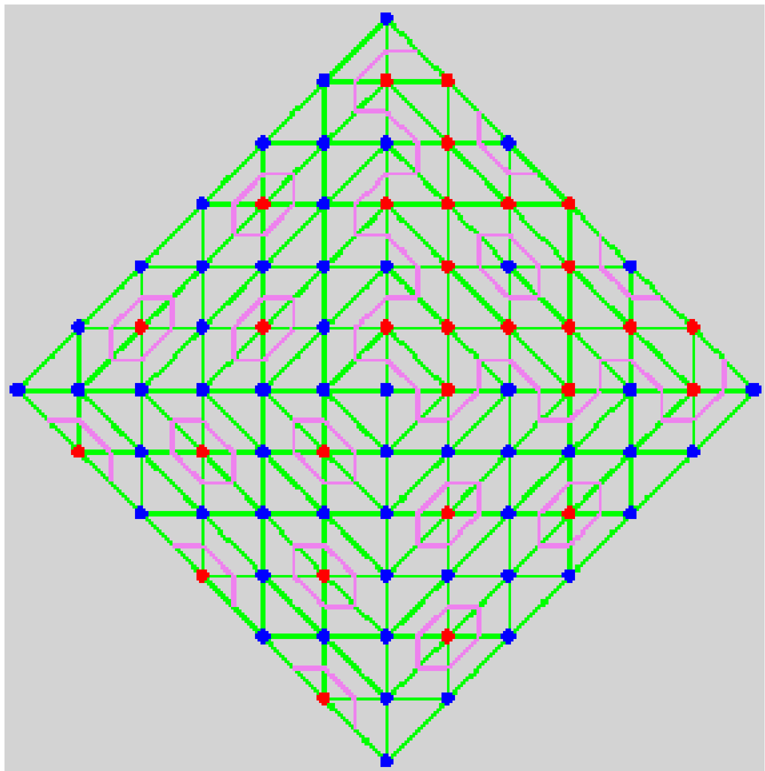


Eight configurations are possible in degree 5. Only the maximal ones are shown.

All that was known when Hilbert posed the problem, but the classification of non-singular real algebraic curves of degree six was not completed until the 1960's [Gudkov]. There are 56 types degree six curves, three with 11 ovals:

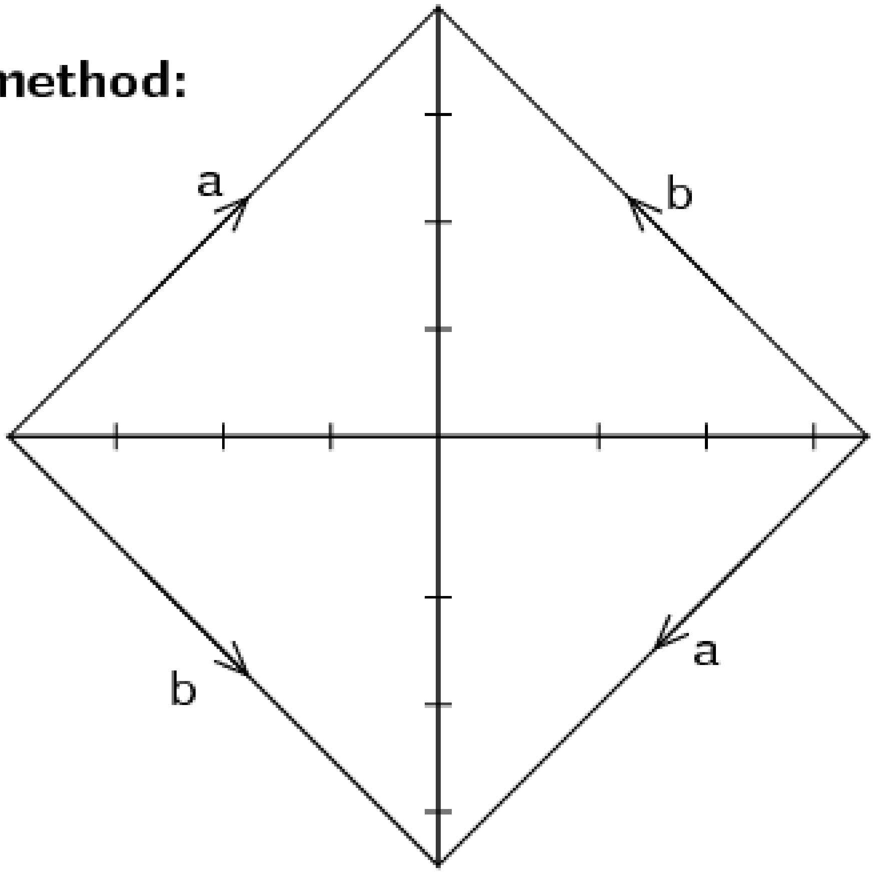


What about dimension 7? It was solved by **Viro**, in 1984 with a method that involves triangulations.



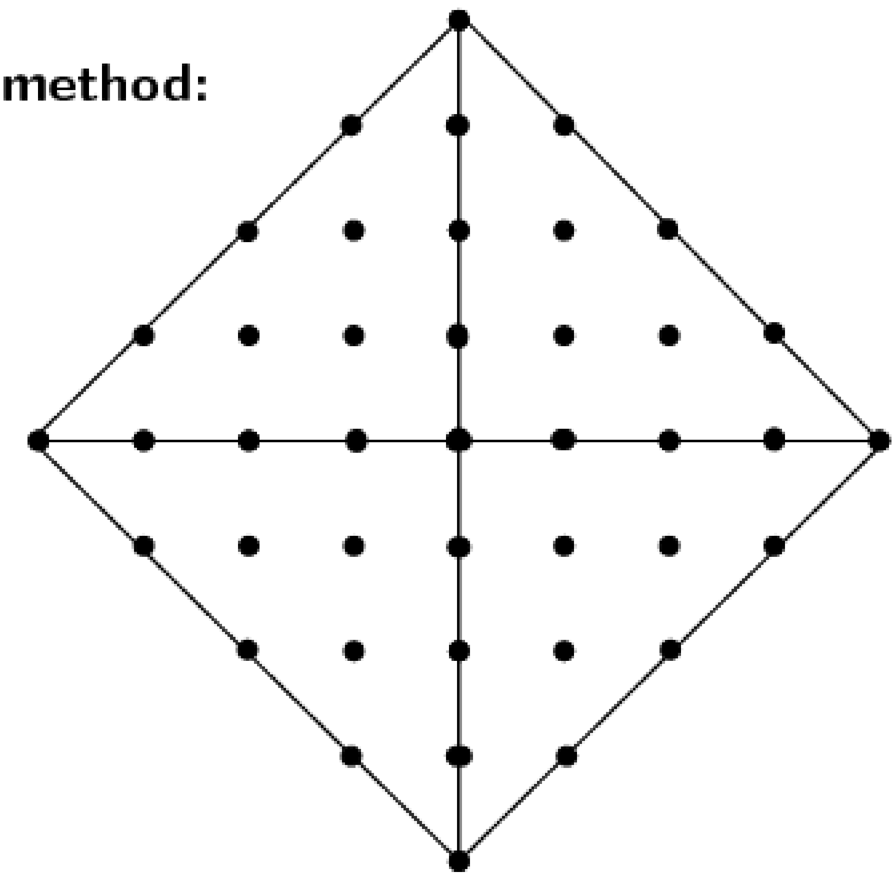
A curve of degree 6 constructed using Viro's method

Viro's method:



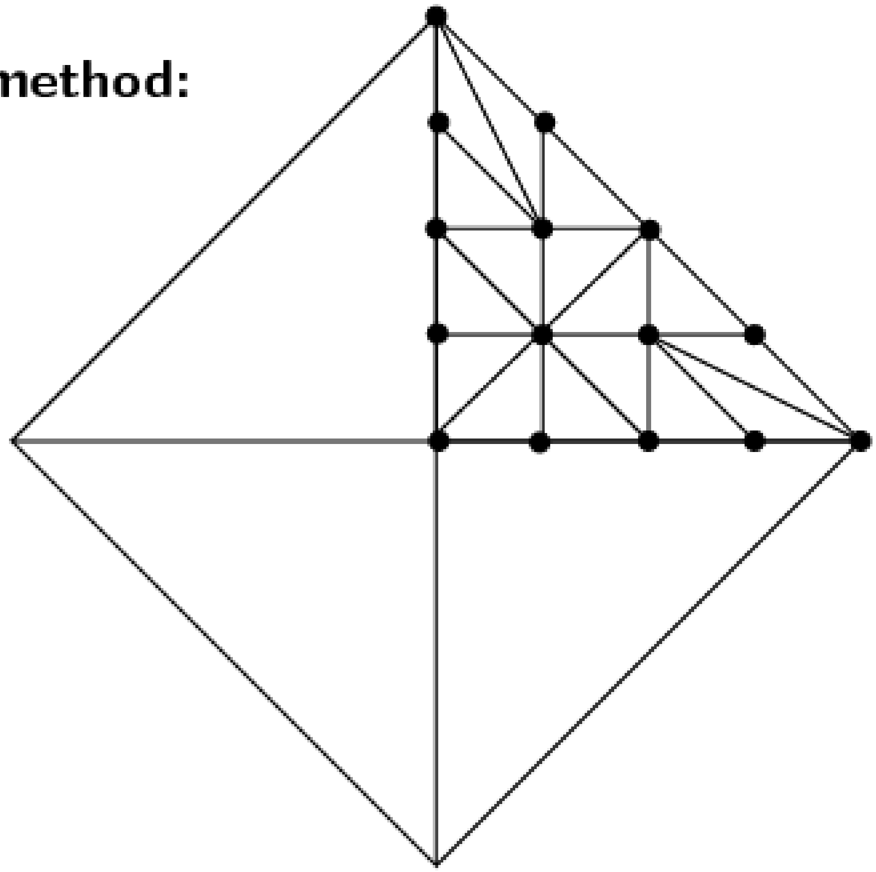
For any given d, construct a topological model of the projective plane by gluing the triangle $(0,0),(d,0),(0,d)$ and its symmetric copies in the other quadrants:

Viro's method:



Consider as point set all the integer points in your rhombus (remark: those in a particular orthant are related to the possible homogeneous monomials of degree d in three variables).

Viro's method:



Triangulate the positive orthant arbitrarily . . .

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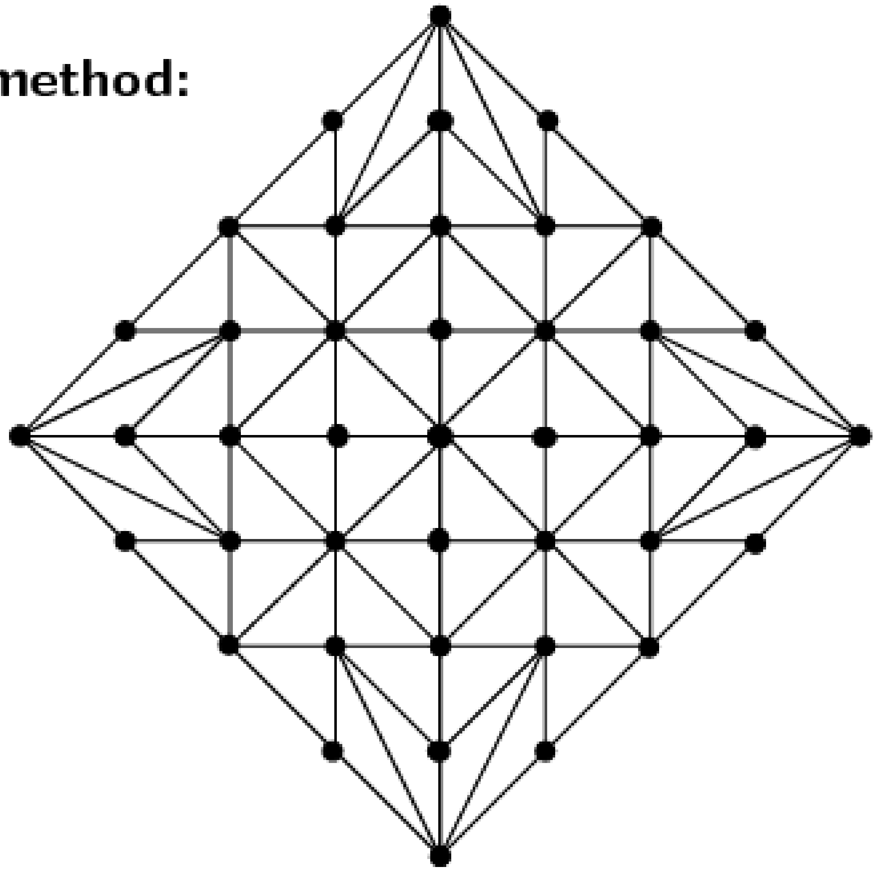
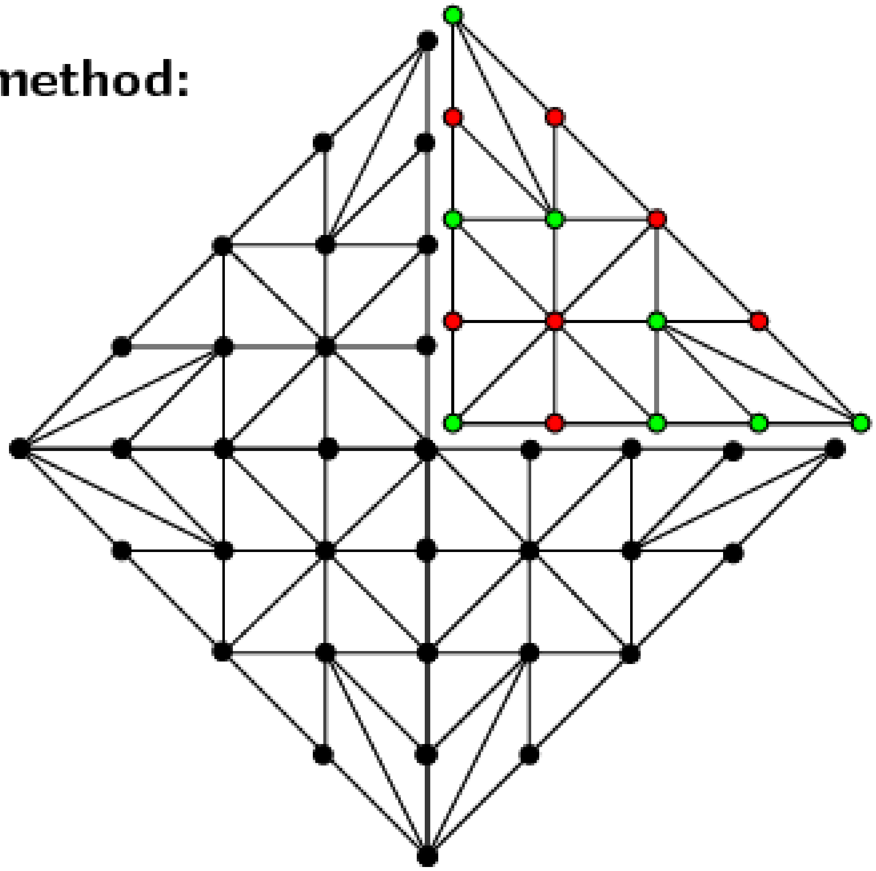


Triangulate the positive quadrant arbitrarily . . .

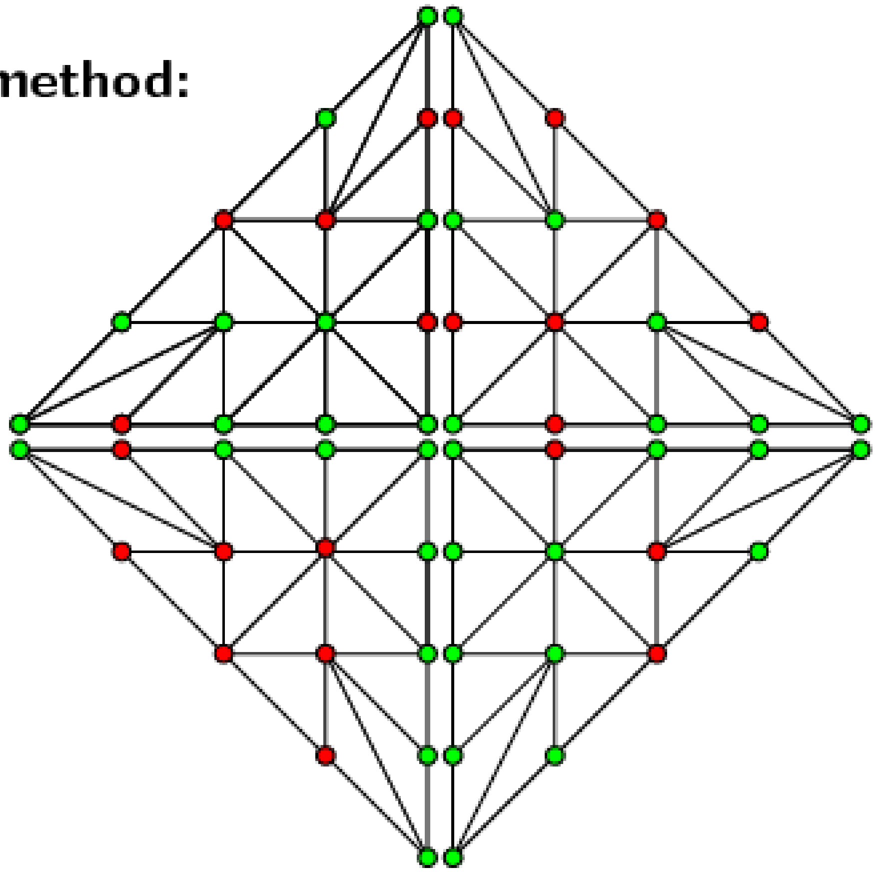
. . . and replicate the triangulation to the other three quadrants by reflection on the axes.

Viro's method:



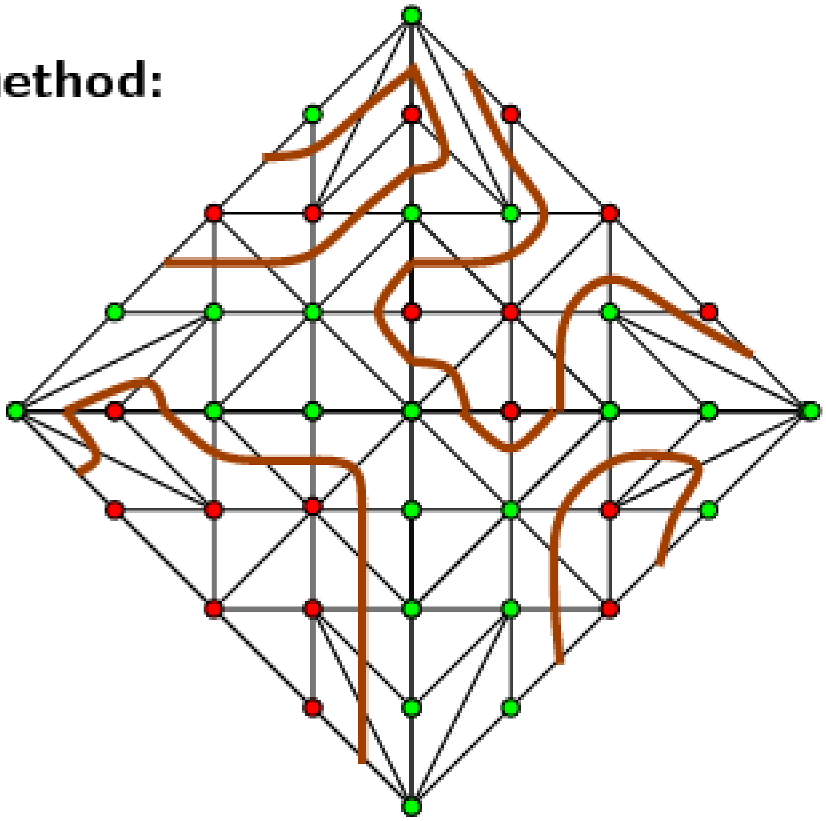
Choose arbitrary signs for the points in the first quadrant

Viro's method:



Choose arbitrary signs for the points in the first quadrant . . . and replicate them to the other three quadrants, taking parity of the corresponding coordinate into account.

Viro's method:



Finally draw your curve in such a way that it separates positive from negative points.

Viro's Theorem

Theorem (Viro, 1987) If the triangulation T chosen for the first quadrant is regular then there is a real algebraic non-singular projective curve f of degree d realizing exactly that topology.

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More precisely, let $w_{i,j}$ $(0 \leq i \leq i + j \leq d)$ denote "weights" ($\leftrightarrow$cost vector$\leftrightarrow$lifting function) producing your triangulation and let $c_{i,j}$ be any real numbers of the sign you've given to the point (i, j).

Then, the polynomial

$$f_t(x, y) = \sum c_{i,j} x^i y^j z^{d-i-j} t^{w(i,j)}$$

for any positive and sufficiently small t gives the curve you're looking for.

Viro's Theorem

- The method works exactly the same in higher dimension (and produces smooth real algebraic projective hypersurfaces).

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Viro's Theorem

• The method works exactly the same in higher dimension (and produces smooth real algebraic projective hypersurfaces).

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• What happens if we do the construction with a non-regular triangulation?
Well, then the formula in the theorem cannot be applied (there is no possible choice of weights). But there is no known example of a curve constructed via Viro's method (with a non-regular triangulation) and which is not isotopic to a real algebraic curve of the corresponding degree. (There are examples of such curves in toric varieties other than the projective plane [Orevkov-Shustin, 2000]).

Viro's Theorem

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• What happens if we do the construction with a non-regular triangulation?
Well, then the formula in the theorem cannot be applied (there is no possible choice of weights).

• Still, the curves constructed with Viro's method (with non-regular triangulations) can be realized as **pseudo-holomorphic** curves in $\mathbb{CP}^2$ [Itenberg-Shustin, 2002].

Ragsdale's conjecture

Let $d = 2k$ be even, so that a non-singular curve of degree d consists only of ovals. An oval is called positive (or even) if it lies inside an even number of other ovals, and negative (or odd) otherwise.

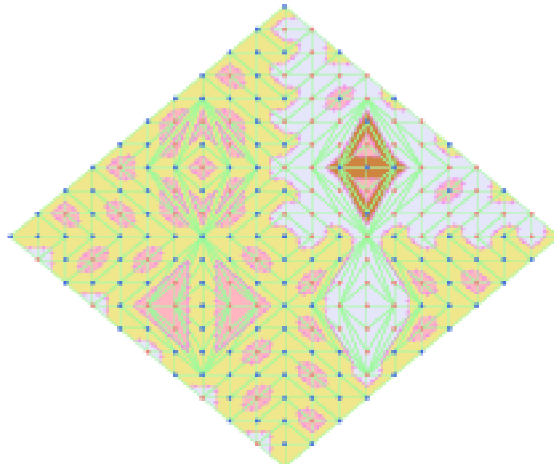
Harnack's inequality says that the total number of ovals cannot exceed $2k^2 \pm O(k)$. Virginia **Ragsdale** conjectured in 1906 (based on empirical evidence) that the numbers of positive ovals could not exceed $3k^2/2 \pm O(k)$.

In the 1930's, **Petrovskii** proved that the difference between the two numbers was bounded by

$$p - n \leq 3(k^2 - k)/2 + 1,$$

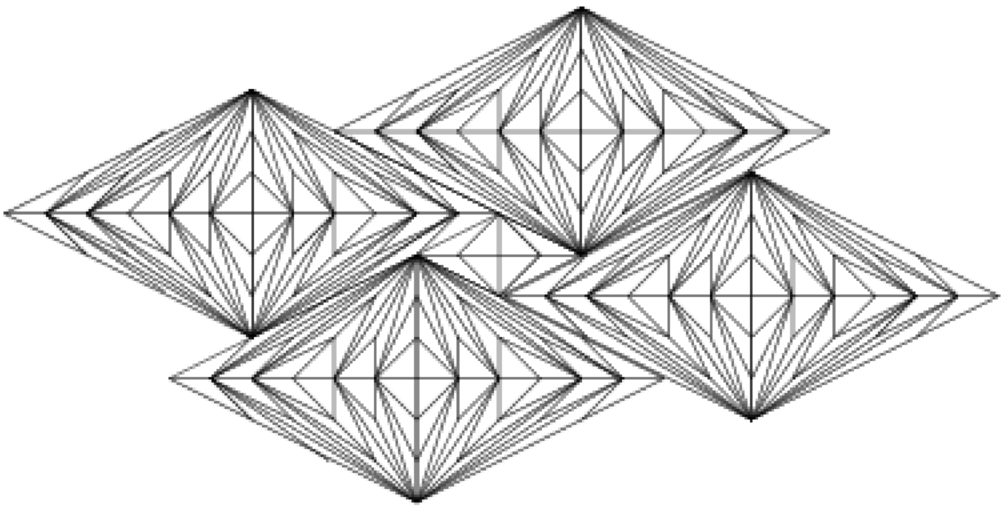
which implies $p \leq 7k^2/4 \pm O(k)$.

In 1993, **Itenberg** (using Viro's method) constructed curves having $13k^2/8 \pm O(k)$ positive ovals.



This was improved by B. **Haas** to $10k^2/6 \pm O(k)$.

Curiously enough, using **non-regular triangulations**, one can construct Viro curves with $17k^2/10 \pm O(k)$ positive ovals [Santos, 1995].



Are these curves realizable algebraically?

For comparison

Ragsdale's conjecture: 180 $k^2/120 \pm O(k)$.

Itenberg construction: 195 $k^2/120 \pm O(k)$.

Haas construction: 200 $k^2/120 \pm O(k)$.

Santos construction: 204 $k^2/120 \pm O(k)$.

Petrovskii inequality: 210 $k^2/120 \pm O(k)$.

Harnack inequality: 240 $k^2/120 \pm O(k)$.

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Remark: Petrovskii inequality is valid for pseudo-holomorphic curves (hence for Viro curves too)

5. Triangulations and optimization

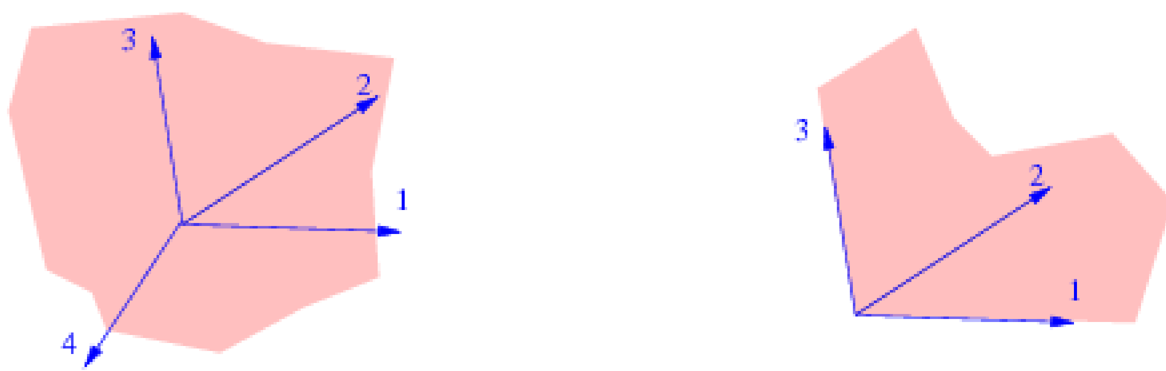
1. Parametric linear programming
2. Sperner Lemma

Parametric linear programming

Triangulations of vector sets

Let $A = \{a_1, \ldots, a_n\}$ be a finite set of real **vectors** (a vector configuration).

The cone of A is $\mathrm{cone}(A) := \{\sum \lambda_i a_i : \alpha_i \geq 0, \forall i = 1, \ldots, n\}$



Two vector configurations, and their cones

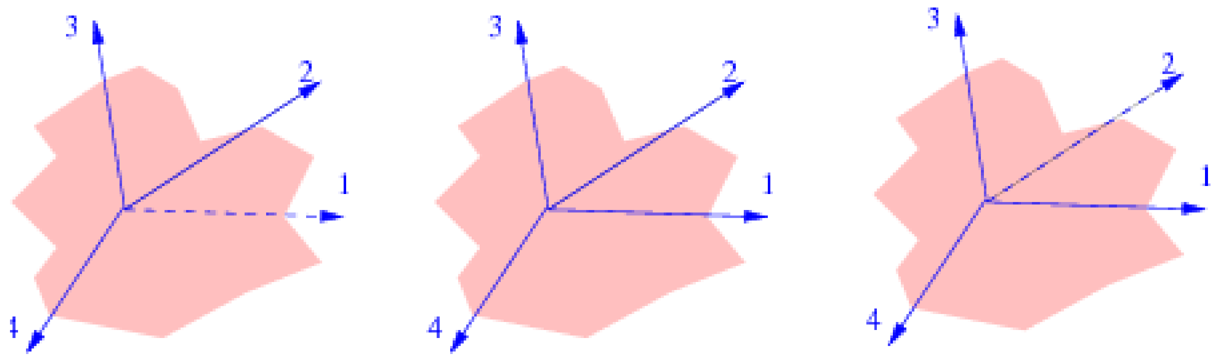
A simplicial cone is one generated by linearly independent vectors.

Triangulations of vector sets

A triangulation of a vector configuration A is a partition of $\operatorname{cone}(A)$ into simplicial cones with generators contained in A and such that:

(UP) The union of all these simplices equals $\operatorname{conv}(A)$. (Union Property.)

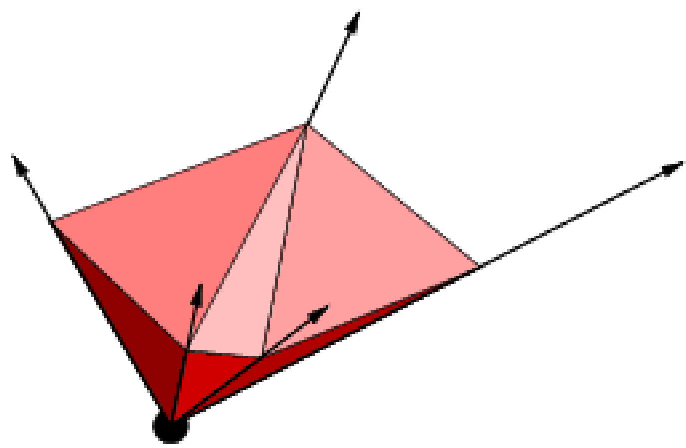
(IP) Any pair of them intersects in a common face (Intersection Property.)

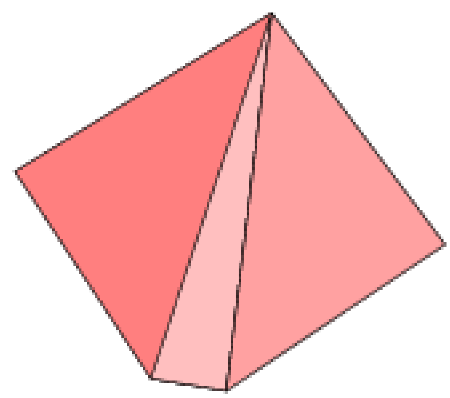


The three triangulations of the first configuration

A cone is pointed if it is contained (except for the origin) in an **open** half-space. If this happens for $\operatorname{cone}(A)$, then A is called acyclic.

Remark: Triangulations of a {pointed/acyclic} {cone/vector set} of dimension d are the same as the triangulations of the {polytope/point} set of dimension $d-1$ obtained cutting by an affine hyperplane:





Linear programming

Let $A = (a_1, \ldots, a_n) \in \mathbb{R}^{d \times n}$ be a matrix. Let $b \in \mathbb{R}^d$ and $c \in \mathbb{R}^n$. To this data one associates the linear programming problem $LP_{A,c}(b) := \min\{c(x) : Ax = b,\ x \geq 0\}$:

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"minimize the cost function $c(x)$

subject to $Ax = b$ and $x \geq 0$"

We say that the linear program $LP_{A,c}(b)$ is feasible if $\{x \in \mathbb{R}^n : Ax = b\}$ is not empty. It is bounded if c has a lower bound in $\{x \in \mathbb{R}^n : Ax = b\}$.

Linear programming

Example: $A = \begin{pmatrix} 0 & 3 & 1 \\ 2 & 1 & 0 \end{pmatrix}, \qquad b = \begin{pmatrix} 6 \\ 4 \end{pmatrix}.$

Then,

$$Ax = b \quad \Leftrightarrow \quad x = (1, 2, 0) + \lambda(1, -2, 6)$$

The linear program is feasible ($(1, 2, 0)$ is a feasible solution). It is bounded for every c, because big and small values of λ will make some coordinate of x negative.

Remarks:

- feasible $\quad \Leftrightarrow \quad b \in \operatorname{cone}(A) := \operatorname{cone}(\{a_1, \ldots, a_n\})$.

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- if b and c are generic, there is (at most) one optimal solution. In this case, the optimal solution has d non-zero coordinates and the corresponding columns of A form a basis of $\text{cone}(A)$. They are called the optimal basis of $LP_{A,c}(b)$.

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- if we knew the optimal basis σ, we could find the optimal solution by just solving a linear system of equations:

$$Ax = b, \qquad \text{and} \qquad x_i = 0 \quad \forall i \notin \sigma.$$

Parametric linear programming

Let us study how the previous linear program depends on the right hand side b. That is, study the family of linear programs

$$LP_{A,c} = \{LP_{A,c}(b) : b \in \mathrm{cone}(A)\}$$

Question: How does the optimal basis depend on b?

Parametric linear programming

Theorem (Walkup-Wets 1969) Let $LP_{A,c}(b)$ denote the linear program

$$\min\{cx : Ax = b, x \geq 0\},$$

where c and A are fixed.

Then, there exists a regular triangulation T of $\mathrm{cone}(A)$ such that the optimal basis of $LP_{A,c}(b)$ for each $b \in \mathrm{cone}(A)$ is precisely the (generators of) the simplicial cone $\mathrm{cone}(\sigma)$ with $\sigma \in T$ and $b \in \mathrm{cone}(\sigma)$.

Idea of proof: Consider the lifted vector configuration $\tilde{A} = \begin{pmatrix} a_1 & \cdots & a_n \\ c_1 & \cdots & c_n \end{pmatrix} \subset \mathbb{R}^{d+1}$. The triangulation of A in question is the lower envelope of $\mathrm{cone}(\tilde{A})$.

Parametric linear programming

Example: $A = \begin{pmatrix} 0 & 3 & 1 \\ 2 & 1 & 0 \end{pmatrix}$, $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, $c = \begin{pmatrix} c_1 & c_2 & c_3 \end{pmatrix}$.

$$Ax = b \quad \Leftrightarrow \quad x = x_0 + \lambda(1, -2, 6)$$

Then:

• if $c_1 - 2c_2 + 6c_3 \geq 0$, then the optimal basis is $(*, *, 0)$ or $(0, *, *)$, and this happens depending on whether $b \in \mathrm{cone}(a_1, a_2)$ or $b \in \mathrm{cone}(a_2, a_3)$

• if $c_1 - 2c_2 + 6c_3 \leq 0$, then the optimal basis is $(*, 0, *)$ for every $b \in \mathrm{cone}(a_1, a_2, a_3)$

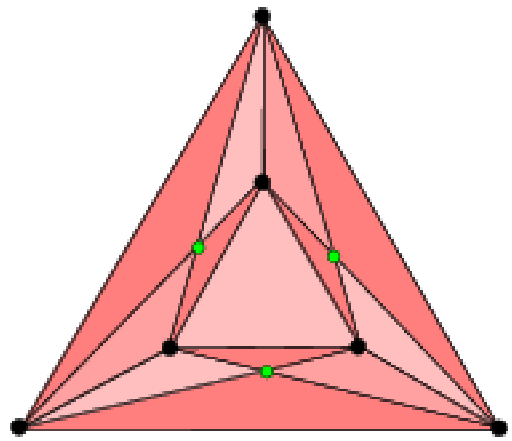
Parametric linear programming (cont.)

Let us get back to the linear programs $LP_{A,c}(b)$, for a fixed matrix A. But suppose that now c varies, too. By the previous theorem, each value of c will provide a different triangulation of $\mathrm{cone}(A)$.

Question: What values $b, b' \in \mathrm{cone}(A)$ are guaranteed to provide the same optimal solution of $LP_{A,c}(b)$ no matter what c?

Answer: clearly, those which are contained in exactly the same bases of A. That is to say, those in the same chamber of the chamber complex of A.

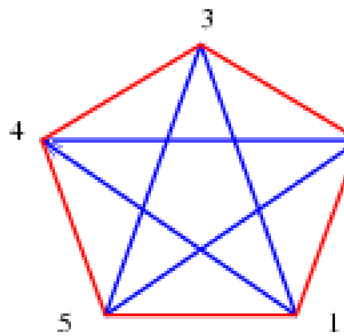
The chamber complex

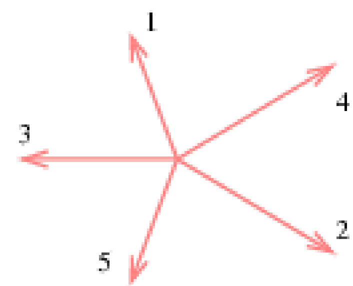


The chamber complex of $\mathrm{cone}(A)$.

. . . curiously enough:

Theorem (Billera, Filliman, Sturmfels 1990) For any vector configuration A there is another vector configuration A^* (its Gale transform) such that the chambers of A correspond to regular triangulations of A^* and viceversa





5 regular triangulations

11 chambers

11 regular triangulations

5 chambers

(The chamber complex of A is the normal fan of the secondary polytope of A^*).

Sperner's lemma and fixed points

Sperner's lemma and fixed points

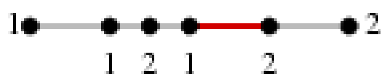
Lemma (Sperner) Let A be a point configuration whose convex hull is a d-dimensional simplex Δ and let T be a triangulation of A. Let $\Delta_1, \dots, \Delta_{d+1}$ denote the $d+1$ facets $\Delta_1, \dots, \Delta_{d+1}$ in the simplex Δ.

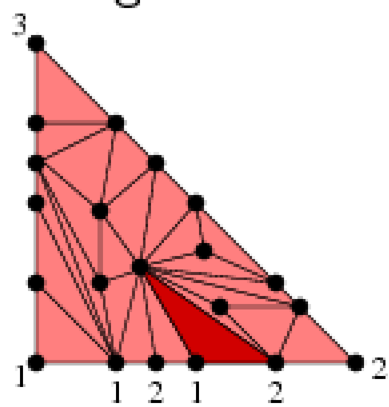
Label all the vertices of T using the numbers $1, 2, \dots, d+1$ in such a way that no vertex that lies on the facet Δ_i receives the label i.

Then there is a simplex in T whose vertices carry all the different $d+1$ labels.

Sperner's lemma and fixed points

Proof By induction on the dimension: start with a fully labeled simplex of one dimension less in the boundary; then dive into the big simplex until you find a fully labeled simplex in the triangulation.





Sperner's lemma and fixed points

Corollary (Brower's fixed point theorem) If C is a topological d-dimensional ball and $f : C \to C$ is a continuous map, then there is a point in C such that $f(x) = x$.

Proof: For any given triangulation T, Sperner Lemma allows you to find a simplex in which the i-th barycentric coordinate of the i-th vertex does does not increase. Doing this for finer and finer triangulations, converges to a fixed point.

Sperner's lemma and fixed points

The algorithmic performance of Sperner Lemma depends heavily on the size (number of simplices) of your triangulations. This raises the question of what is the smallest size of a triangulation. Unfortunately, this is a hard problem:

Theorem (Below, De Loera, Richter-Gebert, 2000) It is NP-complete to compute the smallest size triangulation of a polytope, even in dimension 3.

Remark Even for the d dimensional cube I^d, the smallest size triangulation has only been computed up to $d = 7$, and the asymptotics of the minimum size of a triangulations is not known.