Volumes and Integrals over Polytopes

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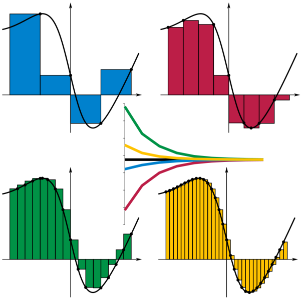
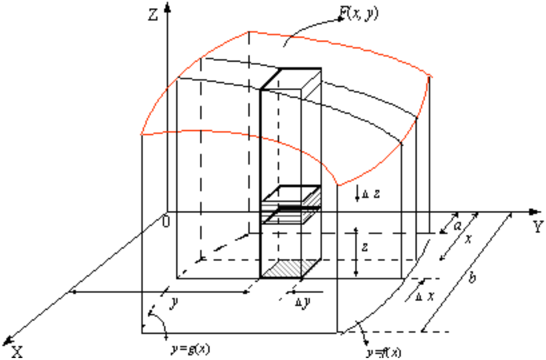
Meet Volume

The (Euclidean) **volume** $V(R)$ of a region of space R is real non-negative number defined via the Riemann integral over the regions.





Meet Volume's Cousins

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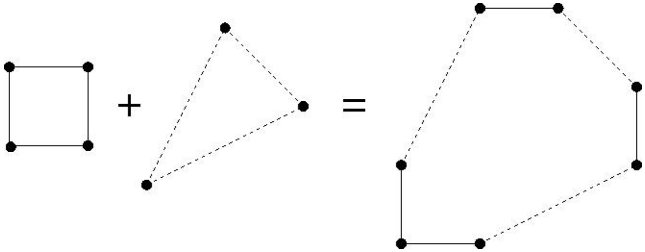
- Given polytopes $P_1, \ldots, P_k \subset \mathbb{R}^n$ and real numbers $t_1, \ldots, t_k \geq 0$ the Minkowski sum is the polytope

$$t_1 P_1 + \cdots + t_k P_k := \{t_1 v_1 + \cdots + t_k v_k : v_i \in P_i\}$$

.

- **EXAMPLE**



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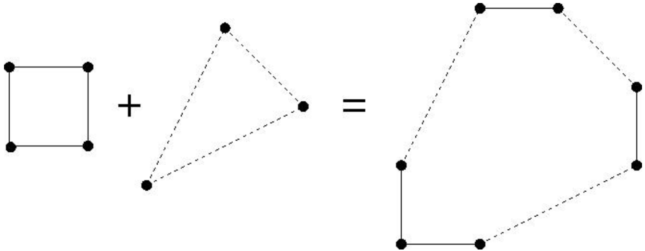

- **Theorem**(H. Minkowski) There exist $MV(P_1^{a_1}, \ldots, P_k^{a_k}) > 0$ (the mixed volumes) such that $V(t_1 P_1 + \cdots + t_k P_k) = \sum_{a_1+\cdots+a_k=n} \binom{n}{a_1,\ldots,a_k} MV(P_1^{a_1}, \ldots, P_k^{a_k}) t_1^{a_1} t_2^{a_2} \cdots t_k^{a_k}$.

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- (for computational algebraic geometers) Let $f_1, \ldots, f_n$ be polynomials in $\mathbb{C}[x_1, \ldots, x_n]$. Let $New(f_j)$ denote the Newton polytope of f_j, If $f_1, \ldots, f_n$ are generic, then the number of solutions of the polynomial system of equations $f_1 = 0, \ldots, f_n = 0$ with no $x_i = 0$ is equal to the normalized mixed volume $n! MV(New(f_1), \ldots, New(f_n))$.

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- (for Combinatorialists) Volumes count things! $CR_m = \{(a_{ij}) : \sum_i a_{ij} = 1, \sum_j a_{ij} = 1,$ with $a_{ij} \geq 0$ but $a_{ij} = 0$ when $j > i+1$ $\}$, then $NV(CR_m) =$ product of first $(m-2)$ Catalan numbers. (D. Zeilberger).
- Many Other applications

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- For convex bodies, deterministic approximation is already hard, but randomized approximation can be done efficiently (work by Barany, Dyer, Elekes, Furedi, Frieze, Kannan, Lovász, Simonovits, others)

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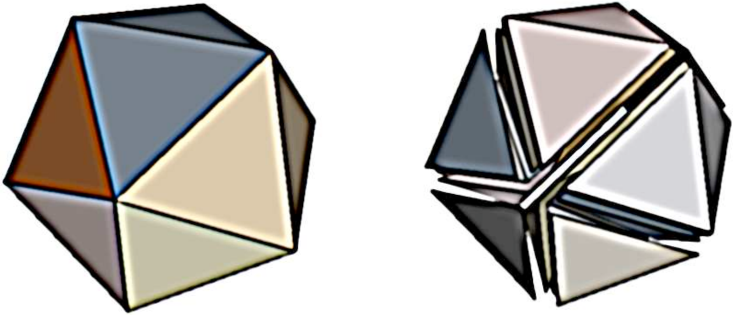
- The volume of a (Euclidean) simplex is given by a fast determinant calculation. To compute the volume of a polytope: divide it as a disjoint union of simplices, calculate volume for each simplex and then add them up!

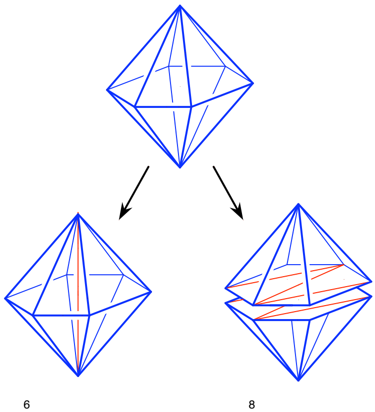
Triangulations: Enough to know how to do it for simplices!



Theorem: For all polytopes in fixed dimension d their whole volume can be computed in polynomial time.

The size of a triangulation changes!

Triangulations of a convex polyhedron come in different sizes! i.e. the number of simplices changes.



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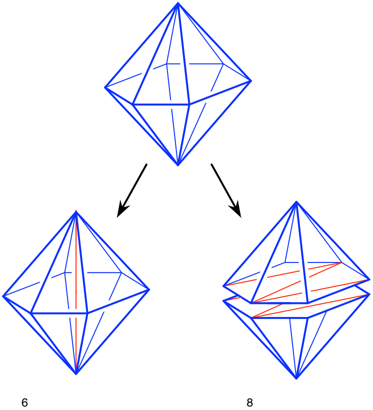
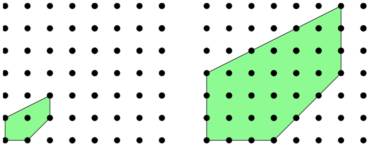


Counting lattice points to approximate volume

- Lattice points are those points with integer coordinates: $\mathbb{Z}^n = \{(x_1, x_2, \ldots, x_n) | x_i \text{ integer}\}$ We wish to count how many lie inside a given polytope!
- Let P be a convex polytope in $\mathbb{R}^d$. For each integer $n \geq 1$, let

$$nP = \{nq | q \in P\}$$



P 3P

- For P a d-polytope, let

$$i(P,n) = \#(nP \cap \mathbb{Z}^d) = \#\{q \in P \,|nq \in \mathbb{Z}^d\}$$

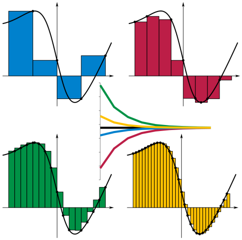
- This is the number of lattice points in the dilation nP.

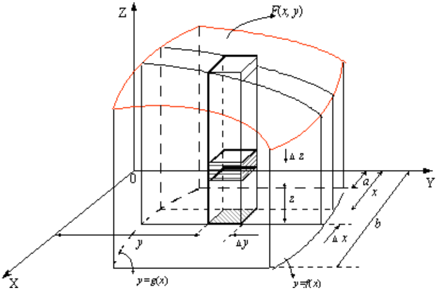
$$\text{Volume of } P = limit_{n\to\infty} \frac{i(P,n)}{n^d}$$

At each dilation we can approximate the volume by placing a small unit cube centered at each lattice point:

Integration of polynomials:

Given P be a d-dimensional rational polytope inside $\mathbb{R}^n$ and let $f \in \mathbb{Q}[x_1, \ldots, x_n]$ be a polynomial with rational coefficients.





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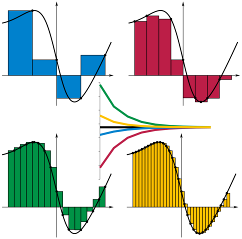

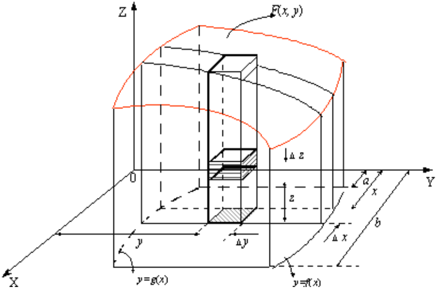

Compute the EXACT value of the integral $\int_P f \; dm$?

Example

If we integrate the monomial $x^{17}y^{111}z^{13}$ over the three-dimensional standard simplex Δ. Then $\int_\Delta x^{17}y^{111}z^{13}dxdydz$ equals exactly

$$\frac{1}{317666399137306017655882907073489948282706281567360000}$$

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- **My point:** Exact integration useful for calibration!!!

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- For this $\mathfrak{d}m$, every integral of a polynomial function with rational coefficients will be a *rational number*.

Polynomials to Powers of Linear forms

- it is well-known that any polynomial of degree M can be decomposed as sums of M-th powers of linear forms.
- Decompose f as a sum $f := \sum_{\ell} c_{\ell} \langle \ell, x \rangle^M$ with at most 2^M terms in the sum, for this one uses the well-known identity

$$x_1^{M_1} x_2^{M_2} \cdots x_n^{M_n} = \frac{1}{|\mathbf{M}|!} \sum_{0 \le p_i \le M_i} (-1)^{|\mathbf{M}|-(p_1+\cdots+p_n)} \binom{M_1}{p_1} \cdots \binom{M_n}{p_n} (p_1 x_1 + \cdots + p_n x_n)$$

where $|\mathbf{M}| = M_1 + \cdots + M_n \le M$.

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From fixed number of linear forms to fixed degree.

- We can also efficiently compute integrals of polynomials of fixed degree M.
- Write a polynomial as a sum of powers of linear forms. Explicit formula with at most 2^M terms.
$x_1^{m_1} x_2^{m_2} \cdots x_d^{m_d} =$
$\frac{1}{|m|!} \sum_{0 \leq p_i \leq m_i} (-1)^{|m|-|p|} \binom{m_1}{p_1} \cdots \binom{m_d}{p_d} (p_1 x_1 + \cdots + p_d x_d)^{|m|}.$
- **Example:**

$$7x^2 + y^2 + 5z2 + 2xy + 9yz =$$

$$\frac{1}{8}(12(2x)^2 - 9(2y)^2 + (2z)^2 2 + 8(x+y)^2 + 36(y+z)^2)$$

Integration of arbitrary powers of quadratic forms is NP-hard

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- **Theorem** [Motzkin-Straus 1965]
  G a graph with clique number $\omega(G)$.
  $Q_G(x) := \frac{1}{2}\sum_{(i,j)\in E(G)} x_i x_j$. Function on standard simplex in $\mathbb{R}^{|V(G)|}$.
  Then $\|Q_G\|_\infty = \frac{1}{2}(1 - \frac{1}{\omega(G)})$.

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  Then $\|Q_G\|_\infty = \frac{1}{2}(1 - \frac{1}{\omega(G)})$.
- **Lemma** Let G a graph with d vertices. For $p \geq 4(e-1)d^3 \ln(32d^2)$, the clique number $\omega(G)$ is equal to $\left\lceil \frac{1}{1-2\|Q_G\|_p} \right\rceil$. ($L^p$-norm, Holder inequality).

Valuations

A function S on polyhedra is a **valuation**. If it is a linear map from the vector space of characteristic functions $\chi(\mathfrak{p}_i)$ of any polyhedra into a field.
Thus if polyhedra $\mathfrak{p}_i$ satisfy a linear relation $\sum_i r_i \chi(\mathfrak{p}_i) = 0$, then

$$\sum_i r_i S(\mathfrak{p}_i) = 0,$$

Example:

$$\chi(\mathfrak{p}_1 \cup p_2) + \chi(\mathfrak{p}_1 \cap p_2) - \chi(\mathfrak{p}_1) - \chi(\mathfrak{p}_2) = 0,$$

Two important valuations for polyhedra

$\mathfrak{p}$ (convex) polyhedron, rational (lattice Λ).

$$S(\mathfrak{p})(\xi) := \sum_{x \in \mathfrak{p} \cap \Lambda} e^{\langle \xi, x \rangle}$$

generating function for lattice points of $\mathfrak{p}$.

$$I(\mathfrak{p})(\xi) := \int_{\mathfrak{p}} e^{\langle \xi, x \rangle} \, dm.$$

when integral and series converge. If $\mathfrak{p}$ contains a line, then $S(\mathfrak{p}) := 0$ and $I(\mathfrak{p}) := 0$.

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IMPORTANT FACT: When $\mathfrak{p}$ is a simplicial cone easy to write.

Sums $S(\mathfrak{p})$ in dim 1

For the real line we have

$$\sum_{n>s} e^{n\xi} + \sum_{n<s} e^{n\xi} = \frac{e^{\xi}}{1-e^{\xi}} + \frac{1}{1-e^{-\xi}} = 0$$

$$\sum_{n=-\infty}^{\infty} e^{n\xi} = 0$$

For the line segment $[a, b]$ we have.

$$\chi([a,b]) = \chi([-\infty, b]) + \chi([a, +\infty]) - \chi(\mathbb{R})$$

$$\sum_{n=a}^{\infty} e^{n\xi} + \sum_{n=-\infty}^{b} e^{n\xi} = \frac{e^{a\xi}}{1-e^{\xi}} + \frac{e^{b\xi}}{1-e^{-\xi}} = \frac{e^{a\xi} - e^{(b+1)\xi}}{1-e^{\xi}}$$

Valuations for simplicial cones

Theorem: $s + \mathfrak{c}$ affine cone with vertex s and integral generators $v_1, \ldots, v_d \in$ lattice Λ. $\mathfrak{c} = \mathbb{R}_+ v_1 + \ldots \mathbb{R}_+ v_d$.

$$I(s+\mathfrak{c})(\xi) = |\det_{\Lambda}(v_j)| \prod_j \frac{-e^{\langle \xi, s \rangle}}{\langle \xi, v_j \rangle}$$

$$S(s+\mathfrak{c})(\xi) = \left(\sum_{x \in (s+\mathfrak{b}) \cap \Lambda} e^{\langle \xi, x \rangle} \right) \prod_j \frac{1}{1 - e^{\langle \xi, v_j \rangle}}$$

where $\mathfrak{b} = \sum_j [0, 1[v_j$, *semi-closed cell.*

Polyhedron $\equiv$ sum of its supporting cones at vertices

Theorem *(Brion-Lawrence-Varchenko)*

$\mathfrak{p}$ convex polyhedron, $s + \mathfrak{c}_s$ supporting cone at vertex s.

$$S(\mathfrak{p}) = \sum_{s \in \text{ vertices}} S(s + \mathfrak{c}_s), \quad I(\mathfrak{p}) = \sum_s I(s + \mathfrak{c}_s)$$

Example:

Let Δ be a simplex. Let ℓ be a linear form which is regular w.r.t. Δ, i.e., $\langle \ell, \mathbf{s}_i \rangle \neq \langle \ell, \mathbf{s}_j \rangle$ for any pair $i \neq j$. Then

$$\int_\Delta e^{<\ell,x>} \mathfrak{d}m = d! \operatorname{vol}(\Delta, \mathfrak{d}m) \sum_{i=1}^{d+1} \frac{e^{\langle \ell, \mathbf{s}_i \rangle}}{\prod_{j \neq i} \langle \ell, \mathbf{s}_i - \mathbf{s}_j \rangle}. \qquad (2)$$

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$$\int_\Delta < \ell, x >^M \mathfrak{d}m = d! \operatorname{vol}(\Delta, \mathfrak{d}m) \frac{M!}{(M+d)!} \left(\sum_{i=1}^{d+1} \frac{\langle \ell, \mathbf{s}_i \rangle^{M+d}}{\prod_{j \neq i} \langle \ell, \mathbf{s}_i - \mathbf{s}_j \rangle} \right). \qquad (3)$$

From Exponentials to Powers of Linear Forms

- To compute $L^M(P)(\ell) = \int_P \langle \ell, x\rangle^M \mathfrak{d}m$ for linear form ℓ such that the integral exists over a polytope P we use valuation property and do it for cones:

$$\int_{s+C} e^{\langle t\ell,x\rangle} \mathfrak{d}m = \mathrm{vol}(\Pi_C) e^{\langle t\ell,s\rangle} \prod_{i=1}^{d} \frac{1}{\langle -t\ell, u_i\rangle}. \tag{4}$$

  The value of this integral is an analytic function of t.
- We wish to recover the value of the integral of $\langle \ell, x\rangle^M$ over the cone. This is the coefficient of t^M in the Taylor expansion in the left side.
- We equate it to the **Laurent series expansion** around $t = 0$ of the right-hand-side expression, which is a meromorphic function of t.

$$\operatorname{vol}(\Pi_C)e^{\langle t\ell,s\rangle}\prod_{i=1}^{d}\frac{1}{\langle -t\ell u_i\rangle}=\sum_{n=0}^{\infty}t^{n-d}\frac{\langle \ell,s\rangle^n}{n!}\cdot\operatorname{vol}(\Pi_C)\prod_{i=1}^{d}\frac{1}{\langle -\ell,u_i\rangle},$$

thus we can conclude the following.

Corollary

For a regular linear form ℓ, a simplicial cone C generated by rays $u_1, u_2, \ldots u_d$ with vertex s

$$L^M(s+C)(\ell)=\frac{M!}{(M+d)!}\operatorname{vol}(\Pi_C)\frac{(\langle \ell,s\rangle)^{M+d}}{\prod_{i=1}^{d}\langle -\ell,u_i\rangle}. \tag{5}$$

Corollary

If $\langle -\ell, u_i\rangle = 0$ *for some* u_i*, then*

$$L^M(s+C)(\ell) = \frac{M!}{(M+d)!}\,\mathrm{vol}(\Pi_C)\,\mathrm{Res}_{\epsilon=0}\,\frac{(\langle \ell+\hat{\epsilon}, s\rangle)^{M+d}}{\epsilon\prod_{i=1}^{d}\langle -\hat{\ell}-\hat{\epsilon}, u_i\rangle}, \quad (6)$$

where $\hat{\epsilon}$ *is a vector in terms of* ϵ *such that* $\langle -\ell-\hat{\epsilon}, u_i\rangle \neq 0$ *for all* u_i*,*

Corollary

For any triangulation $\mathcal{D}_s$ *of the feasible cone* C_s *at each of the vertices* s *of the polytope* P *we have*

$$L^M(P)(\ell) = \sum_{s\in V(P)}\sum_{C\in\mathcal{D}_s} L(s+C_s)(\ell). \quad (7)$$

EXAMPLE: integrate x over the unit square

The polytope has four vertices that we need to consider, and each tangent cone is already simple.

Vertex $s_1 = (0,0)$: Because $\langle \ell, s_1 \rangle^{1+2} = 0$ the integral on this cone is zero.

Vertex $s_2 = (0,1)$: For the same reason as s_1, the integral on this cone is zero.

Vertex $s_3 = (1,0)$: The rays are $u_1 = (0,1), u_2 = (-1,0)$. Because $\langle \ell, u_1 \rangle = 0$, we need a perturbation vector $\hat{\epsilon}$ so that when $\ell := \ell + \hat{\epsilon}$, we do not divide by zero on any cone (we have to check this cone and the next one). Pick $\hat{\epsilon} = (\epsilon, \epsilon)$. Then the integral on this cone is

$$\frac{M!}{(M+d)!} \operatorname{vol}(\Pi_C) \operatorname{Res}_{\epsilon=0} \frac{(1+\epsilon)^{1+2}}{\epsilon(\epsilon)(-1-\epsilon)} = \frac{1!}{(1+2)!} \times 1 \times -2 = -2/6.$$

Vertex $s_4 = (1,1)$: The rays are $u_1 = (-1,0), u_2 = (0,-1)$.
Again, we perturbate ℓ by the same $\hat{\epsilon}$. The integral on this cone is

$$\frac{M!}{(M+d)!} \operatorname{vol}(\Pi_C) \operatorname{Res}_{\epsilon=0} \frac{(1+2\epsilon)^{1+2}}{\epsilon(-\epsilon)(-1-\epsilon)} = \frac{1!}{(1+2)!} \times 1 \times 5 = 5/6.$$

The integral $\int_P x \mathfrak{d} x \mathfrak{d} y = 0 + 0 - 2/6 + 5/6 = 1/2$ as it should be.

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- Integration of polynomials of fixed degree is OK too.
- Integration of arbitrary powers of quadratic forms is NP-hard.
- Algorithms run nicely in practice! TONIGHT's demo:

Latte Integrale

Thank you