Lab 1: Monday, July 20

1. Compute a random 4-dimensional polytope P as the convex hull of 10 random points using `rand_sphere(4,10)`. Run `VISUAL` to see a Schlegel diagram. How many 3-dimensional polytopes do you see? How many facets does P have?

2. Using a for-loop in polymake, generate 100 random 3-dimensional polytopes and 5-dimensional polytopes using the `polymake` function `rand_sphere`. Compute their f-vectors. What pattern (called Euler's relation) do you notice?

3. The graph of a soccer ball (below) is 3-regular and has exactly 32 faces, some of which are pentagonal (5-gonal) and some of which are hexagonal (6-gonal). Let p_5 (respectively p_6) be the number of pentagonal (respectively hexagonal) faces. Write p_5 in terms of p_6 using Euler's relation. Then determine the values of p_5 and p_6.

4. For each $4 \geq d < n \leq 9$, examine the graph of the cyclic polytope $C_d(n)$. The function `cyclic(d,n)` constructs the polytope $C_d(n)$. Use the function `VISUAL_GRAPH` to view the graph. How many edges are in each graph?

5. Try finding coordinates for eight points $s_1, \ldots, s_8$ in $\mathbb{R}^3$ such that $\mathrm{conv}(s_1, \ldots, s_8)$ has the same face lattice as an honest cube, but it has fewer coplanarities. How well can you do? Can you make the only 4-point coplanarities on the six facets?

Extra questions

1. Starting in dimension three, construct a polytope as the convex hull of a collection of points using the `polymake` function `POINTS`. How many vertices (use `N_VERTICES`) can you get? Try to get the largest number possible. How many facets can you get?

2. Compute the dual of the cyclic polytope $C_3(9)$. What does its graph look like?

3. Let P be the 3×4 transportation polytope with margin sizes $(4, 4, 4)$ and $(3, 3, 3, 3)$. How many vertices does P have? How many facets does P have?

Lab 2: Tuesday, July 21

1. Prove Gale's Theorem, using Fourier-Motzkin elimination. In particular, show how to find z in the case **(b)**.

   Gale's Theorem. For a matrix $A \in \mathbb{R}^{m \times d}$ and a vector $b \in \mathbb{R}^m$, exactly one of the following holds true:
   (a) There is a $x \in \mathbb{R}^d$ such that $Ax \leq b$;
   (b) There is a $z \in \mathbb{R}^m$ such that $z \geq 0, z^T A = 0$ and $z^T b < 0$.

2. Consider the polytope P defined by the following system of inequalities:

$$\begin{aligned} -x - 4y &\leq 9 \\ -2x - y &\leq -4 \\ x - 2y &\leq 0 \\ x &\leq 4 \\ 2x + y &\leq 11 \\ -2x + 6y &\leq 17 \\ -6x - y &\leq -6. \end{aligned}$$

   Use Fourier-Motzkin elimination to eliminate the variable y. What are the smallest and largest values of x? Draw the polytope P to confirm this. Enter the polytope P into `polymake`. How do (the coordinates of) the vertices confirm this same information?

3. Consider the vector configuration $(1, 0), (-1, 0), (0, 1), (0, -1), (1, 1), (-1, -1)$. Use `topcom`'s command line program `points2triangs` to compute the number of triangulations. Draw the triangulations. How many are there? How many are regular?

4. How many different ways are there to triangulate a pentagon? How many different ways are there to triangulate a hexagon?

Additional Exercises

1. How many faces are there in a d-dimensional cube? That is, what is the value of $f_0 + f_1 + f_2 + \cdots + f_d$? How many faces are there in a d-dimensional cross-polytope? How many faces are there in a d-dimensional simplex?

Lab 3: Wednesday, July 22

1. Euro coins come in 1 cent, 2 cent, 5 cent, 10 cent, 20 cent, 50 cent, 1 euro, and 2 euro pieces. (100 cents is equal to 1 euro.) How many ways are there to split a 5 euro bill into these smaller pieces? (Hint: Use `LattE` and describe a 7-dimensional polytope in 8-dimensional space using eight inequalities and one equation.)

2. An $n \times n$ magic square with sum S is a filling of an $n \times n$ table with non-negative integers so that the n numbers in each row and the n numbers in each column add up to S. How many 4×4 magic squares with sum 20 are there?

3. Consider the 3-dimensional cube C with vertices $(\pm 1, \pm 1, \pm 1)$. Use TOPCOM to consider the triangulations of C. How many triangulations are there? Do they all use the same number of tetrahedra? Pick one triangulation: Compute the volume of each tetrahedron in the triangulation. What is the sum of these volumes?

4. Let

$$P = conv(\{(-1,3,1,2),(-1,3,-1,1),(-1,-1,1,1),(-1,-1,-1,0),$$
$$(3,-1,1,-3),(3,-1,-1,4)\}).$$

   Use polymake to: Determine the vertices of the polar polytope and draw the graphs of P and its polar. What are the facets? How many are there?

5. Compute the volume of the polytope given by the five inequalities

$$\begin{aligned} -x_1 + x_2 &\leq 2 \\ x_2 &\leq 4 \\ 3x_1 + 2x_2 &\leq 15 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

   using the Lawrence method.

Lab 4: Thursday, July 23

1. Determine the graph of a 3-polytope with 13 vertices and 25 edges. Can you find vertices for this polytope and verify your work in polymake?

2. Draw the Schlegel diagram of an icosahedron.

3. Show that four points (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4) lie on a circle if and only if the four points of the form $(x_i, y_i, x_i^2 + y_i^2)$ lie on a plane.

4. Experiment with polymake to find values for the numbers a and b to make the linear program

$$\begin{array}{ll} \text{maximize} & x_1 + x_2 \\ \text{subject to} & \begin{cases} ax_1 + bx_2 \leq 1 \\ x_1 \geq 0 \\ x_2 \geq 0 \end{cases} \end{array}$$

   (a) have an optimal solution.
   (b) be infeasible.
   (c) be unbounded.

5. Prove that there are only five Platonic solids.

6. Construct a non-degenerate 3×4 transportation polytope P in polymake and pick (and store) a linear functional c for it. Use polymake to find a vertices of P that maximize and minimize c on P.