Volumes and Integrals over Polytopes

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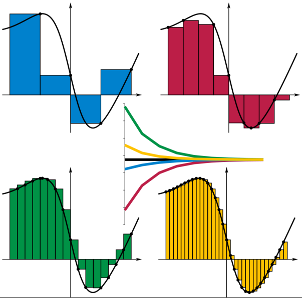
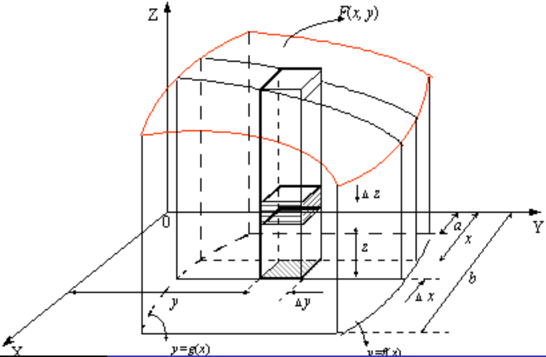
Meet Volume

The (Euclidean) **volume** $V(R)$ of a region of space R is real non-negative number defined via the Riemann integral over the regions.





Meet Volume's Cousins

- In the case when P is an n-dimensional *lattice polytope* (i.e., all vertices have integer coordinates) we can naturally define a **normalized volume** of P, $NV(P)$ to be $n!V(P)$.
- EXAMPLE: $P = \{(x, y) : 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$

$$NV(P) = 2! \cdot 1 = 2.$$

- Given polytopes $P_1, \ldots, P_k \subset \mathbb{R}^n$ and real numbers $t_1, \ldots, t_k \geq 0$ the Minkowski sum is the polytope

$$t_1 P_1 + \cdots + t_k P_k := \{t_1 v_1 + \cdots + t_k v_k : v_i \in P_i\}$$

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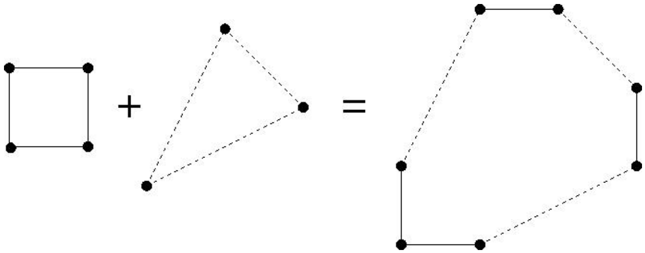
- **EXAMPLE**

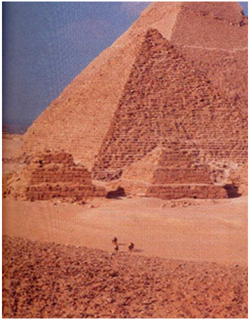


- **Theorem**(H. Minkowski) There exist $MV(P_1^{a_1}, \ldots, P_k^{a_k}) > 0$ (the mixed volumes) such that $V(t_1 P_1 + \cdots + t_k P_k) = \sum_{a_1+\cdots+a_k=n} \binom{n}{a_1,\ldots,a_k} MV(P_1^{a_1}, \ldots, P_k^{a_k}) t_1^{a_1} t_2^{a_2} \cdots t_k^{a_k}$.

A few reasons to compute them

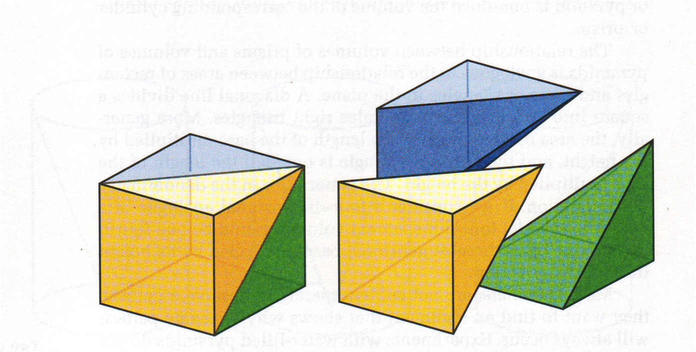
- (for algebraic geometers) If P is an integral polytope, then the normalized volume of P is the degree of the toric variety associated to P.
- (for computational algebraic geometers) Let $f_1, \ldots, f_n$ be polynomials in $\mathbb{C}[x_1, \ldots, x_n]$. Let $New(f_j)$ denote the Newton polytope of f_j, If $f_1, \ldots, f_n$ are generic, then the number of solutions of the polynomial system of equations $f_1 = 0, \ldots, f_n = 0$ with no $x_i = 0$ is equal to the normalized mixed volume $n!MV(New(f_1), \ldots, New(f_n))$.
- (for Combinatorialists) Volumes count things! $CR_m = \{(a_{ij}) : \sum_i a_{ij} = 1, \sum_j a_{ij} = 1,$ with $a_{ij} \geq 0$ but $a_{ij} = 0$ when $j > i+1$ $\}$, then $NV(CR_m) =$ product of first $(m-2)$ Catalan numbers. (D. Zeilberger).

Do we need limits to define volumes of polytopes?

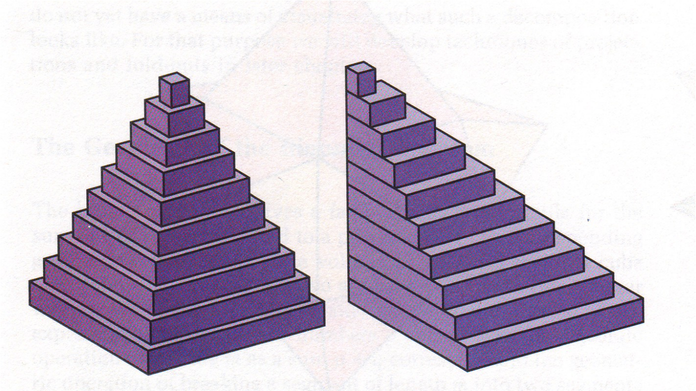


$$\text{volume of egyptian pyramid} = \frac{1}{3}(\text{area of base}) \times \text{height}$$

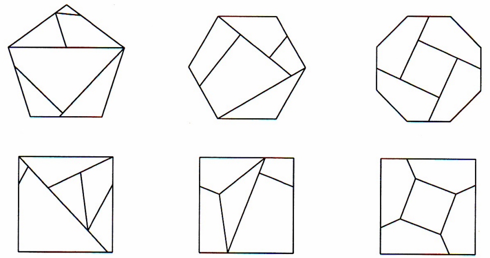
Easy and pretty in some cases...



In general, proofs seem to rely on an infinite process!



But not necessary in dimension two!



Polygons of the same area are equidecomposable, i.e., one can be partitioned into pieces that can be reassembled into the other.

Hilbert's Third Problem

Are any two convex 3-dimensional polytopes of the same volume equidecomposable?

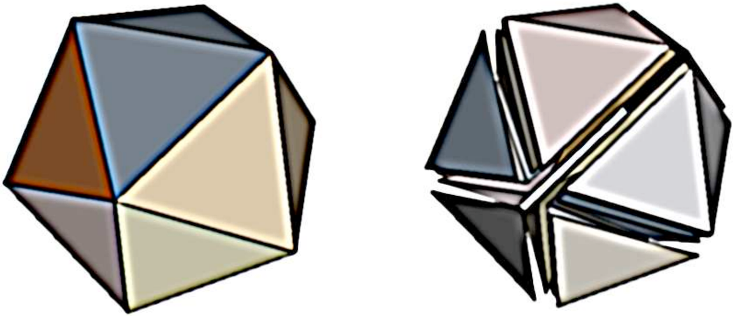
NOT always!!! We need calculus to define the volume of high-dimensional polytopes.

- It is hard to compute the volume of a vertex presented polytopes (Dyer and Frieze 1988, Khachiyan 1989).
- Number of digits necessary to write the volume of a rational polytope P cannot always be bounded by a polynomial on the input size. *(J. Lawrence 1991)*.
- **Theorem** (Brightwell and Winkler 1992) It is $\#P$-hard to compute the volume of a d-dimensional polytope P represented by its facets.
- We even know that it is hard to compute the volume of *zonotopes* (Dyer, Gritzmann 1998). Thus computing mixed volumes, even for Minkowski sums of line segments, is already hard!
- For convex bodies, deterministic approximation is already hard, but randomized approximation can be done efficiently (work by Barany, Dyer, Elekes, Furedi, Frieze, Kannan

simplices

- SIMPLICES are d-dimensional polytopes with $d+1$ vertices. E.g., triangles, tetrahedra, etc.
- The volume of a (Euclidean) simplex is given by a fast determinant calculation.
  To compute the volume of a polytope: divide it as a disjoint union of simplices, calculate volume for each simplex and then add them up!

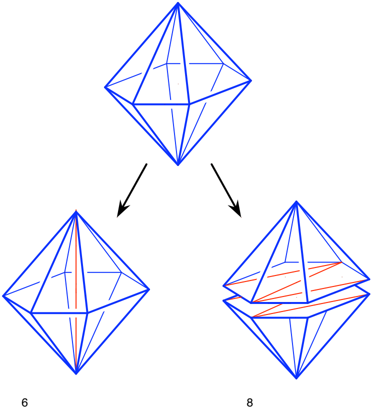
Triangulations: Enough to know how to do it for simplices!



Theorem: For all polytopes in fixed dimension d their whole volume can be computed in polynomial time.

The size of a triangulation changes!

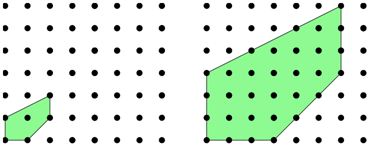
Triangulations of a convex polyhedron come in different sizes! i.e. the number of simplices changes.



Counting lattice points to approximate volume

- Lattice points are those points with integer coordinates: $\mathbb{Z}^n = \{(x_1, x_2, \ldots, x_n) | x_i \text{ integer}\}$ We wish to count how many lie inside a given polytope!
- Let P be a convex polytope in $\mathbb{R}^d$. For each integer $n \geq 1$, let

$$nP = \{nq | q \in P\}$$



P 3P

- For P a d-polytope, let

$$i(P,n) = \#(nP \cap \mathbb{Z}^d) = \#\{q \in P \mid nq \in \mathbb{Z}^d\}$$

- This is the number of lattice points in the dilation nP.

$$\text{Volume of } P = limit_{n\to\infty} \frac{i(P,n)}{n^d}$$

At each dilation we can approximate the volume by placing a small unit cube centered at each lattice point:

Lawrence's Style Volume Formulas

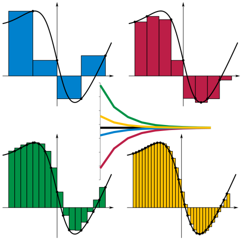
Theorem (J. Lawrence 1991) Let P be a simple d-polytope given by $\{x \in \mathbb{R}^d : b_i - a_i^t x \geq 0,\ i = 1 \dots m\}$. Suppose that c is a vector such that the dot produt of c with any edge of P is non-zero. Then the volume of P equals

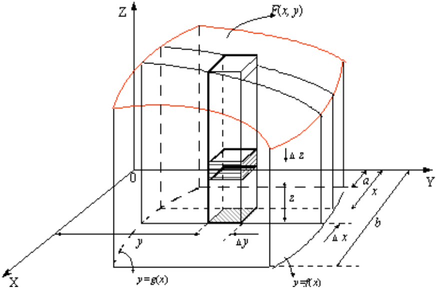
$$vol(P) = \frac{1}{d!} \sum_{v \in V(P)} \frac{(\langle c, v \rangle)^d}{\delta_v \gamma_1 \gamma_2 \cdots \gamma_d}$$

where if indices of the constraints that are binding at v are $i_1, \dots, i_d$ then γ_i's are such
$c = \gamma_1 a_{i_1} + \gamma_2 a_{i_2} + \cdots + \gamma_n a_{i_d}$ and $\delta_v = |det([a_{i_1}, a_{i_2}, \dots, a_{i_d}])|$.

Integration of polynomials:

Given P be a d-dimensional rational polytope inside $\mathbb{R}^n$ and let $f \in \mathbb{Q}[x_1, \ldots, x_n]$ be a polynomial with rational coefficients.





Compute the EXACT value of the integral $\int_P f \; dm$?

Example

If we integrate the monomial $x^{17}y^{111}z^{13}$ over the three-dimensional standard simplex Δ. Then $\int_\Delta x^{17}y^{111}z^{13}dxdydz$ equals exactly

$$\frac{1}{317666399137306017655882907073489948282706281567360000}$$

Why exact integration?

- Integrals over polytopes arise in probability, statistics, algebraic geometry, combinatorics, symplectic geometry. Already computing volumes is a very important subroutine.
- Despite the success of APPROXIMATE integration, still EXACT integration is necessary.
- **Example:** Computation of marginal likelihood integrals in model selection.
- **Example:** Statisticians used BIC, Laplace, Montecarlo approximations in concrete 6 variable problems. They say "Problems are too hard for exact methods". approximation leads to model answer.
- **My point:** Exact integration useful for calibration!!!

TECHNICAL DETAILS...

- **The input simplex Δ:** encoding of Δ is given by the number of the dimension d, and the largest binary encoding size of the coordinates among vertices.
- For simplicity assume the polytope P is full dimension n, in $\mathbb{R}^n$ $\mathfrak{d}m$ is the standard Lebesgue measure, which gives volume 1 to the fundamental domain of the lattice $\mathbb{Z}^n$.
- For this $\mathfrak{d}m$, every integral of a polynomial function with rational coefficients will be a *rational number*.

How to represent a polynomial in a computer?

- **The input polynomial:** requires that one specifies concrete data structures for reading the input polynomial and to carry on the calculations. Three main possibilities:
- **dense representation:** polynomials are given by a list of the coefficients of all monomials up to a given total degree M.
- **sparse representation:** Polynomials are specified by a list of exponent vectors of monomials with non-zero coefficients, together with their coefficients.
- **straight-line program** Φ if polynomial is a finite sequence of polynomial functions of $\mathbb{Q}[x_1, \dots, x_n]$, namely $q_1, \dots, q_k$, such that each q_i is either a variable $x_1, \dots, x_n$, an element of $\mathbb{Q}$, or either the sum or the product of two preceding polynomials in the sequence and such that $q_k = f$.

Best News: Fast Integration for powers of linear forms

Theorem: There exists a polynomial-time algorithm for the following problem.

Input:

- numbers $d, M \in \mathbb{N}$.
- affinely independent rational vectors $\mathbf{s}_1, \ldots, \mathbf{s}_{d+1} \in \mathbb{Q}^d$ in binary encoding,
- a power of a linear form $\langle \ell, x \rangle^M$

Output:, in binary $\int_\Delta \langle \ell, \mathbf{x} \rangle^M \mathfrak{d}m$.

From fixed number of linear forms to fixed degree.

- We can also deal with arbitrary polynomials of fixed degree.
- Write a polynomial as a sum of powers of linear forms. Explicit formula with at most 2^M terms.
$x_1^{m_1} x_2^{m_2} \cdots x_d^{m_d} =$
$\frac{1}{|m|!} \sum_{0 \leq p_i \leq m_i} (-1)^{|m|-|p|} \binom{m_1}{p_1} \cdots \binom{m_d}{p_d} (p_1 x_1 + \cdots + p_d x_d)^{|m|}$.

Integration of arbitrary powers of quadratic forms is NP-hard

- The clique problem (does G contain a clique of size $\geq n$) is NP-complete. (Karp 1972).
- **Theorem** [Motzkin-Straus 1965]
  G a graph with clique number $\omega(G)$.
  $Q_G(x) := \frac{1}{2}\sum_{(i,j)\in E(G)} x_i x_j$. Function on standard simplex in $\mathbb{R}^{|V(G)|}$.
  Then $\|Q_G\|_\infty = \frac{1}{2}(1 - \frac{1}{\omega(G)})$.
- **Lemma** Let G a graph with d vertices. For $p \geq 4(e-1)d^3 \ln(32d^2)$, the clique number $\omega(G)$ is equal to $\left\lceil \frac{1}{1-2\|Q_G\|_p} \right\rceil$. ($L^p$-norm, Holder inequality).

Valuations

A function S on polyhedra is a **valuation**. If it is a linear map from the vector space of characteristic functions $\chi(\mathfrak{p}_i)$ of any polyhedra into a field.
Thus if polyhedra $\mathfrak{p}_i$ satisfy a linear relation $\sum_i r_i \chi(\mathfrak{p}_i) = 0$, then

$$\sum_i r_i S(\mathfrak{p}_i) = 0,$$

Example:

$$\chi(\mathfrak{p}_1 \cup p_2) + \chi(\mathfrak{p}_1 \cap p_2) - \chi(\mathfrak{p}_1) - \chi(\mathfrak{p}_2) = 0,$$

Two important valuations for polyhedra

$\mathfrak{p}$ (convex) polyhedron, rational (lattice Λ).

$$S(\mathfrak{p})(\xi) := \sum_{x \in \mathfrak{p} \cap \Lambda} e^{\langle \xi, x \rangle}$$

generating function for lattice points of $\mathfrak{p}$.

$$I(\mathfrak{p})(\xi) := \int_{\mathfrak{p}} e^{\langle \xi, x \rangle} \, dm.$$

when integral and series converge. If $\mathfrak{p}$ contains a line, then $S(\mathfrak{p}) := 0$ and $I(\mathfrak{p}) := 0$.
IMPORTANT FACT: When $\mathfrak{p}$ is a simplicial cone easy to write.

Sums $S(\mathfrak{p})$ in dim 1

For the real line we have

$$\sum_{n>s} e^{n\xi} + \sum_{n<s} e^{n\xi} = \frac{e^{\xi}}{1-e^{\xi}} + \frac{1}{1-e^{-\xi}} = 0$$

$$\sum_{n=-\infty}^{\infty} e^{n\xi} = 0$$

For the line segment $[a, b]$ we have.

$$\chi([a,b]) = \chi([-\infty, b]) + \chi([a, +\infty]) - \chi(\mathbb{R})$$

$$\sum_{n=a}^{\infty} e^{n\xi} + \sum_{n=-\infty}^{b} e^{n\xi} = \frac{e^{a\xi}}{1-e^{\xi}} + \frac{e^{b\xi}}{1-e^{-\xi}} = \frac{e^{a\xi} - e^{(b+1)\xi}}{1-e^{\xi}}$$

Case of a simplicial affine cone

$s+\mathfrak{c}$ affine cone with vertex s and integral generators $v_1, \ldots, v_d \in$ lattice Λ. $\mathfrak{c} = \mathbb{R}_+ v_1 + \ldots \mathbb{R}_+ v_d$.

$$I(s+\mathfrak{c})(\xi) = |\det_{\Lambda}(v_j)| \prod_j \frac{-e^{\langle \xi, s\rangle}}{\langle \xi, v_j\rangle}$$

$$S(s+\mathfrak{c})(\xi) = \left(\sum_{x \in (s+\mathfrak{b})\cap\Lambda} e^{\langle \xi, x\rangle} \right) \prod_j \frac{1}{1-e^{\langle \xi, v_j\rangle}}$$

where $\mathfrak{b} = \sum_j [0,1[v_j$, *semi-closed cell.*

Polyhedron $\equiv$ sum of its supporting cones at vertices

Theorem *(Brion-Lawrence-Varchenko)*
$\mathfrak{p}$ convex polyhedron, $s + \mathfrak{c}_s$ supporting cone at vertex s.

$$S(\mathfrak{p}) = \sum_{s \in \text{ vertices}} S(s + \mathfrak{c}_s), \quad I(\mathfrak{p}) = \sum_s I(s + \mathfrak{c}_s)$$

Example:
Let Δ be a simplex. Let ℓ be a linear form which is regular w.r.t. Δ, i.e., $\langle \ell, \mathbf{s}_i \rangle \neq \langle \ell, \mathbf{s}_j \rangle$ for any pair $i \neq j$. Then

$$\int_\Delta e^\ell \mathfrak{d}m = d!\,\mathrm{vol}(\Delta, \mathfrak{d}m) \sum_{i=1}^{d+1} \frac{e^{\langle \ell, \mathbf{s}_i \rangle}}{\prod_{j \neq i} \langle \ell, \mathbf{s}_i - \mathbf{s}_j \rangle}. \qquad (1)$$

$$\int_\Delta \ell^M \mathfrak{d}m = d!\,\mathrm{vol}(\Delta, \mathfrak{d}m) \frac{M!}{(M+d)!} \left(\sum_{i=1}^{d+1} \frac{\langle \ell, \mathbf{s}_i \rangle^{M+d}}{\prod_{j \neq i} \langle \ell, \mathbf{s}_i - \mathbf{s}_j \rangle} \right). \qquad (2)$$

Summary

- Integration of arbitrary powers of linear forms can be done efficiently over simplices
- Not only over simplices! Also good over simple polytopes with polynomially many vertices, simplicial polytopes with polynomially many facets.
- Integration of polynomials of fixed degree is OK too.
- Integration of arbitrary powers of quadratic forms is already hard.
- Algorithms run nicely in practice. Have been useful to check results.

Thank you