Optimization (168)

Lecture 10-11-12

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LAST EPISODE...

- **Theorem** V. Klee and G. Minty (1972) showed that there are explicit linear programs with *n* constraints for which the largest-coefficient pivot rule can take $2^n - 1$ pivots to reach the optimal solution.
- But we know today there are many alternative methods to the SIMPLEX METHOD:
- Fourier-Motzkin Elimination (Goes back to Fourier, rediscovered by T. Motzkin in 1930's. Interesting in theory but much slower than Simplex)
- Relaxation Methods (invented by T. Motzkin and his students in the 1950's. Interesting in theory but much slower than Simplex).
- Kachiyan's Ellipsoid Method (invented in the late 1970's. First ever polynomial time algorithm for solving linear programs. SLOW!)
- Karmarkar's Interior Point Methods (invented in the late 1980's. Good theoretical and practical performance!! Competes with Simplex!).
- Many other algorithms exist, but they are mostly variations or improvements of those mentioned so far.

DUALITY IN LINEAR PROGRAMMING

Suppose you know nothing about simplex, but wish to estimate

$$
\begin{aligned}
\max \quad & 4x_1 + x_2 + 5x_3 + 3x_4 \\
\text{s.t.} \quad & x_1 - x_2 - x_3 + 3x_4 \leq 1 \\
& 5x_1 + x_2 + 3x_3 + 8x_4 \leq 55 \\
& -x_1 + 2x_2 + 3x_3 - 5x_4 \leq 3 \\
& x_1, x_2, x_3, x_4 \geq 0
\end{aligned}
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Clearly, if you GUESS a feasible solution we obtain a **lower bound** for the maximum.

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Suppose we multiply the second constraint by $\frac{5}{3}$ we obtain the inequality:

$$\frac{25}{3}x_1 + \frac{5}{3}x_2 + 5x_3 + \frac{40}{3}x_4 \leq \frac{275}{3}$$

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Note that the objective function has a nice relation with this new inequality!

$$4x_1 + x_2 + 5x_3 + 3x_4 \leq \frac{25}{3}x_1 + \frac{5}{3}x_2 + 5x_3 + \frac{40}{3}x_4 \leq \frac{275}{3}$$

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Thus the maximum value must be $z^* \leq \frac{275}{3}$. We can be more clever:
If we add the second and third constraints we have

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THIS method can be generalized!

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IDEA: We need to construct *linear combinations* of the constraints of the original LP.
We multiply the i-th constraint of the LP by $y_i \ge 0$ and add them up!! In our example:

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Regroup in x_j's

$$(y_1 + 5y_2 - y_3)x_1 + (-y_1 + y_2 + 2y_3)x_2 + (-y_1 + 3y_2 + 3y_3)x_3 + (3y_1 + 8y_2 - 5y_3)x_4$$
$$\leq y_1 + 55y_2 + 3y_3.$$

We wish to use the LEFT-hand side as an upper bound to the objective function!!!
$4x_1 + x_2 + 5x_3 + 3x_4$ this means

$$y_1 + 5y_2 - y_3 \geq 4$$
$$-y_1 + y_2 + 2y_3 \geq 1$$
$$-y_1 + 3y_2 + 3y_3 \geq 5$$
$$3y_1 + 8y_2 - 5y_3 \geq 3$$

If all these is true, then we must have that $4x_1 + x_2 + 5x_3 + 3x_4 \leq y_1 + 55y_2 + 3y_3$

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If all these is true, then we must have that $4x_1 + x_2 + 5x_3 + 3x_4 \leq y_1 + 55y_2 + 3y_3$
Since we want the BEST upper bound for our maximization problem we need to find the SMALLEST value of $y_1 + 55y_2 + 3y_3$

Minimize $y_1 + 55y_2 + 3y_3$

$$y_1 + 5y_2 - y_3 \geq 4$$
$$-y_1 + y_2 + 2y_3 \geq 1$$
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Economic Interpretation

Given a cache of raw materials and a factory for turning these raw materials into a variety of finished products, how many of each product type should we make so as to maximimze profit?

This is a resource allocation problem ($m = 2$, $n = 3$):

$$\begin{array}{lrcrcrcl} \text{maximize} & c_1x_1 & + & c_2x_2 & + & c_3x_3 & & \\ \text{subject to} & a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & \leq & b_1 \\ & a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & \leq & b_2 \\ & & & & & x_1,\ x_2,\ x_3 & \geq & 0, \end{array}$$

where

$$\begin{array}{rcl} c_j & = & \text{profit per unit of product } j \text{ produced} \\ b_i & = & \text{units of raw material } i \text{ on hand} \\ a_{ij} & = & \text{units raw material } i \text{ required to produce } 1 \text{ unit of prod } j. \end{array}$$

Economic Interpretation

If we produce one unit less of product j, then we free up:

- a_{1j} units of raw material 1 and
- a_{2j} units of raw material 2.

Selling these unused raw materials for y_1 and y_2 dollars/unit yields $a_{1j}y_1 + a_{2j}y_2$ dollars.

Only interested if this exceeds lost profit on each product j:

$$a_{1j}y_1 + a_{2j}y_2 \geq c_j, \qquad j = 1, 2, 3.$$

Consider a buyer offering to purchase our entire inventory.

Subject to above constraints, buyer wants to minimize cost:

$$\begin{array}{lrcrcl} \text{minimize} & b_1y_1 & + & b_2y_2 & & \\ \text{subject to} & a_{11}y_1 & + & a_{21}y_2 & \geq & c_1 \\ & a_{12}y_1 & + & a_{22}y_2 & \geq & c_2 \\ & a_{13}y_1 & + & a_{23}y_2 & \geq & c_3 \\ & & & y_1,\ y_2 & \geq & 0 \end{array}.$$

This remarkable observation gives you a construction: For each LP (PRIMAL)

$$\begin{aligned} \text{maximize} \quad & \sum_{j=1}^{n} c_j x_j \\ \text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j \le b_i \qquad i = 1, 2, \ldots, m \\ & x_j \ge 0 \qquad j = 1, 2, \ldots, n, \end{aligned}$$

We have a DUAL LINEAR PROGRAM

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{m} b_i y_i \\ \text{subject to} \quad & \sum_{i=1}^{m} y_i a_{ij} \ge c_j \qquad j = 1, 2, \ldots, n \\ & y_i \ge 0 \qquad i = 1, 2, \ldots, m. \end{aligned}$$

Lemma: If we rewrite the dual in standard maximization form it equals the negative transpose of the primal problem. Moreover the dual of the dual is the original LP.

Rewriting the dual in the maximization form

$$\begin{aligned} -\text{maximize} \quad & \sum_{i=1}^{m} -b_i y_i \\ \text{subject to} \quad & \sum_{i=1}^{m} -a_{ij} y_i \leq -c_j \quad j = 1, \ldots, n \\ & y_i \geq 0 \qquad i = 1, \ldots, m. \end{aligned}$$

Dual is "negative transpose" of primal.

Thus we get the original back if we dualize.

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There is a nice way to write the primal and the dual in matrix form:

$$\begin{aligned} & \max \ c^T x \\ & Ax \le b \\ & x \ge 0 \end{aligned} \qquad \qquad \begin{aligned} & \min b^T y \\ & A^T y \ge c \\ & y \ge 0 \end{aligned}$$

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$\max\ c^T x$	$\min b^T y$
$Ax \leq b$	$A^T y \geq c$
$x \geq 0$	$y \geq 0$

Theorem: (Weak Duality theorem) if P is the primal linear program in standard maximization form and D is the dual in the minimization form. For each pair of feasible solutions, x, y we have

$$\sum_j c_j x_j = c^T x \leq b^T y = \sum_i b_i y_i$$

QUESTION: Is there a gap in between the largest **primal** solution and the smallest **dual** solution??

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Theorem (von Neumann Strong Duality Theorem) A linear program has an optimal solution if and only if its dual has an optimal solution too and $c^T x = b^T y$. Thus for a primal and its dual we have only 4 allowable situations:

	Finite optimum	**Unbounded**	**Infeasible**
Finite optimum	Possible	Impossible	Impossible
Unbounded	Impossible	Impossible	Possible
Infeasible	Impossible	Possible	Possible

Let us see why we have all the options...

What is the dual of this LP?

$$\textbf{Maximize } 2x_1 - x_2$$
$$x_1 - x_2 \leq 1$$
$$-x_1 + x_2 \leq -2$$
$$x_1, x_2 \geq 0$$

GOOD NEWS: can use the duality theorem to check whether we really found an optimal solution!! True solutions (x^*, y^*) must satisfy $c^T x^* = b^T y^*$

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GOOD NEWS: The simplex method is not just solving the primal problem! **It simultaneously solves the dual problem too**!

The optimal solution of the dual problem can be read off the objective function row of the final dictionary and vice versa.

original variables of the primal	**MATCHED with**	dual slack variables
slack variables of the primal	**MATCHED with**	dual variables

Take a (primal) LP and its dual

$$\begin{aligned} \textbf{Maximize } & -3x_1 + 2x_2 + x_3 \\ & -x_2 + 2x_3 \le 0 \\ & -3x_1 + 4x_2 - x_3 \le 3 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

$$\begin{aligned} \textbf{Minimize } & 3y_2 \\ & -3y_2 \ge -3 \\ & -y_1 + 4y_2 \ge 2 \\ & 2y_1 - y_2 \ge 1 \\ & y_1, y_2 \ge 0 \end{aligned}$$

But if we rewrite both as dictionaries with the maximization forms we have

$$\begin{aligned} \mathbf{z} &= -3x_1 + 2x_2 + x_3 \\ w_1 &= x_2 - 2x_3 \\ w_2 &= 3 + 3x_1 - 4x_2 - x_3 \\ & x_1, x_2, x_3, w_1, w_2 \ge 0 \end{aligned}$$

$$\begin{aligned} z &= -3y_2 \\ z_1 &= 3 - 3y_2 \\ z_2 &= -2 - y_1 + 4y_2 \\ z_3 &= -1 + 2y_1 - y_2 \\ & y_1, y_2, z_1, z_2, z_3 \ge 0 \end{aligned}$$

KEY POINT: As we pivot the dictionaries will be negative transpose of each other!

obj	=	0	+	-3	x1	+	2	x2	+	1	x3		
w1	=	0	-	0	x1	-	-1	x2	-	2	x3		
w2	=	3	-	-3	x1	-	4	x2	-	1	x3		

Its Dual:

obj	=	0	+	0	y1	+	-3	y2
z1	=	3	-	0	y1	-	3	y2
z2	=	-2	-	1	y1	-	-4	y2
z3	=	-1	-	-2	y1	-	-1	y2

Notes:

- Dual is negative transpose of primal.
- Primal is feasible, dual is not.

Use primal to choose pivot: x_2 enters, w_2 leaves.

Make analogous pivot in dual: z_2 leaves, y_2 enters.

After First Pivot:

Primal (feasible):

obj	=	3/2	+	−3/2	x1	+	−1/2	w2	+	1/2	x3
w1	=	3/4	−	−3/4	x1	−	1/4	w2	−	9/4	x3
x2	=	3/4	−	−3/4	x1	−	1/4	w2	−	1/4	x3

Dual (still not feasible):

obj	=	−3/2	+	−3/4	y1	+	−3/4	z2
z1	=	3/2	−	3/4	y1	−	3/4	z2
y2	=	1/2	−	−1/4	y1	−	−1/4	z2
z3	=	−1/2	−	−9/4	y1	−	−1/4	z2

Note: *negative transpose property intact.*

Again, use primal to pick pivot: x_3 enters, w_1 leaves.

Make analogous pivot in dual: z_3 leaves, y_1 enters.

Primal:

- Is *optimal*.

obj	=	5/3	+	-4/3	x1	+	-5/9	w2	+	-2/9	w1
x3	=	1/3	−	-1/3	x1	−	1/9	w2	−	4/9	w1
x2	=	2/3	−	-2/3	x1	−	2/9	w2	−	-1/9	w1

Dual:

- Negative transpose property remains intact.
- Is *optimal*.

obj	=	-5/3	+	-1/3	z3	+	-2/3	z2
z1	=	4/3	−	1/3	z3	−	2/3	z2
y2	=	5/9	−	-1/9	z3	−	-2/9	z2
y1	=	2/9	−	-4/9	z3	−	1/9	z2

Conclusion

Simplex method applied to primal problem (two phases, if necessary), solves both the primal and the dual.