h-VECTORS OF SMALL MATROID COMPLEXES

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ABSTRACT. Stanley conjectured in 1977 that the h-vector of a matroid simplicial complex is a pure O-sequence. We give simple constructive proofs that the conjecture is true for matroids of rank less than or equal to 3, and co-rank 2. We also used computers to verify Stanley's conjecture holds for all matroids on at most nine elements.

1. INTRODUCTION AND BACKGROUND

Before stating the key goal of our investigations and stating our results we will briefly review some relevant background material on matroids and simplicial complexes. For further information, we refer the reader to the books of Oxley [?], White [?], and Stanley [?].

Recall that a matroid $M = (E(M), \mathcal{I}(M))$ consists of a ground set E(M) and a family of subsets $\mathcal{I}(M) \subseteq 2^E$ called *independent sets* such that

- (I1): $\emptyset \in \mathcal{I}(M);$
- (I2): If $A \in \mathcal{I}(M)$ and $A' \subset A$, then $A' \in \mathcal{I}(M)$; and

(I3): If $A, A' \in \mathcal{I}(M)$ with |A| < |A'|, then there is some $e \in A' \setminus A$ such that $A \cup e \in \mathcal{I}(M)$.

Equivalently, the independent sets of a matroid M on ground set E(M) form a simplicial complex with the property that the restriction $\mathcal{I}(M)|_{E'}$ is pure for any subset $E' \subseteq E(M)$. A *basis* of M is a maximal independent set under inclusion, and the *rank* of M is the cardinality of a basis. Given a matroid M on ground set E(M) with bases $\mathcal{B}(M)$, we define its *dual matroid*, M^* , to be the matroid on E(M) whose bases are $\mathcal{B}(M^*) = \{E \setminus B : B \in \mathcal{B}(M)\}$.

If M is a matroid of rank d, the *f*-vector of M is $f(M) := (f_{-1}(M), f_0(M), \dots, f_{d-1}(M))$, whose entries are $f_{i-1}(M) := |\{A \in \mathcal{I}(M) : |A| = i\}|$. Oftentimes, it is more convenient to study the *h*-vector $h(M) := (h_0(M), \dots, h_d(M))$ whose entries are defined by the relation

$$\sum_{j=0}^{d} h_j(M)\lambda^j = \sum_{i=0}^{d} f_{i-1}(M)\lambda^i (1-\lambda)^{d-i}.$$

See [?] for more on *h*-vectors and the combinatorics of simplicial complexes.

It should not be expected that the *h*-numbers of a general simplicial complex are nonnegative; however, the *h*-numbers of a matroid M may be interpreted combinatorially in terms of certain invariants of M. Fix a total ordering $\{v_1 < v_2 < \ldots < v_n\}$ on E(M). Given a basis $B \in \mathcal{I}(M)$, an element $v_j \in B$ is *internally passive in* B if there is some $v_i \in E(M) \setminus B$

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such that $v_i < v_j$ and $(B \setminus v_j) \cup v_i$ is a basis of M. Dually, $v_j \in E(M) \setminus B$ is externally passive in B if there is an element $v_i \in B$ such that $v_i < v_j$ and $(B \setminus v_i) \cup v_j$ is a basis. (Alternatively, v_j is externally passive in B if it is internally passive in $E(M) \setminus B$ in M^* .) It is well known ([?, Equation (7.12)]) that

(1.1)
$$\sum_{j=0}^{d} h_j(M) \lambda^j = \sum_{B \in \mathcal{B}(M)} \lambda^{ip(B)}$$

where ip(B) counts the number of internally passive elements in B. This proves that the h-numbers of a matroid complex are nonnegative. Alternatively,

(1.2)
$$\sum_{j=0}^{d} h_j(M)\lambda^j = \sum_{B \in \mathcal{B}(M^*)} \lambda^{ep(B)},$$

where ep(B) counts the number of externally passive elements in B. Since the f-numbers (and hence the h-numbers) of a matroid depend only on its independent sets, Equations (1.1) and (1.2) hold for *any* ordering of the ground set of M. It is worth remarking that the h-polynomial above is actually an specialization of the well-known *Tutte polynomial* of the corresponding matroid (see [?]).

A longstanding conjecture of Stanley [?] suggests that matroid h-vectors are highly structured.

Conjecture 1.1. For any matroid M, h(M) is a pure O-sequence.

An order ideal \mathcal{O} is a family of monomials (say of degree at most r) with the property that if $\mu \in \mathcal{O}$ and $\nu | \mu$, then $\nu \in \mathcal{O}$. Let \mathcal{O}_i denote the collection of monomials in \mathcal{O} of degree i. Let $F_i(\mathcal{O}) := |\mathcal{O}_i|$ and $F(\mathcal{O}) = (F_0(\mathcal{O}), F_1(\mathcal{O}, \ldots, F_r(\mathcal{O})))$. We say that \mathcal{O} is pure if all of its maximal monomials (under divisibility) have the same degree. A vector $h = (h_0, \ldots, h_d)$ is a pure O-sequence if there is a pure order ideal \mathcal{O} such that $h = F(\mathcal{O})$.

Conjecture 1.1 is known to hold for several families of matroid complexes, such as paving matroids [?], cographic matroids [?], cotransversal matroids [?], lattice path matroids [?], and matroids of rank at most three [?, ?]. The purpose of this paper is to present a proof of Stanley's conjecture for all matroids of with at most nine elements, all matroids of corank two and all matroids of rank at most three. While Stanley's conjecture is known to hold for matroids of rank two [?] and rank three [?], we use the geometry of the independence complexes of matroids of small rank to provide much simpler shorter proofs for these cases. Our results show that any counterexample to Stanley's conjecture must have at least 10 elements and rank at least four.

This article will use several ideas from the theory of multicomplexes and monomial ideals. Although a general classification of matroid h-vectors or pure O-sequences seems to be an incredibly difficult problem, some properties are known and will be used in the proofs below: **Theorem 1.2.** [?, ?, ?] Let $\mathbf{h} = (h_0, h_1, \dots, h_d)$ be a matroid h-vector or a pure O-sequence with $h_d \neq 0$. Then

(1) $h_0 \leq h_1 \leq \cdots \leq h_{\lfloor \frac{d}{2} \rfloor},$ (2) $h_i \leq h_{d-i} \text{ for all } 0 \leq i \leq \lfloor \frac{d}{2} \rfloor, \text{ and}$ (3) for all $0 \leq s \leq d$ and $\alpha \geq 1$, we have

(1.3)
$$\sum_{i=0}^{s} (-\alpha)^{s-i} h_i \ge 0.$$

Inequality (1.3) is known as the Brown-Colbourn inequality [?, Theorem 3.1].

2. Rank-2 Matroids

Let M be a loopless matroid of rank 2. The independence complex of M is a complete multipartite graph whose partite sets E_1, \ldots, E_t are the parallelism classes of M. Let $s_i := |E_i|$. Choose one representative $e_i \in E_i$ from each parallelism class of M so that the simplification of M is a complete graph on $\{e_1, \ldots, e_t\}$, and let $\widetilde{E}_i = E_i \setminus e_i$. Clearly

$$f_0(M) = \sum_{i=1}^t (s_i - 1) + t$$

and $f_1(M) = \sum_{1 \le i < j \le t} (s_i - 1)(s_j - 1) + (t - 1) \sum_{i=1}^t (s_i - 1) + {t \choose 2},$

and hence,

$$h_1(M) = \sum_{i=1}^t (s_i - 1) + (t - 2)$$

and $h_2(M) = \sum_{1 \le i < j \le t} (s_i - 1)(s_j - 1) + (t - 2) \sum_{i=1}^t (s_i - 1) + {t - 1 \choose 2}.$

Consider the pure O-sequence O with

$$\mathcal{O}_1 = \{x_1, \dots, x_{t-2}\} \cup \{x_e : e \in E_i, 1 \le i \le t\}$$

$$\mathcal{O}_2 = \{x_e x_{e'} : e \in \widetilde{E}_i, e' \in \widetilde{E}_j, 1 \le i < j \le t\}$$

$$\cup \{x_i x_e : e \in \widetilde{E}_j, 1 \le i \le t-2, 1 \le j \le t\}$$

$$\cup \{\text{degree 2 monomials in } x_1, \dots, x_{t-2}\}.$$

We see that $h(M) = F(\mathcal{O})$, which proves the following theorem.

Theorem 2.1. Let M be a matroid of rank 2. Then h(M) is a pure O-sequence.

3. CORANK-2 MATROIDS

In this section, we aim to prove Conjecture 1.1 for corank-2 matroids.

Theorem 3.1. Let M be a matroid of rank 2. Then $h(M^*)$ is a pure O-sequence.

Proof. As before, let E_1, \ldots, E_t denote the parallelism classes of the independence complex of M. Order the ground set E(M) so that $v_i < v_j$ for all $v_i \in E_k$ and $v_j \in E_\ell$ and all $1 \le k < \ell \le t$.

For each basis $B = \{v_i, v_j\}$ of M with $v_i \in E_k, v_j \in E_\ell$, and $k < \ell$, let

$$a_1(B) := \#\{i' > i : v_{i'} \in E_k \cup \ldots \cup E_{\ell-1}\}$$

and $a_2(B) := \#\{j' > j : v_{j'} \in E_\ell \cup \ldots \cup E_t\},$

and set $\mu_B := x_1^{a_1(B)} x_2^{a_2(B)}$. We claim that $\mathcal{O} := \{\mu_B : B \in \mathcal{B}(M)\}$ is a pure order ideal and that $F(\mathcal{O}) = h(M^*)$.

FIGURE 1. The bases $B = \{v_i, v_j\}$ (left) and $\widetilde{B} = \{u_1, u_\ell\}$ (right) with their externally passive elements shaded.

We see that $a_1(B)$ counts the number of elements $v \in E(M) \setminus B$ that are externally passive in B for which $v_i < v < v_j$ (shown in Figure 1 (left) shaded with lines of slope 1); and $a_2(B)$ counts the number of elements $v \in E(M) \setminus B$ that are externally passive in B for which $v_j < v \leq v_n$ (shown in Figure 1 (left) shaded with lines of slope -1). Since $a_1(B) + a_2(B)$ counts the number of externally passive elements in B, Equation (1.2) shows that $h(M^*) = F(\mathcal{O})$.

To see that \mathcal{O} is an order ideal, we need only show that if $\nu|\mu_B$ and $\deg(\nu) = \deg(\mu_B) - 1$, then $\nu \in \mathcal{O}$. Let $B = \{v_i, v_j\}$ as before. If $a_1(B) > 0$, consider $B' = \{v_{i+1}, v_j\} \in \mathcal{I}(M)$. Clearly $a_1(B') = a_1(B) - 1$ and $a_2(B') = a_2(B)$ so that $\mu_{B'} \in \mathcal{O}$ and $\deg(\mu_{B'}) = \deg(\mu_B) - 1$. If $a_2(B) > 0$, we must consider two possible cases. If $v_{j+1} \in \mathcal{E}_\ell$, then consider $B'' = \{v_i, v_{j+1}\} \in \mathcal{I}(M)$. Again $a_1(B'') = a_1(B)$ and $a_2(B'') = a_2(B) - 1$ so that $\mu_{B''} = x_1^{a_1(B)} x_2^{a_2(B)} - 1$. On the other hand, if $v_{j+1} \in \mathcal{E}_{\ell+1}$, then $v_{j-a_1(B)} \in \mathcal{E}_{k'}$ for some $k' \leq \ell$, and so $B''' = \{v_{j-a_1(B)}, v_{j+1}\} \in \mathcal{I}(M)$. Again we see that $\mu_{B'''} = x_1^{a_1(B)} x_2^{a_2(B)-1}$. This establishes that \mathcal{O} is an order ideal.

Finally, we must show that \mathcal{O} is pure. For each $1 \leq i \leq t$, let u_i denote the smallest element of E_i . For any basis $B = \{v_i, v_j\}$ as above, let $\widetilde{B} = \{u_1, u_\ell\}$. As Figure 1 (right) indicates, $a_1(B) \leq a_1(\widetilde{B})$ and $a_2(B) \leq a_2(\widetilde{B})$, and hence $\mu_B | \mu_{\widetilde{B}}$. Moreover, we see that $\deg(\mu_{\widetilde{B}}) = |E_1| + \cdots + |E_t| - 2$, and hence each such monomial $\mu_{\widetilde{B}}$ has the same degree. \Box

The techniques used to prove Theorem 3.1 can be easily extended to prove that $h(M^*)$ is a pure \mathcal{O} -sequence for any matroid M whose simplification is a uniform matroid. The

reader may easily check, however, that these techniques may not be used to prove Stanley's conjecture when M is the Fano matroid.

4. Rank-3 Matroids

Our goal for this section is to give a simple geometric-combinatorial proof of the following theorem which was first proved in [?] in the case that d = 3 using the language of commutative algebra.

Theorem 4.1. Let M be a loopless matroid of rank $d \ge 3$. The vector $(1, h_1(M), h_2(M), h_3(M))$ is a pure O-sequence.

Lemma 4.2. For any positive integers s_1, \ldots, s_t , the vector $\mathbf{h} = (1, h_1, h_2, h_3)$ with

$$h_{1} = \sum_{i=1}^{t} (s_{i} - 1) + (t - d),$$

$$h_{2} = \sum_{1 \le i < j \le t} (s_{i} - 1)(s_{j} - 1) + (t - d) \sum_{i=1}^{t} (s_{i} - 1) + {\binom{t - d + 1}{2}},$$

$$h_{3} = \sum_{1 \le i < j < k \le t} (s_{i} - 1)(s_{j} - 1)(s_{k} - 1) + (t - d) \sum_{1 \le i < j \le t} (s_{i} - 1)(s_{j} - 1),$$

$$+ {\binom{t - d + 1}{2}} \sum_{i=1}^{t} (s_{i} - 1) + {\binom{t - d + 2}{3}},$$

is a pure O-sequence.

Proof. Consider disjoint sets $\widetilde{E}_1, \ldots, \widetilde{E}_t$ with $|\widetilde{E}_i| = s_i - 1$ for all *i*. We will construct a pure order ideal \mathcal{O} with $F(\mathcal{O}) = \mathbf{h}$ whose degree-one terms are

$$\mathcal{O}_1 = \{x_1, \dots, x_{t-d}\} \cup \{x_e : e \in \widetilde{E}_i\}_{i=1}^t.$$

We explicitly construct such an order ideal by setting

$$\mathcal{O}_2 = \{ x_e x_{e'} : e \in \widetilde{E}_i, e' \in \widetilde{E}_j, 1 \le i < j \le t \}$$
$$\cup \{ \text{all degree 2 monomials in } x_1, \dots, x_{t-d} \}$$

and

$$\mathcal{O}_3 = \{x_e x_{e'} x_{e''} : e \in \widetilde{E}_i, e' \in \widetilde{E}_j, e'' \in \widetilde{E}_k, 1 \le i < j < k \le t\}$$
$$\cup \{x_k x_e x_{e'} : e \in \widetilde{E}_i, e' \in \widetilde{E}_j, 1 \le k \le t - d, 1 \le i < j \le t\}$$
$$\cup \{x_j x_k x_e : e \in \widetilde{E}_i, 1 \le j < k \le t - d, 1 \le i \le t\}$$
$$\cup \{x_j^2 x_e : e \in \widetilde{E}_i, 1 \le i \le t, 1 \le j \le t - d\}$$
$$\cup \{\text{all degree 3 monomials in } x_1, \dots, x_{t-d}\}$$

Lemma 4.3. For any positive integers s_1, \ldots, s_t , the vector $\mathbf{h}' = (1, h_1, h_2, h_3)$ with

$$h_{1} = \sum_{i=1}^{t} (s_{i} - 1) + (t - d),$$

$$h_{2} = \sum_{1 \le i < j \le t} (s_{i} - 1)(s_{j} - 1) + (t - d) \sum_{i=1}^{t} (s_{i} - 1) + {t - d - 1 \choose 2},$$

$$h_{3} = \sum_{1 \le i < j \le t} (s_{i} - 1)(s_{j} - 1) + (t - d - 1) \sum_{i=1}^{t} (s_{i} - 1) + {t - d \choose 2} + 1,$$

is a pure O-sequence.

Proof. As in the proof of Lemma 4.2, let $\widetilde{E}_1, \ldots, \widetilde{E}_t$ be disjoint sets with $|\widetilde{E}_i| = s_i - 1$. Recall the order ideal \mathcal{O} constructed in the proof of Lemma 4.2 We will construct a pure order ideal $\widetilde{\mathcal{O}}$ with $F(\widetilde{\mathcal{O}}) = \mathbf{h}'$ such that $\widetilde{\mathcal{O}}_1 = \mathcal{O}_1, \widetilde{\mathcal{O}}_2 = \mathcal{O}_2$, and $\widetilde{\mathcal{O}}_3 \subseteq \mathcal{O}_3$. Indeed, we set

$$\widetilde{\mathcal{O}}_{3} = \{x_{1}x_{e}x'_{e} : e \in \widetilde{E}_{i}, e' \in \widetilde{E}_{j}, 1 \leq i < j \leq t\} \\
\cup \{x_{j}^{2}x_{e} : e \in \widetilde{E}_{i}, 1 \leq i \leq t, 2 \leq j \leq t - d\} \\
\cup \{x_{i}^{2}x_{j} : 1 \leq i < j \leq t - d\} \cup \{\mu_{0}\},$$

where μ_0 is a monomial defined as follows. If $\widetilde{E}_1 \cup \cdots \cup \widetilde{E}_t$ is nonempty, choose some $e_0 \in \widetilde{E}_1 \cup \cdots \cup \widetilde{E}_t$ and set $\mu_0 = x_1^2 x_{e_0}$; otherwise, set $\mu_0 = x_1^3$.

Proof: (Theorem 4.1)

Let $E_1, \ldots, E_t \subseteq E(M)$ denote the parallel classes of M, and set $s_i := |E_i|$. Choose one representative e_i from each class E_i , and let $W = \{e_1, \ldots, e_t\}$. Observe that $\Delta := M|_W$ is a simple matroid of rank d. Let $\widetilde{E}_i = E_i \setminus \{e_i\}$, and notice that for any choices of

$$\begin{split} \widetilde{e}_{ij} \in E_{ij}, \{\widetilde{e}_{i1}, \dots, \widetilde{e}_{ik}\} \in \mathcal{I}(M) \text{ if and only if } \{e_{i1}, \dots, e_{ik}\} \in \Delta. \text{ Thus} \\ f_0(M) &= \sum_{i=1}^t s_i \text{ and hence} \\ h_1(M) &= \sum_{i=1}^t (s_i - 1) + (t - d); \\ f_1(M) &= \sum_{1 \leq i < j \leq t} s_i s_j \\ &= \sum_{1 \leq i < j \leq t} (s_i - 1)(s_j - 1) + (t - 1) \sum_{i=1}^t (s_i - 1) + \binom{t}{2} \text{ and hence} \\ h_2(M) &= \sum_{1 \leq i < j \leq t} (s_i - 1)(s_j - 1) + (t - d) \sum_{i=1}^t (s_i - 1) + \binom{t - d + 1}{2}; \\ f_2(M) &\leq \sum_{1 \leq i < j < k \leq t} s_i s_j s_k \text{ and hence} \\ h_3(M) &\leq \sum_{1 \leq i < j < k \leq t} (s_i - 1)(s_j - 1)(s_k - 1) + (t - d) \sum_{1 \leq i < j \leq t} (s_i - 1)(s_j - 1) \\ &+ \binom{t - d + 1}{2} \sum_{i=1}^t (s_i - 1) + \binom{t - d + 2}{3}. \end{split}$$

On the other hand, by the Brown-Colbourn inequality (1.3), we have

$$h_3(M) \ge h_2(M) - h_1(M) + h_0(M)$$

= $\sum_{1 \le i < j \le t} (s_i - 1)(s_j - 1) + (t - d - 1) \sum_{i=1}^t (s_i - 1) + {t - d \choose 2} + 1.$

We construct a pure order ideal \mathcal{O}' with $F(\mathcal{O}') = h(M)$ as follows. Following the notation used in Lemmas 4.2 and 4.3, we set $\mathcal{O}'_1 = \mathcal{O}_1$; $\mathcal{O}'_2 = \mathcal{O}_2$, and choose $\widetilde{\mathcal{O}}_3 \subseteq \mathcal{O}'_3 \subseteq \mathcal{O}_3$ with $|\mathcal{O}'_3| = h_3(M)$.

5. Matroids with at most 9 elements

This part of our paper is mostly experimental and is crucially based on the data provided to us by Dillon Mayhew and Gordon Royle. They constructed a computer database of matroids on at most nine elements [?]. We use this data to generate the possible h-vectors of matroid complexes, then we proceeded to search for each of them a corresponding pure *O*-sequence that match those numbers. The key idea is simple, for given rank and given number of elements we know the number of monomials of top degree that must be present. So by sampling the space of monomials of given degree we can generate thousands of pure *O*-sequences that are candidates to be *h*-vectors of matroid complexes. Of course we often generated pure *O*-sequences that did not match any matroid, for example (1, 5, 15, 27, 22) and (1, 5, 15, 27, 35) are both valid pure *O*-sequences we generated, but we also know the the only h-vectors of matroid complexes with initial value (1, 5, 15, 27, *) are

(1	5	15	27	0)	(1	5	15	27	19)	(1	5	15	27	20)	(1	5	15	27	21)	(1	5	15	27	24)
(1	5	15	27	25)	(1	5	15	27	26)	(1	5	15	27	27)	(1	5	15	27	30)	(1	5	15	27	36)

The code we use was mostly a Perl code available at XXXXXX.

In addition to generating large numbers of O-sequences, we made use of previous work, such as [?], to eliminate certain h-vectors as already verified. For instance, paving matroids are easily identified, and therefore we have not included monomials for these matroids.

Further, we only consider co-loopless matroids. If a matroid has j co-loops, it has an h-vector of the form: $(h_0, h_1, \ldots, h_{r-j}, 0, \ldots, 0)$. The truncated sequence: $(h_0, h_1, \ldots, h_{r-j})$ is the h-vector of the same matroid but with all co-loops contracted. This object is a matroid of rank r - j on n - j elements, and therefore appears in our list of matroids and has already been shown to satisfy Stanley's conjecture.

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