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GRÖBNER BASES AND TRIANGULATIONS OF THE SECOND HYPERSIMPLEX*

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The algebraic technique of Gröbner bases is applied to study triangulations of the second hypersimplex $\Delta(2,n)$. We present a quadratic Gröbner basis for the associated toric ideal $I(K_n)$. The simplices in the resulting triangulation of $\Delta(2,n)$ have unit volume, and they are indexed by subgraphs which are linear thrackles [28] with respect to a circular embedding of K_n. For $n \geq 6$ the number of distinct initial ideals of $I(K_n)$ exceeds the number of regular triangulations of $\Delta(2,n)$; more precisely, the secondary polytope of $\Delta(2,n)$ equals the state polytope of $I(K_n)$ for $n \leq 5$ but not for $n \geq 6$. We also construct a non-regular triangulation of $\Delta(2,n)$ for $n \geq 9$. We determine an explicit universal Gröbner basis of $I(K_n)$ for $n \leq 8$. Potential applications in combinatorial optimization and random generation of graphs are indicated.

1. Introduction

Let $\mathcal{A}_n = \{e_i + e_j : 1 \leq i < j \leq n\}$ be the set of column vectors of the vertex-edge incidence matrix of the complete graph K_n. The convex hull of $\mathcal{A}_n$ is called the *second hypersimplex* of order n and is denoted $\Delta(2,n)$. The second hypersimplex is an $(n-1)$-dimensional polytope in $\mathbb{R}^n$, with $\binom{n}{2}$ vertices. In this paper we investigate triangulations and Gröbner bases associated with $\Delta(2,n)$. Its toric ideal $I(K_n)$ is the kernel of the ring map $\Phi : k[\ y[i,j] : 1 \leq i < j \leq n] \to k[t_1, \ldots, t_n]$ induced by $y[i,j] \mapsto t_i t_j$, where k is any field. The variables $y[i,j]$ are indexed by the edges in K_n. This family of ideals has been studied in [18] and [27].

With any point configuration $\mathcal{B}$ of lattice points we can associate two special polytopes. The *secondary polytope* $\sum(\mathcal{B})$ is a convex polytope whose faces are in bijection with the regular polyhedral subdivisions of $\mathcal{B}$. We refer the reader to [3], [4], [10] and [15] for details on regular triangulations, secondary polytopes and non-regular triangulations. The *state polytope* of a homogeneous ideal I is a convex polytope whose vertices are in bijection with the distinct initial ideals of I (see [1]). The relations among triangulations, state polytopes and secondary polytopes of a point configuration and its toric ideal can be found in [14], [21] and [22]. We refer to [13] for graph-theoretical concepts used in this paper and to [6] for the theory of Gröbner bases.

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This paper is organized as follows. In Section 2 we describe a quadratic, square-free Gröbner basis for $I(K_n)$ and the associated triangulation of $\Delta(2,n)$ into unit simplices. In Section 3, we compare the state polytope of $I(K_n)$ with the secondary polytope of $\mathcal{A}_n$. A non-regular triangulation of $\Delta(2,n)$ for $n \geq 9$ is presented in Section 4. Section 5 describes a universal Gröbner basis of $I(K_n)$ for $n \leq 8$. We conclude with some applications of the above theory to the generation of random graphs and to the perfect f-matching problem.

2. A quadratic Gröbner basis and its triangulation

We identify the vertices of K_n with the vertices of a regular n-gon labeled clockwise from 1 to n. The edges are closed line segments joining any two vertices. We define the *weight* of the variable $y[i,j]$ as the number of edges of K_n which do not meet the edge (i,j). (We say that two edges meet if they either share an endpoint or a point in their relative interiors.) In what follows $\succ$ denotes any monomial order that refines the partial order specified by these weights. Between any two vertices of K_n there are two paths that use only edges of the n-gon. We define the *circular distance* between two vertices to be the length of the shorter path. For example, vertices 1 and n are at a circular distance 1 inside K_n and the pair $\{1,6\}$ is at a distance 3 inside K_8. Given any pair of disjoint, non-intersecting edges $(i,j),(k,l)$ of K_n, one of the pairs $(i,k),(j,l)$ or $(i,l),(j,k)$ meets in a point. With the disjoint edges $(i,j),(k,l)$, we associate the binomial $y[i,j]y[k,l]-y[i,l]y[j,k]$ where $(i,l),(j,k)$ is the intersecting pair. We denote by $\mathcal{C}$ the set of all binomials obtained in this fashion and by $in_{\succ}(\mathcal{C})$ the finite set of their initial monomials with respect to $\succ$.

Theorem 2.1. *The set $\mathcal{C}$ is the reduced Gröbner basis of the ideal $I(K_n)$ with respect to the monomial order $\succ$.*

Proof. For each binomial $y[i,j]y[k,l] - y[i,l]y[j,k]$ in $\mathcal{C}$, the initial term with respect to $\succ$ corresponds to the disjoint edges. This follows from the convex embedding of K_n prescribed above and the definition of the weights. We recall from [27] that the binomials of $I(K_n)$ are in bijection with even closed walks in K_n. Throughout this paper we assume that all walks are closed. With an even walk $\Gamma = (i_1, i_2, \ldots, i_{2k-1}, i_{2k}, i_1)$ we associate the binomial $b_\Gamma = \prod_{l=1}^{k} y[i_{2l-1}, i_{2l}] - \prod_{l=1}^{k} y[i_{2l}, i_{2l+1}]$. Similarly one can recover an even walk from its binomial. The binomials associated with even closed walks in K_n form a universal Gröbner basis for the ideal $I(K_n)$ (see [21], [27]). Therefore in order to prove that $\mathcal{C}$ is a Gröbner basis of $I(K_n)$ with respect to $\succ$, it is enough to prove that the initial monomial of a binomial b_Γ with respect to $\succ$ is always divisible by some monomial $y[i,j]y[k,l]$ where $(i,j),(k,l)$ is a pair of disjoint edges.

Suppose on the contrary there exists a binomial b_Γ that contradicts our assertion. This implies that each pair of edges appearing in the initial monomial of b_Γ intersects. We will assume that b_Γ is a minimal counterexample in the sense that n is minimal and that b_Γ has minimal weight. The *weight of a binomial* is the sum of weights of its two terms. The walk Γ is spanning in K_n by minimality of n. All edges in the walk are visited with a unique parity since otherwise one can factor out the variable associated with an odd-even edge, which contradicts minimality of weight. If $b_\Gamma = y^\alpha - y^\beta$ and $in_\succ(b_\Gamma) = y^\alpha$, then we can assume that each pair of edges appearing in the trailing monomial y^β intersects. Otherwise if $(i,j),(k,l)$ is a non-intersecting pair of edges then we can reduce y^β modulo $\mathscr{C}$ to obtain a counterexample of smaller weight.

Let (s,t) be an edge of the walk Γ such that the circular distance between s and t is smallest possible. The edge (s,t) separates the vertices of K_n except s and t into two disjoint sets P and Q where $\|P\| \geq \|Q\|$. Let us start Γ at $(s,t) = (i_1,i_2)$. The walk is then a sequence of vertices and edges $\Gamma = (i_1,(i_1,i_2),i_2,(i_2,i_3),\ldots,(i_{2k-1},i_{2k}),i_{2k},(i_{2k},i_1))$. Each pair of odd (even) edges intersect. The odd edges are of type (i_{2r-1},i_{2r}) and the even edges of type (i_{2r},i_{2r+1}). Since the circular distance of i_1,i_2 is minimal, the vertex i_3 cannot be in Q. Otherwise the edge (i_2,i_3) would have smaller circular distance. We claim that if P contains an odd vertex i_{2r-1}, then it also contains the subsequent odd vertices $i_{2r+1},i_{2r+3},\ldots,i_{2k-1}$. The edge (i_1,i_2) is the common boundary of the two regions P and Q. Any odd edge intersects it (at least by having an end in $\{i_1,i_2\}$) and thus i_{2r} is in $Q \cup \{i_1,i_2\}$. Since any even edge must intersect (i_2,i_3), the vertex i_{2r+1} lies in $P \cup \{i_2\}$. To complete the proof of the claim we show that $i_{2r+1} \neq i_2$. The equality $i_{2r+1} = i_2$ would imply either $i_{2r} = i_1$ or $i_{2r} \in Q$. If $i_{2r} = i_1$ then (i_1,i_2) is both odd and even. On the other hand if $i_{2r} \in Q$ then (i_{2r},i_2) has smaller circular distance than (i_1,i_2). Thus i_{2r+1} belongs to P. The claim is proved by repeating this argument.

Since i_3 was shown to be in P it follows that all odd vertices except i_1 lie in P and the even vertices in $Q \cup \{i_1,i_2\}$. The final vertex i_{2k} is thus in Q. The even edge (i_{2k},i_1) must be a closed line segment contained in the region Q of the n-gon. Therefore (i_2,i_3) and (i_{2k},i_1) are two even edges that do not intersect which is a contradiction. This proves that $\mathscr{C}$ is a Gröbner basis of $I(K_n)$ with respect to $\succ$.

By construction, no monomial in an element of $\mathscr{C}$ is divisible by the leading term of an element in $\mathscr{C}$. Hence $\mathscr{C}$ is also the reduced Gröbner basis of $I(K_n)$ with respect to $\succ$. ∎

We use the above theorem to give an explicit triangulation and determine the normalized volume of $\Delta(2,n)$. We recall that the *normalized volume* of a $(k-1)$-dimensional lattice polytope $Q = conv(\{b_1,\ldots,b_q\})$ is given by the unique volume form on the affine hull of Q which is normalized by requiring that the non-zero simplex volumes $Vol(conv(b_{i_1},\ldots,b_{i_k}))$ with $1 \leq i_1 < \ldots < i_k \leq n$, are relatively prime integers. By Theorem 3.1 in [21] the square-free monomial ideal $\langle in_\succ(\mathscr{C})\rangle = in_\succ(I(K_n))$ is the Stanley-Reisner ideal of a regular triangulation $T_\succ$ of $\Delta(2,n)$. The simplices in $T_\succ$ are the supports of standard monomials with respect to the Gröbner basis $\mathscr{C}$. All maximal simplices in $T_\succ$ have unit normalized volume. This follows from Theorem 5.3 in [14] (see also Proposition 3.11 in [23]). We observed before that the elements of $in_\succ(\mathscr{C})$, *i.e.*, the minimally non-standard monomials, are supported on pairs of disjoint edges.

Corollary 2.2. *The simplices of $T_\succ$ are the subgraphs of K_n with the property that any pair of edges intersect in the given convex embedding of the graph.* ∎

Here and in what follows we identify subgraphs of K_n with subpolytopes of $\Delta(2,n)$; the subgraph H is identified with the convex hull of the column vectors of its vertex-edge incidence matrix.

Theorem 2.3. *The maximal simplices in the triangulation $T_{\succ}$ of $\Delta(2,n)$ are spanning subgraphs on n edges with the property that any pair of edges intersect. Every such subgraph is connected and contains a unique odd cycle. The number of such subgraphs and hence the normalized volume of $\Delta(2,n)$ is $2^{n-1}-n$.*

The formula for the volume of $\Delta(2,n)$ is well known. It was first computed by Laplace (see [20]). It can be shown that the triangulation $T_{\succ}$ is isomorphic to the $k=2$ case of the *Eulerian triangulation* in [20]. A different triangulation of the second hypersimplex was constructed by Gel'fand, Kapranov and Zelevinsky in [11]. The difference between the two triangulations will be discussed in Example 3.2. For the proof of Theorem 2.3 we need the following lemma.

Lemma 2.4. *A subpolytope σ of $\Delta(2,n)$ is an $(n-1)$-dimensional simplex if and only if the corresponding subgraph H satisfies the following properties.*
(i) *H is a spanning subgraph with n edges,*
(ii) *all cycles in H are odd,*
(iii) *every component contains at least one odd cycle.*
Furthermore the normalized volume of σ is $2^{q(H)-1}$ where $q(H)$ is the number of disjoint cycles in H.

Proof of Lemma 2.4. Suppose H supports an $(n-1)$-simplex. Let M_H be the $(0,1)$ incidence matrix of H. This matrix is non-singular which implies properties (i) and (ii). Suppose there exists a component C of H with no odd cycles. By property (ii), C is a tree. By induction on the number of edges in the tree one can prove that M_H is singular. The converse direction can be shown similarly.

Under the hypothesis (i), (ii) and (iii) we will now show that the subpolytope supported by H is a simplex having the stated volume. Let M_H be its vertex-edge incidence matrix. We prove that the absolute value of the determinant of M_H is equal to $2^{q(H)}$. If all vertices of H have degree two, then H is a disjoint union of odd cycles C_i and the matrix M_H (up to permutation of columns) is the direct sum of the matrices M_{C_i}. Therefore the determinant of M_H is the product of the determinants of the matrices M_{C_i}. The determinant of the incidence matrix of an odd cycle is 2 or -2. Therefore the absolute value of the determinant of M_H is $2^{q(H)}$. If the set of vertices with degree distinct from two is non-empty, then there is a vertex v of H of degree one. The row associated with v has 1 in some column and 0 elsewhere. Therefore the absolute values of the determinants of M_H and M_{H-v} are equal. Using this repeatedly we can reduce to the first case. The g.c.d. of $\det(M_H)$ where H ranges over all subgraphs of the specified kind is two. Hence the normalized volume of a simplex σ is $2^{q(H)-1}$. ∎

Proof of Theorem 2.3. The characteristics of the subgraphs follow from Corollary 2.2 and Lemma 2.4. Since the normalized volume of a maximal simplex in the triangulation $T_{\succ}$ was seen to be one, it follows that there is a unique odd cycle in the corresponding subgraph. Recall that the vertices of the graph are the vertices of a regular n-gon numbered in a clockwise manner and the edges are closed line segments joining two vertices. Consider an odd cycle C in K_n with $2k-1$ edges, $k \in \{2,\dots,\lceil n/2 \rceil\}$. We assume C is drawn such that each pair of edges in C intersect. There are $l = n-(2k-1)$ vertices that are not in C. We need to introduce l new edges in order to obtain a spanning subgraph. Let v be a vertex outside C. Due to the convex embedding of K_n and the requirement that the new edge should intersect all existing edges, there exists a unique vertex w in C such that (v, w) is one of the new edges. Therefore there is exactly one way to complete an odd cycle to be a graph with the above properties. There are $\binom{n}{2k-1}$ odd cycles for each $k \in \{2, \ldots, \lceil n/2 \rceil\}$ and the total number of such graphs is $\sum_{k=2}^{\lceil n/2 \rceil} \binom{n}{2k-1} = 2^{n-1} - n$. ∎

Remarks 2.5.

1. The graphs appearing as simplices in $T_{\succ}$ are known as *thrackles* in the combinatorics literature [28]. The fact that our *thrackle complex* $T_{\succ}$ is pure of dimension $n-1$ reflects (the easy linear version of) Conway's famous *Thrackle Conjecture*, stating that a thrackle on n vertices can have at most n edges.

2. The standard monomials modulo our Gröbner basis are precisely the *multi-thrackles*. In other words, a monomial m does not lie in $in_{\succ}(I(K_n))$ if and only if the support of m is a thrackle. This is equivalent to $m = y[i_1, j_1] y[i_2, j_2] \cdots y[i_r, j_r]$, where

$$(2.1) \qquad i_1 \leq i_2 \leq \cdots \leq i_r \leq j_1 \leq j_2 \leq \cdots \leq j_r, \ \ i_1 < j_1, \, i_2 < j_2, \, \ldots, i_r < j_r.$$

Note that m is recovered from $\Phi(m) = t_{i_1} t_{i_2} \cdots t_{i_r} t_{j_1} \cdots t_{j_r}$ by simply sorting indices.

3. The number of sequences which satisfy (2.1) equals the Hilbert polynomial of $k[y]/I(K_n)$, which is the Erhart polynomial of $\Delta(2, n)$:

$$(2.2) \qquad H_n(r) \ = \ card\big(r \cdot \Delta(2, n) \cap \mathbf{Z}^n\big) = \binom{n+2r-1}{n-1} - n \cdot \binom{n+r-2}{n-1}.$$

(This formula was shown to us by Richard Stanley.) Using the methods in [19], one can derive the h-vector and the f-vector of the triangulation $T_{\succ}$ from (2.2).

4. The easier fact that $I(K_n)$ is generated by quadrics is a special instance ($V = \wedge_2 \mathbf{C}^n$) of Theorem 4 in [9]. Flaschka and Haine proved that the ideal of the generic torus orbit in any irreducible $GL(n)$-module V is generated by quadrics.

3. On Secondary and State Polytopes

In this section we compare the state polytope of the homogeneous ideal $I(K_n)$ with the secondary polytope of $\mathcal{A}_n$. Both polytopes lie in $\mathbb{R}^p$ and are $(p-n)$-dimensional where $p = \binom{n}{2}$ is the number of edges in K_n. The secondary polytope of a general point configuration $\mathcal{B}$ is a Minkowski summand of the state polytope of the corresponding toric ideal. We refer the reader to [1], [14], [21] and [22] for the general theory. Our main result in this section is:

Theorem 3.1. *The state polytope of $I(K_n)$ and the secondary polytope of $\mathcal{A}_n$*
(A) *coincide up to $n = 5$ and*
(B) *are distinct for $n \geq 6$.*

Proof of (A). A point configuration $\mathcal{B}$ of lattice points is called *unimodular* if any full dimensional simplex with vertices in $\mathcal{B}$ has unit normalized volume. It can be seen by inspection that the configurations $\mathcal{A}_3$, $\mathcal{A}_4$ and $\mathcal{A}_5$ are unimodular. By results in Section 5 of [22] it follows that the secondary polytope of $\mathcal{A}_n$ and the state polytope of $I(K_n)$ coincide up to $n=5$. This proves statement (A). ∎

Example 3.2. We discuss the cases $n = 4$ and $n = 5$ in detail. The polytope $\Delta(2,4)$ is a regular octahedron in $\mathbb{R}^4$. It has three distinct regular triangulations. Therefore the secondary polytope, which is the same as the state polytope of $I(K_4)$, is a triangle in $\mathbb{R}^6$. The distinct initial ideals are $\langle y[1,4]y[2,3],\ y[1,3]y[2,4]\rangle$, $\langle y[1,3]y[2,4],\ y[1,2]y[3,4]\rangle$ and $\langle y[1,4]y[2,3],\ y[1,2]y[3,4]\rangle$.

The polytope $\Delta(2,5)$ has dimension four with 10 vertices and 10 facets (5 tetrahedra and 5 octahedra). Its secondary polytope, $\sum(\mathcal{A}_5)$, is a five dimensional polytope with 102 vertices, 255 edges, 240 two-faces, 105 three-faces and 20 facets. Under the natural S_5-action on the vertices of $\Delta(2,5)$, the regular triangulations fall into three distinct orbits. We list a representative from each orbit. Each triangulation consists of eleven subgraphs with five edges each. Let $[ij]$ be short for edge (i,j). The triangulations are:

1. {{[13], [23], [34], [35], [45]}, {[13], [23], [24], [34], [45]}, {[13], [14], [24], [34], [45]}, {[13], [23], [25], [35], [45]},{[13], [15], [25], [35], [45]}, {[13], [14], [15], [25], [45]}, {[12], [13], [14], [15], [25]}, {[13], [23], [24], [25], [45]}, {[12], [13], [23], [24], [25]}, {[13], [14], [24], [25], [45]}, {[12], [13], [14], [24], [25]}}
2. {{[13], [23], [34], [35], [45]}, {[12], [23], [24], [34], [45]}, {[12], [14], [24], [34], [45]}, {[12], [23], [25], [35], [45]}, {[12], [15], [25], [35], [45]}, {[12], [23], [24], [25], [45]}, {[12], [13], [23], [35], [45]}, {[12], [13], [23], [34], [45]}, {[12], [13], [14], [34], [45]}, {[12], [13], [15], [35], [45]}, {[12], [13], [14], [15], [45]}}
3. {{[12], [15], [24], [25], [35]}, {[15], [24], [25], [35], [45]}, {[14], [15], [24], [34], [45]}, {[12], [13], [15], [34], [35]}, {[12], [13], [23], [34], [35]}, {[12], [23], [24], [34], [35]}, {[12], [23], [24], [25], [35]}, {[12], [13], [14], [15], [34]}, {[12], [14], [15], [24], [34]}, {[15], [24], [34], [35], [45]}, {[12], [15], [24], [34], [35]}}

The third triangulation is our thrackle triangulation. The second triangulation is the one constructed by Gel'fand, Kapranov and Zelevinsky in [11]. To distinguish these symmetry classes of triangulations we list the following features. The minimal non-faces of triangulations 1 and 3 have two vertices, while triangulation 2 has a minimal non-face with three vertices: {[12], [34], [35]}. (Thus the reduced Gröbner basis of triangulation 2 is not quadratic.) Triangulations 1 and 2 have the property that all maximal simplices have a common vertex. (These are respectively [13] and [45].) The thrackle triangulation 3 has more symmetry: each vertex is contained in either eight or three maximal simplices.

Proof of (B). Consider the subgraph G of K_6 shown in Figure 1.

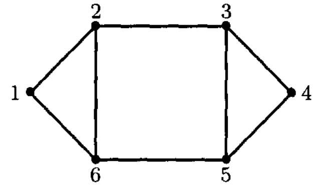

Fig. 1. Subgraph G

Let $\mathcal{A}_G$ be the set of columns of the vertex-edge incidence matrix of G. The configuration $\mathcal{A}_G$ consists of eight vectors spanning a 6-dimensional space. The Gale transform of $\mathcal{A}_G$ is thus 2-dimensional and is shown in Figure 2.

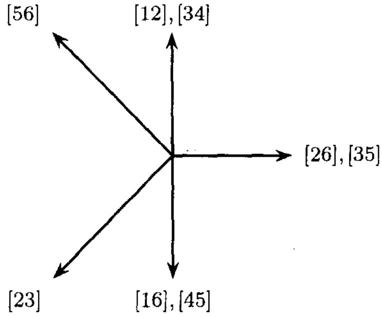

Fig. 2. Gale transform of $\mathcal{A}_G$

Using this Gale transform and the methods in [26], we see that the following four binomials form a universal Gröbner basis for the toric ideal $I(G)$:

$$y[2,3]y[5,6]-y[2,6]y[3,5], \quad y[1,2]y[5,6]y[3,4]-y[1,6]y[2,3]y[4,5],$$
$$y[1,2]y[5,6]^2y[3,4]-y[1,6]y[2,6]y[3,5]y[4,5],$$
$$y[1,2]y[2,6]y[3,5]y[3,4]-y[1,6]y[2,3]^2y[4,5].$$

The first two binomials suffice to generate $I(G)$, which is therefore a complete intersection.

Let $P(G)$ denote the convex hull of $\mathcal{A}_G$ and Y_G the set of eight variables $y[i,j]$ appearing in $I(G)$. We briefly recall some known general facts ([15], [21], [22]) about polyhedral subdivisions and toric ideals. Every function $\omega : Y_G \to \mathbb{R}$ defines a regular polyhedral subdivision Δ_ω of $P(G)$, and it defines an initial ideal $in_\omega(I(G))$ of $I(G)$. For special choices of ω, the ideal $in_\omega(I(G))$ need not be a monomial ideal, and Δ_ω need not be a triangulation. However, by Theorem 3.1 in [21], whenever $in_\omega(I(G))$ is a monomial ideal for some ω, then Δ_ω must be a triangulation. The converse statement holds if and only if the state polytope $St(I(G))$ equals the secondary polytope $\Sigma(\mathcal{A}_G)$.

For the graph G in Figure 1 we define ω to be the function which maps $y[2,3]$ and $y[5,6]$ to 1, and the other six variables to 0. The initial ideal $in_\omega\big(I(G)\big)$ is generated by

$$\big\{y[2,3]y[5,6], \quad y[1,2]y[5,6]y[3,4]-y[1,6]y[2,3]y[4,5],$$
$$y[1,2]y[5,6]^2y[3,4], \qquad y[1,6]y[2,3]^2y[4,5]\big\}.$$

We see that $in_\omega\big(I(G)\big)$ is not a monomial ideal, and that it has precisely two initial ideals, both having the same radical

$$J=\langle\, y[2,3]y[5,6],\ y[1,2]y[5,6]y[3,4],\ y[1,6]y[2,3]y[4,5]\rangle.$$

Hence both these initial ideals correspond to the same triangulation and we conclude that Δ_ω is a triangulation, namely it is the simplicial complex whose non-faces are the square-free monomials in J (Section 3, [21]).

Combining the discussion in the last two paragraphs, we have shown that the secondary polytope $\Sigma(\mathcal{A}_G)$ does not equal the state polytope $St(I(G))$. The Gale transform in Figure 2 gives more precise information: using the methods in Section 4 of [3] we see that $\Sigma(\mathcal{A}_G)$ is a pentagon, and by direct computation we see that $St(I(G))$ is a hexagon.

We next show the same result, $\Sigma(\mathcal{A}_{K_6}) \neq St(I(K_6))$, for the complete graph on six nodes. Let Y_{K_6} be the set of variables in $I(K_6)$ and $v : Y_{K_6} \to \mathbb{Z}$ be any function which is zero on Y_G, with G as above, positive elsewhere, and such that each maximal cell in the regular polyhedral subdivision Δ_v of $\Delta(2,6)$ is either a simplex or equal to $P(G)$. Let $\omega : Y_{K_6} \to \mathbb{Z}$ be the function whose restriction to Y_G equals ω as above and which is zero elsewhere.

Let us now consider the weight function $M \cdot v + \omega$ where M is a very large positive integer. The polyhedral subdivision $\Delta_{M \cdot v + \omega}$ is a regular triangulation of $\Delta(2,6)$: outside the subpolytope $P(G)$ it has precisely the same simplices as Δ_v, while inside $P(G)$ it has the same simplices as Δ_ω above.

To prove our result, it suffices to show that the ideal $in_{M \cdot v + \omega}(I(K_6))$ is not a monomial ideal. If it were a monomial ideal, then also its elimination ideal

$$\begin{aligned} in_{M\cdot v+\omega}(I(K_6)) \cap k[Y_G] &= in_{M\cdot v+\omega}\big(I(K_6) \cap k[Y_G]\big) \\ &= in_{M\cdot v+\omega}\big(I(G)\big) = in_\omega\big(I(G)\big) \end{aligned}$$

would be a monomial ideal. This contradicts our result above, and it hence completes the proof for K_6. To establish the result for K_n, $n > 6$, we can use the exact same technique, which is to write $I(K_6)$ as an elimination ideal of $I(K_n)$. This proves statement (B) of Theorem 3.1. ∎

4. A Non-Regular Triangulation

We complement our discussion on regular triangulations by showing that $\Delta(2,n)$ for $n \geq 9$ admits a non-regular triangulation. Our technique is similar to that in Section 4 of [4]. We exhibit a non-regular triangulation of the point configuration associated with a subgraph of K_9. This is then extended to a non-regular triangulation to $\Delta(2,9)$ and finally to a non-regular triangulation of $\Delta(2,n)$ for $n \geq 9$. Consider the subgraph G of K_9 shown in Figure 3.

Let the point configuration associated with G be $\mathcal{A}_G \subset \mathbb{R}^9$. The convex hull $P(G)$ of the points in $\mathcal{A}_G$ is an 8-dimensional polytope with 12 vertices. Let $\mathcal{B}_G = \{z_a, z_b, \ldots, z_l\} \subset \mathbb{R}^3$ be the following Gale transform of $\mathcal{A}_G$ (we note that the rows of this matrix are even walks in G).

$$\begin{array}{cccccccccccc} z_a & z_b & z_c & z_d & z_e & z_f & z_g & z_h & z_i & z_j & z_k & z_l \end{array}$$
$$\begin{pmatrix} 0 & 0 & 0 & 1 & -1 & 1 & -2 & 1 & -1 & 1 & 0 & 0 \\ -1 & 1 & -2 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

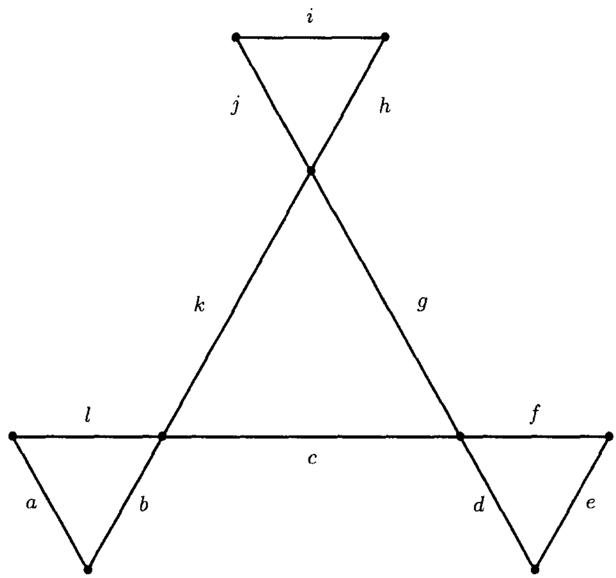

Fig. 3. Subgraph of K_9

Figure 4 shows an affine Gale diagram (see [24]) of $\mathcal{A}_G$. We think of Figure 4 as a configuration in the northern hemisphere of the 2-sphere. The points a, c, e, g, i, k are contained in the northern hemisphere while the points b, d, f, l, h, j are contained in the southern hemisphere. These six southern points are represented on the northern hemisphere by their antipodal points $\bar{b}, \bar{d}, \bar{f}, \bar{l}, \bar{h}, \bar{j}$. The *chamber complex* associated with the vector configuration $\mathcal{B}_G$ equals the normal fan of the secondary polytope of $\mathcal{A}_G$ by the results of [3], [4] and [10]. The part of this complex which lies in the northern hemisphere is drawn in Figure 4.

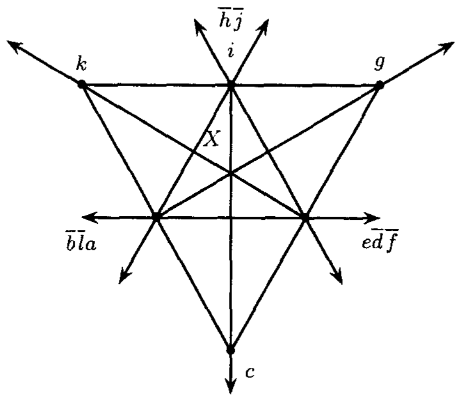

Fig. 4. Affine Gale diagram of the configuration $\mathcal{A}_G$

We consider the maximal cell X in Figure 4 which is the intersection of the simplicial cones $\mathrm{pos}(\{b_i,b_j,b_k\})$ where ijk ranges over the triples in the following list:

$$\begin{array}{ccccc} c,k,g & e,i,k & c,i,k & e,k,g & a,c,i \\ c,i,d & c,f,i & a,e,i & b,e,k & e,k,l \\ \underline{a,f,g} & \underline{a,k,g} & \underline{a,i,g} & \underline{a,g,d}. & \end{array}$$

The regular triangulation T_X corresponding to the region X has as its maximal simplices the complements of all triples in the above list. Let C be the signed circuit of $\mathcal{A}_G$ given by $C=(0,0,+,-,+,-,0,+,-,+,-,0)$. The tetrahedra $\{c,e,h,j\}$ and $\{d,f,i,k\}$ intersect in their relative interior. Hence we can geometrically perform a generalized bistellar operation [10] (also known as a *modification* in Chapter 7, page 231, of [12]) for the triangulation T_X to obtain a new triangulation T_Y. The maximal simplices in T_Y are the complements of the non-underlined triples in the above list and the new triples $\{a,c,g\}$,$\{a,e,g\}$,$\{a,h,g\}$ and $\{a,j,g\}$.

Lemma 4.1. *The triangulation T_Y is a non-regular triangulation of $P(G)$.*

Proof. If T_Y was a regular triangulation of $P(G)$ then the intersection of the interiors of the open cones $\mathrm{pos}(\{b_i,b_j,b_k\})$ where ijk ranges over the triples in the first list that are not underlined and those in the second list would be non-empty. In Figure 4 it can be seen that the intersection of the cones $\{a,c,i\}$, $\{e,k,g\}$ and $\{a,e,g\}$ is empty and hence T_Y is a non-regular triangulation of $P(G)$. ∎

Theorem 4.2. *There exists a non-regular triangulation for $\Delta(2,n)$, $n\geq 9$.*

Proof. The polytope $P(G)$ is a subpolytope of $\Delta(2,9)$. We can complete the triangulation T_Y of $P(G)$ to be a triangulation T of $\Delta(2,9)$ by *placing* (see [15]) the remaining vertices. Since T_Y was a non-regular triangulation of $P(G)$ there does not exist any piecewise linear convex function on $\Delta(2,9)$ that induces T_Y. Any such function would have induced locally a regular triangulation of $P(G)$. This extends to $\Delta(2,n)$, $n\geq 9$ since $\Delta(2,9)$ is a subpolytope of $\Delta(2,n)$ and the same argument applies. ∎

Remark 4.3. In the above proof we made use of two general facts:
- Let $P\subset Q$ be polytopes and let T be any triangulation of P. Then there exists a triangulation T' of Q that extends T.
- If T is non-regular, then T' is non-regular.

5. Universal Gröbner bases

In this section we describe minimal universal Gröbner bases for the ideals $I(K_5)$,$I(K_6)$, $I(K_7)$ and $I(K_8)$. We remark that an infinite universal Gröbner basis for the ideal $I(K_n)$ consists of the binomials associated with all possible even closed walks in K_n. An n-tuple $(\lambda_1,\dots,\lambda_n)$ is an *elementary vector* of the point configuration $\mathcal{A}=\{a_1,\dots,a_n\}\subset\mathbb{R}^d$ if it defines an integral affine dependency $\lambda_1 a_1+\ldots+\lambda_n a_n=0$ with the properties: *(a)* At least one λ_i is distinct from zero. *(b)* The greatest common divisor of $\{\lambda_1,\dots,\lambda_n\}$ is one. *(c)* The support $\{i|\lambda_i\neq 0\}$ is minimal with respect to inclusion. In the case of the specific point configuration

$\mathcal{A}_n$ the elementary vectors correspond to either even cycles or two disjoint odd cycles joined by a path [27]. We call the binomials of $I(K_n)$ associated with the elementary vectors, the *circuits* of $I(K_n)$.

Theorem 5.1. *The circuits form a universal Gröbner basis of $I(K_n)$ for $n \leq 7$. The same statement is not true for $n \geq 8$.*

Proof. For these computations we used the Lawrence construction introduced in [25] to calculate universal Gröbner bases of toric ideals. All computations were done using MACAULAY [2]. The elements of the universal Gröbner basis are (up to a relabeling using the natural S_n-action) presented in Table 1. The numbers give the cardinality of the S_n-orbit of each binomial.

Types of binomials	K_5	K_6	K_7	K_8
$y[ab]y[cd]-y[ac]y[bd]$	15	45	105	210
$y[ab]y[cd]y[ec]-y[bc]y[de]y[ac]$	15	90	315	840
$y[ab]y[cd]^2y[ef]-y[bc]y[df]y[de]y[ac]$	0	90	630	2520
$y[ab]y[cd]y[ef]-y[bc]y[de]y[af]$	0	60	420	1680
$y[ab]y[cd]y[ef]y[cg]-y[bc]y[de]y[fg]y[ac]$	0	0	1260	10080
$y[ab]y[cd]^2y[eg]y[ef]-y[bc]y[de]^2y[fg]y[ac]$	0	0	630	2520
$y[ab]y[cd]y[ef]y[gh]-y[bc]y[de]y[fg]y[ah]$	0	0	0	2520
$y[ab]y[cb]^2y[ef]^2y[gh]-y[bc]y[de]^2y[fg]y[fh]y[ac]$	0	0	0	5040
$y[ab]y[cd]^2y[ef]y[gh]-y[bc]y[de]y[fg]y[dh]y[ac]$	0	0	0	10080
$y[ab]y[cd]y[ef]y[eg]y[ch]-y[bc]y[de]y[fg]y[eh]y[ac]$	0	0	0	2520
$y[ab]y[cd]y[ef]y[dg]y[ch]-y[bc]y[de]y[df]y[gh]y[ac]$	0	0	0	5040
TOTAL	30	285	3360	45570

Table 1. Universal Gröbner basis for K_n, $n \leq 8$

Notice that for $n \leq 7$ the elements in the universal Gröbner bases in Table 1 are precisely the circuits of the corresponding $I(K_n)$. Therefore the circuits of $\mathcal{A}_n$ form a universal Gröbner basis for $I(K_n)$ for $n \leq 7$. However this property does not hold for $I(K_8)$ and hence for all $I(K_n)$, $n \geq 8$. The universal Gröbner basis of $I(K_8)$ needs, up to labeling, two extra types of binomials. These are the binomials supported in the subgraphs of K_8 shown in Figure 5.

The computation of the universal Gröbner basis for $I(K_8)$ was a more challenging enterprise than the direct Lawrence calculations of the previous cases. We computed a universal Gröbner basis H for the ideal $I(K_8 - \{(1,8),(1,7),(1,6),(1,5),(1,4)\})$ using the Lawrence technique. The set H contains precisely the eleven types of binomials listed in Table 1. If a walk Γ needs to be added to H in order to complete a universal Gröbner basis of $I(K_8)$, then Γ cannot be supported in any subgraph of K_8 isomorphic to $K_8 - \{(1,8),(1,7),(1,6),(1,5),(1,4)\}$. This implies that Γ must visit every vertex twice and be supported in a subgraph of minimal degree three. We call this type of even walks *saturated*. An

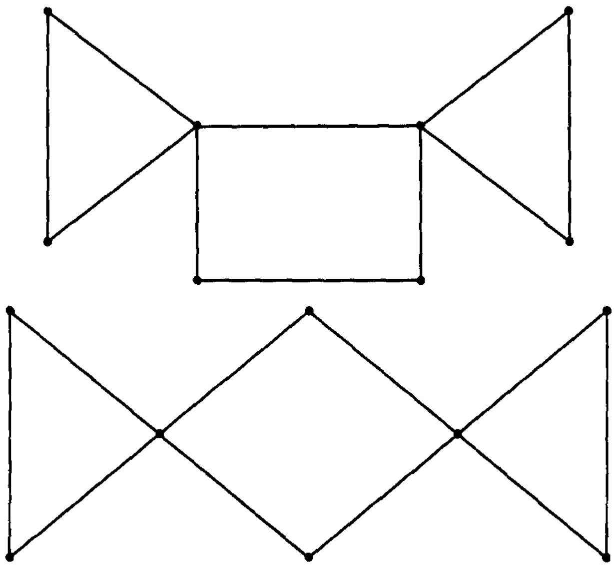

Fig. 5. Non-circuits necessary in a universal Gröbner basis of $I(K_8)$

even walk is *primitive* if there is no proper subset of its edges that supports an even closed subwalk. As a consequence of Proposition 5.1 in [8] (see also [25]), the set of primitive even walks in K_n defines binomials that constitute a universal Gröbner basis for $I(K_n)$. Finally, a detailed case analysis (omitted here) shows that a saturated even walk in K_8 cannot be primitive. All primitive even walks in K_8 are then supported in a subgraph isomorphic to $K_8 - \{(1,8),(1,7),(1,6),(1,5),(1,4)\}$. Therefore, the union of the S_8-orbits of the eleven binomials is a universal Gröbner basis for $I(K_8)$. ∎

As an estimate for the size of a universal Gröbner basis for $I(K_n)$ we can bound the degree of the binomials involved. General bounds, valid for arbitrary toric ideals, can be found in [8] and [21].

Proposition 5.2. *The largest degree d_n of a binomial in a minimal universal Gröbner basis for $I(K_n)$, satisfies $n-2 \leq d_n \leq \binom{n}{2}$.*

Proof. The lower bound holds because any universal Gröbner basis of $I(K_n)$ must contain all circuits and among the circuits there exist elements of degree $n-2$. The associated subgraphs are two disjoint triangles joined by a path of length $n-5$. We already mentioned that the primitive even walks define the binomials of a universal Gröbner basis for $I(K_n)$. We observe that an even walk that visits an edge more than twice with the same parity cannot be primitive. The binomials coming from primitive even walks are then forced to have at most degree two for each of the appearing variables. Finally, we remark that there are at most $\binom{n}{2}/2$ variables in each term of any binomial of $I(K_n)$. ∎

6. Applications

Toric ideals have recently been applied to problems in integer programming [5], [26] and in computational statistics [7] (see also Section 5 in [8]). The specific family of toric ideals $I(K_n)$ studied in this paper has the following interpretations in these two domains of application.

Consider the problem of generating a random multigraph on n nodes with fixed degree at each node. This problem has been studied in detail by Sinclair and Jerrum (see [17]). A natural algorithm follows from the approach in [7]: the idea is to start with any legal multigraph and then to perform a random walk with respect to a fixed finite set of local moves which alters multigraphs while maintaining the vertex degrees. The possible moves are precisely the even walks, that is, binomials in the ideal $I(K_n)$. It follows from Theorem 3.2 in [7] that a set of moves gives an irreducible Markov chain for all degree sequences if and only if the corresponding binomials generate $I(K_n)$. Thus Theorem 2.1 gives an explicit list of connecting moves: each move is quadratic and involves only four edges.

The task of constructing irreducible Markov chains becomes much harder if structural zeros are prescribed (see Section 4.E in [7]), that is, if certain edges are prohibited during the random walk. The algebraic counterpart is to find a generating set for the ideal $I(G)$, where G is a subgraph of K_n. This problem is solved simultaneously for all subgraphs of K_n by finding a universal Gröbner basis for $I(K_n)$. This is shown in Corollary 4.2 of [7]. Note in particular that Proposition 4.3 in [7] deals with the easier case of bipartite graphs. Our results in Section 5 give a complete answer for all graphs with $n \leq 8$ nodes and bounds for the general case.

The set of all multigraphs on n nodes with fixed vertex degrees is the set of feasible solutions of an important problem of combinatorial optimization, namely, the *perfect f-matching problem* [16]. Let f be a positive integer valued function on the n vertices of K_n such that $f(i)$ specifies the degree of the vertex i. An assignment of a non-negative integer $p[i,j]$ to the edge (i,j) such that $\sum_{\{(i,j):j\neq i\}} p[i,j] = f(i)$ holds for each vertex i is called a *perfect f-matching*. Suppose in addition to the degree of each vertex we are also given a cost c_{ij} for each edge (i,j), and the objective is to find a perfect f-matching on n nodes with minimum total cost. Then we have the following integer program which is called the *minimum weight perfect f-matching* problem.

$$\text{Minimize } \sum_{i,j} c_{ij} \cdot p[i,j]$$

subject to

$$\sum_{\{(i,j):j\neq i\}} p[i,j] = f(i) \qquad i = 1,\ldots,n$$

$$p[i,j] \geq 0, \qquad \text{integer}$$

The coefficient matrix of the above integer program is the vertex-edge incidence matrix of K_n and any feasible solution is a perfect f-matching of K_n. Consider the family of integer programs obtained by varying the function f but with the above coefficient matrix and costs c_{ij} fixed. The results of [5] and [26] show that the reduced Gröbner basis of $I(K_n)$ with respect to a term order refining the cost function $c = (c_{ij})$ is a minimal *test set* for the above family. This implies that given any feasible integer solution to a program in this family, one of the following happens. Either there exists a binomial in the corresponding Gröbner basis of $I(K_n)$ that reduces the monomial representation of this solution to give a new solution of smaller cost or there is no such element which implies that the current solution is optimal.

As an example, consider the quadratic Gröbner basis in Section 2. The corresponding optimization problem asks for the minimum weight perfect f-matching of K_n where the cost of a variable $p[i,j]$ is the number of edges in K_n that do not meet (i,j). The function f specifies the degree of each vertex and thus provides the right hand side vector of this integer program. Under the special embedding of K_n prescribed in Section 2, an optimal solution is a perfect f-matching in K_n that maximizes the total number of crossings among edges. In other words, it is a multi-thrackle of K_n.

As an illustration of the technique of solving integer programs using test sets, consider the reduced Gröbner basis $\mathscr{C}$ of Theorem 2.1. We begin with any perfect f-matching of K_n. If no two edges of this multigraph are disjoint, then the corresponding monomial is reduced with respect to the above quadratic Gröbner basis and we conclude that the current solution is optimal. On the other hand, if there exists two edges (i,j) and (k,l) in this multigraph that are disjoint, then the corresponding monomial is divisible by $y[i,j]y[i,k]$, the leading term of $y[i,j]y[k,l]-y[i,l]y[j,k]$ which is an element of the quadratic Gröbner basis $\mathscr{C}$. Reducing the current feasible solution by this element in $\mathscr{C}$ provides the new feasible solution in which the disjoint edges (i,j) and (k,l) have been replaced by the intersecting edges (i,l) and (j,k). Therefore the new solution has strictly smaller weight compared to the old solution. By repeating this procedure we obtain the optimal solution to the integer program after finitely many steps. Therefore the reduced Gröbner basis $\mathscr{C}$ provides a minimal set of directions that allow a monotone path from any non-optimal perfect f-matching of K_n to an optimal perfect f-matching. We emphasize that the specific integer program corresponding to $\mathscr{C}$ has a trivial solution, which is given by the sorting procedure in Remarks 2.5, (2).

A universal Gröbner basis of $I(K_n)$ provides a test set with respect to *every* cost function and right hand side vector for the minimum weight f-matching problem on K_n (see Section 3.2 in [26]). Hence, the universal Gröbner bases of $I(K_n)$ for $n \leq 8$ provides an explicit set of moves by which one can solve the minimum weight f-matching problem on K_n for $n \leq 8$ with respect to any cost function c and function f.

In closing we remark that the work of Gel'fand, Kapranov and Zelevinsky was motivated by yet another application of toric ideals: the study of generalized hypergeometric functions (see [10], [11], [12] and [22]). Gröbner bases and their regular triangulations provide tools for constructing series solutions for the $\mathscr{A}$-hypergeometric systems of differential equations.

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