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FPTAS for optimizing polynomials over the mixed-integer points of polytopes in fixed dimension

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Abstract We show the existence of a fully polynomial-time approximation scheme (FPTAS) for the problem of maximizing a non-negative polynomial over mixed-integer sets in convex polytopes, when the number of variables is fixed. Moreover, using a weaker notion of approximation, we show the existence of a fully polynomial-time approximation scheme for the problem of maximizing or minimizing an arbitrary polynomial over mixed-integer sets in convex polytopes, when the number of variables is fixed.

Keywords Mixed-integer nonlinear programming · Integer programming in fixed dimension · Computational complexity · Approximation algorithms · FPTAS

Mathematics Subject Classification (2000) 90C11 · 90C30 · 90C60 · 90C57

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1 Introduction

A well-known result by H.W. Lenstra Jr. states that integer *linear* programming problems with a fixed number of variables can be solved in polynomial time on the input size [12]. Likewise, mixed integer linear programming problems with a fixed number of integer variables can be solved in polynomial time. It is a natural question to ask what is the computational complexity, when the number of variables (or the number of integer variables) is fixed, of the *non-linear* mixed integer problem

$$
\begin{aligned}
\max \quad & f(x_1,\ldots,x_{d_1},z_1,\ldots,z_{d_2}) && \text{(1a)}\\
\text{s.t.} \quad & A\mathbf{x}+B\mathbf{z} \leq \mathbf{b} && \text{(1b)}\\
& x_i \in \mathbf{R} \qquad \text{for } i=1,\ldots,d_1, && \text{(1c)}\\
& z_i \in \mathbf{Z} \qquad \text{for } i=1,\ldots,d_2, && \text{(1d)}
\end{aligned}
$$

where f is a polynomial function of maximum total degree D with rational coefficients, and $A \in \mathbf{Z}^{p\times d_1}$, $B \in \mathbf{Z}^{p\times d_2}$, $\mathbf{b} \in \mathbf{Z}^p$. We are interested in general polynomial objective functions f *without any convexity assumptions*.

Throughout the paper we assume that the inequality system $A\mathbf{x}+B\mathbf{z} \leq \mathbf{b}$ describes a convex polytope, i.e., a *bounded* polyhedron, which we denote by P. The reason for this restriction are fundamental noncomputability results for problems involving polynomials and integer variables. Indeed, when we permit unbounded feasible regions, there cannot exist any algorithm to decide whether there exists a feasible solution to (1) with $f(\mathbf{x},\mathbf{z}) \geq \alpha$ (for a prescribed bound α), ruling out the existence of an optimization algorithm or any approximation scheme. This is due to the negative answer to Hilbert's tenth problem by Matiyasevich [13, 14]. Due to Jones' strengthening of this negative result [10], there also cannot exist any such algorithm for the cases of unbounded feasible regions for any fixed number of integer variables $d_2 \geq 10$.

For the purpose of complexity analysis, we assume that the data A, B, and $\mathbf{b}$ are given by the binary encoding scheme, and that the objective function f is given as a list of monomials, where the coefficients are encoded using the binary encoding scheme and the exponent vectors are encoded using the unary encoding scheme. In other words, the running times are permitted to grow polynomially not only in the binary encoding of all the problem data, but also in the maximum total degree D of the objective function f.

It is well-known that pure continuous polynomial optimization over polytopes ($d_2 = 0$) in varying dimension is NP-hard and that a fully polynomial time approximation scheme (FPTAS) is not possible (unless P = NP). Indeed the max-cut problem can be modeled as minimizing a quadratic form over the cube $[-1,1]^d$. Håstad [9] proved that the max-cut problem cannot be approximated to a ratio better than 1.0625 (unless P = NP). This excludes the possibility of a polynomial time approximation scheme for (1) in varying dimension, even when the number of integer variables is fixed.

On the other hand, pure continuous polynomial optimization problems over polytopes ($d_2 = 0$) can be solved in polynomial time when the dimension d_1 is fixed. This follows from a much more general result on the computational complexity of approximating the solutions to general algebraic formulae over the real numbers by Renegar [19]; see also [16, 17, 18].

However, when we permit integer variables ($d_2 > 0$), it turns out that, even for fixed dimension $d_1 + d_2 = 2$ and objective functions f of maximum total degree $D = 4$, problem (1) is an NP-hard problem [6]. Thus the best we can hope for, even when the number of both the continuous and the integer variables is fixed, is an approximation result. This paper presents the best possible such result:

Theorem 1 (Fully polynomial-time approximation scheme) *Let the dimension $d = d_1 + d_2$ be fixed.*

(a) *There exists a fully polynomial time approximation scheme (FPTAS) for the optimization problem* (1) *for all polynomial functions $f(x_1, \dots, x_{d_1}, z_1, \dots, z_{d_2})$ with rational coefficients that are non-negative on the feasible region* (1b–1d).
(b) *Moreover, the restriction to non-negative polynomials is necessary, as there does not even exist a polynomial time approximation scheme (PTAS) for the maximization of* arbitrary *polynomials over mixed-integer sets in polytopes, even for fixed dimension $d \geq 2$, unless* P = NP.

The proof of Theorem 1 is presented in section 5. As we will see, Theorem 1 is a non-trivial consequence of the existence of FPTAS for the problem of maximizing a non-negative polynomial with integer coefficients over the lattice points of a convex rational polytope. That such FPTAS indeed exist was recently settled in our paper [6]. The knowledge of paper [6] is not necessary to understand this paper but, for convenience of the reader, we include a short summary in section 2. Our arguments, however, are independent of which FPTAS is used in the integral case.

Our main approach is to use grid refinement in order to approximate the mixed-integer optimal value via auxiliary pure integer problems. One of the difficulties on constructing approximations is the fact that not every sequence of grids whose widths converge to zero leads to a convergent sequence of optimal solutions of grid optimization problems. This difficulty is addressed in section 3. In section 4 we develop techniques for bounding differences of polynomial function values. Section 5 contains the proof of Theorem 1.

Finally, in section 6, we study a different notion of approximation. The usual definition of an FPTAS uses the notion of ε-approximation that is common when considering combinatorial optimization problems, where the approximation error is compared to the optimal solution value,

$$\left|f(\mathbf{x}_\varepsilon, \mathbf{z}_\varepsilon) - f(\mathbf{x}_{\max}, \mathbf{z}_{\max})\right| \leq \varepsilon f(\mathbf{x}_{\max}, \mathbf{z}_{\max}), \tag{2}$$

where $(\mathbf{x}_\varepsilon, \mathbf{z}_\varepsilon)$ denotes an approximate solution and $(\mathbf{x}_{\max}, \mathbf{z}_{\max})$ denotes a maximizer of the objective function. In section 6, we now compare the approximation error to the *range* of the objective function on the feasible region,

$$\left|f(\mathbf{x}_\varepsilon, \mathbf{z}_\varepsilon) - f(\mathbf{x}_{\max}, \mathbf{z}_{\max})\right| \leq \varepsilon \left|f(\mathbf{x}_{\max}, \mathbf{z}_{\max}) - f(\mathbf{x}_{\min}, \mathbf{z}_{\min})\right|, \tag{3}$$

where additionally $(\mathbf{x}_{\min}, \mathbf{z}_{\min})$ denotes a *minimizer* of the objective function on the feasible region. This notion of approximation was proposed by various authors [20, 3, 11]. It enables us to study objective functions that are not restricted to be non-negative on the feasible region. We remark that, when the objective function can take negative values on the feasible region, (3) is weaker than (2). Therefore Theorem 1 (b) does not rule out the existence of an FPTAS with respect to this notion of approximation. Indeed we prove:

Theorem 2 (Fully polynomial-time weak-approximation scheme) *Let the dimension $d = d_1 + d_2$ be fixed. Let f be an arbitrary polynomial function with rational coefficients and maximum total degree D, and let $P \subset \mathbf{R}^d$ be a rational convex polytope.*

(a) *In time polynomial in the input size and D, it is possible to decide whether f is constant on $P \cap \left(\mathbf{R}^{d_1} \times \mathbf{Z}^{d_2}\right)$.*
(b) *In time polynomial in the input size, D, and $\frac{1}{\varepsilon}$ it is possible to compute a solution $(\mathbf{x}_\varepsilon, \mathbf{z}_\varepsilon) \in P \cap \left(\mathbf{R}^{d_1} \times \mathbf{Z}^{d_2}\right)$ with*

$$\left|f(\mathbf{x}_\varepsilon, \mathbf{z}_\varepsilon) - f(\mathbf{x}_{\max}, \mathbf{z}_{\max})\right| \leq \varepsilon \left|f(\mathbf{x}_{\max}, \mathbf{z}_{\max}) - f(\mathbf{x}_{\min}, \mathbf{z}_{\min})\right|.$$

Notation. As usual, we denote by $\mathbf{Q}[x_1, \ldots, x_{d_1}, z_1, \ldots, z_{d_2}]$ the ring of multivariate polynomials with rational coefficients. For writing multivariate polynomials, we frequently use the *multi-exponent notation*, $\mathbf{z}^{\boldsymbol{\alpha}} = z_1^{\alpha_1} \cdots z_d^{\alpha_d}$.

2 An FPTAS for the integer case

The first fully polynomial-time approximation scheme for the integer case appeared in our paper [6]. It is based on Alexander Barvinok's theory for encoding all the lattice points of a polyhedron in terms of short rational functions [1,2]. The set $P \cap \mathbf{Z}^d$ is represented by a Laurent polynomial $g_P(\mathbf{z}) = \sum_{\boldsymbol{\alpha} \in P \cap \mathbf{Z}^d} \mathbf{z}^{\boldsymbol{\alpha}}$. From Barvinok's theory this exponentially-large sum of monomials $g_P(\mathbf{z})$ can be written as a polynomial-size sum of rational functions (assuming the dimension d is fixed) of the form:

$$g_P(\mathbf{z}) = \sum_{i \in I} E_i \frac{\mathbf{z}^{\mathbf{u}_i}}{\prod_{j=1}^{d} (1 - \mathbf{z}^{\mathbf{v}_{ij}})}, \tag{4}$$

where I is a polynomial-size indexing set, and where $E_i \in \{1, -1\}$ and $\mathbf{u}_i, \mathbf{v}_{ij} \in \mathbf{Z}^d$ for all i and j. There is a polynomial-time algorithm for computing this representation [1,2,5,7].

By symbolically applying differential operators to the representation (4), we can compute a short rational function representation of the Laurent polynomial

$$g_{P,f}(\mathbf{z}) = \sum_{\boldsymbol{\alpha} \in P \cap \mathbf{Z}^d} f(\boldsymbol{\alpha}) \mathbf{z}^{\boldsymbol{\alpha}}. \tag{5}$$

In fixed dimension, the size of the expressions occuring in the symbolic calculation can be bounded polynomially:

Lemma 3 ([6], Lemma 3.1) *Let the dimension d be fixed. Let $g_P(\mathbf{z}) = \sum_{\boldsymbol{\alpha} \in P \cap \mathbf{Z}^d} \mathbf{z}^{\boldsymbol{\alpha}}$ be the Barvinok representation of the generating function of $P \cap \mathbf{Z}^d$. Let $f \in \mathbf{Z}[x_1, \ldots, x_d]$ be a polynomial of maximum total degree D. We can compute, in time polynomial in D and the input size, a Barvinok representation $g_{P,f}(\mathbf{z})$ for the generating function $\sum_{\boldsymbol{\alpha} \in P \cap \mathbf{Z}^d} f(\boldsymbol{\alpha}) \mathbf{z}^{\boldsymbol{\alpha}}$.*

Now we present the algorithm to obtain bounds U_k, L_k that reach the optimum. We make use of the elementary fact that, for a set $S = \{s_1, \ldots, s_r\}$ of non-negative real numbers,

$$\max\{s_1, \ldots, s_r\} = \lim_{k \to \infty} \sqrt[k]{s_1^r + \cdots + s_k^r}. \tag{6}$$

Algorithm 4 (Computation of bounds)

Input: A rational convex polytope $P \subset \mathbf{R}^d$, a polynomial objective $f \in \mathbf{Z}[x_1, \dots, x_d]$ of maximum total degree D that is non-negative over $P \cap \mathbf{Z}^d$.

Output: A nondecreasing sequence of lower bounds L_k, and a nonincreasing sequence of upper bounds U_k, both reaching the maximal function value f^* of f over $P \cap \mathbf{Z}^d$ in a finite number of steps.

1. Compute a short rational function expression for the generating function $g_P(\mathbf{z}) = \sum_{\boldsymbol{\alpha} \in P \cap \mathbf{Z}^d} \mathbf{z}^{\boldsymbol{\alpha}}$. Using residue techniques, compute $|P \cap \mathbf{Z}^d| = g_P(\mathbf{1})$ from $g_P(\mathbf{z})$.
2. From the rational function $g_P(\mathbf{z})$ compute the rational function representation of $g_{P,f^k}(\mathbf{z})$ of $\sum_{\boldsymbol{\alpha} \in P \cap \mathbf{Z}^d} f^k(\boldsymbol{\alpha}) \mathbf{z}^{\boldsymbol{\alpha}}$ by Lemma 3. Using residue techniques, compute
$$L_k := \left\lceil \sqrt[k]{g_{P,f^k}(\mathbf{1}) / g_{P,f^0}(\mathbf{1})} \right\rceil \quad \text{and} \quad U_k := \left\lfloor \sqrt[k]{g_{P,f^k}(\mathbf{1})} \right\rfloor.$$

Theorem 5 ([6], Lemma 3.3 and Theorem 1.1) *Let the dimension d be fixed. Let $P \subset \mathbf{R}^d$ be a rational convex polytope. Let f be a polynomial with integer coefficients and maximum total degree D that is non-negative on $P \cap \mathbf{Z}^d$.*

(i) *Algorithm 4 computes the bounds L_k, U_k in time polynomial in k, the input size of P and f, and the total degree D. The bounds satisfy the following inequality:*
$$U_k - L_k \le f^* \cdot \left(\sqrt[k]{|P \cap \mathbf{Z}^d|} - 1 \right).$$

(ii) *For $k = (1 + 1/\varepsilon) \log(|P \cap \mathbf{Z}^d|)$ (a number bounded by a polynomial in the input size), L_k is a $(1 - \varepsilon)$-approximation to the optimal value f^* and it can be computed in time polynomial in the input size, the total degree D, and $1/\varepsilon$. Similarly, U_k gives a $(1 + \varepsilon)$-approximation to f^*.*

(iii) *With the same complexity, by iterated bisection of P, we can also find a feasible solution $\mathbf{x}_\varepsilon \in P \cap \mathbf{Z}^d$ with*
$$\left| f(\mathbf{x}_\varepsilon) - f^* \right| \le \varepsilon f^*.$$

3 Grid approximation results

An important step in the development of an FPTAS for the mixed-integer optimization problem is the reduction of the mixed-integer problem (1) to an auxiliary optimization problem over a lattice $\frac{1}{m}\mathbf{Z}^{d_1} \times \mathbf{Z}^{d_2}$. To this end, we consider the *grid problem* with grid size m,

$$\begin{aligned} \max \quad & f(x_1, \dots, x_{d_1}, z_1, \dots, z_{d_2}) \\ \text{s.t.} \quad & A\mathbf{x} + B\mathbf{z} \le \mathbf{b} \\ & x_i \in \tfrac{1}{m}\mathbf{Z} \qquad \text{for } i = 1, \dots, d_1, \\ & z_i \in \mathbf{Z} \qquad \text{for } i = 1, \dots, d_2. \end{aligned} \tag{7}$$

We can solve this problem approximately using the integer FPTAS (Theorem 5):

Corollary 6 *For fixed dimension $d = d_1 + d_2$ there exists an algorithm with running time polynomial in* $\log m$*, the encoding length of f and of P, the maximum total degree D of f, and $\frac{1}{\varepsilon}$ for computing a feasible solution $(\mathbf{x}^m_\varepsilon, \mathbf{z}^m_\varepsilon) \in P \cap \left(\frac{1}{m}\mathbf{Z}^{d_1} \times \mathbf{Z}^{d_2}\right)$ to the grid problem* (7) *with an objective function f that is non-negative on the feasible region, with*

$$f(\mathbf{x}^m_\varepsilon, \mathbf{z}^m_\varepsilon) \geq (1-\varepsilon) f(\mathbf{x}^m, \mathbf{z}^m), \tag{8}$$

where $(\mathbf{x}^m, \mathbf{z}^m) \in P \cap \left(\frac{1}{m}\mathbf{Z}^{d_1} \times \mathbf{Z}^{d_2}\right)$ is an optimal solution to (7)*.*

Proof We apply Theorem 5 to the pure integer optimization problem:

$$\begin{aligned} \max \quad & \tilde{f}(\tilde{\mathbf{x}}, \mathbf{z}) \\ \text{s.t.} \quad & A\tilde{\mathbf{x}} + mB\mathbf{z} \leq m\mathbf{b} \\ & \tilde{x}_i \in \mathbf{Z} \qquad\qquad \text{for } i = 1, \dots, d_1, \\ & z_i \in \mathbf{Z} \qquad\qquad \text{for } i = 1, \dots, d_2, \end{aligned} \tag{9}$$

where $\tilde{f}(\tilde{\mathbf{x}}, \mathbf{z}) := m^D f(\frac{1}{m}\tilde{\mathbf{x}}, \mathbf{z})$ is a polynomial function with integer coefficients. Clearly the binary encoding length of the coefficients of $\tilde{f}$ increases by at most $\lceil D \log m \rceil$, compared to the coefficients of f. Likewise, the encoding length of the coefficients of mB and $m\mathbf{b}$ increases by at most $\lceil \log m \rceil$. By Theorem 1.1 of [6], there exists an algorithm with running time polynomial in the encoding length of $\tilde{f}$ and of $A\mathbf{x} + mB\mathbf{z} \leq m\mathbf{b}$, the maximum total degree D, and $\frac{1}{\varepsilon}$ for computing a feasible solution $(\mathbf{x}^m_\varepsilon, \mathbf{z}^m_\varepsilon) \in P \cap \left(\frac{1}{m}\mathbf{Z}^{d_1} \times \mathbf{Z}^{d_2}\right)$ such that $\tilde{f}(\mathbf{x}^m_\varepsilon, \mathbf{z}^m_\varepsilon) \geq (1-\varepsilon)\tilde{f}(\mathbf{x}^m, \mathbf{z}^m)$, which implies the estimate (8). □

One might be tempted to think that for large-enough choice of m, we immediately obtain an approximation to the mixed-integer optimum with arbitrary precision. However, this is not true, as the following example demonstrates.

Example 7 Consider the mixed-integer optimization problem

$$\begin{aligned} \max \quad & 2z - x \\ \text{s.t.} \quad & z \leq 2x \\ & z \leq 2(1-x) \\ & x \in \mathbf{R}_{\geq 0},\ z \in \{0,1\}, \end{aligned} \tag{10}$$

whose feasible region consists of the point $(\frac{1}{2}, 1)$ and the segment $\{\,(x,0) : x \in [0,1]\}$. The unique optimal solution to (10) is $x = \frac{1}{2}$, $z = 1$. Now consider the sequence of grid approximations of (10) where $x \in \frac{1}{m}\mathbf{Z}_{\geq 0}$. For even m, the unique optimal solution to the grid approximation is $x = \frac{1}{2}$, $z = 1$. However, for odd m, the unique optimal solution is $x = 0$, $z = 0$. Thus the full sequence of the optimal solutions to the grid approximations does not converge since it has two limit points; see Figure 1.

Even though taking the limit does not work, taking the upper limit does. More strongly, we can prove that it is possible to construct, in polynomial time, a subsequence of finer and finer grids that contain a lattice point $(\lfloor \mathbf{x}^* \rceil_\delta, \mathbf{z}^*)$ that is arbitrarily close to the mixed-integer optimum $(\mathbf{x}^*, \mathbf{z}^*)$. This is the central statement of this section and a basic building block of the approximation result.

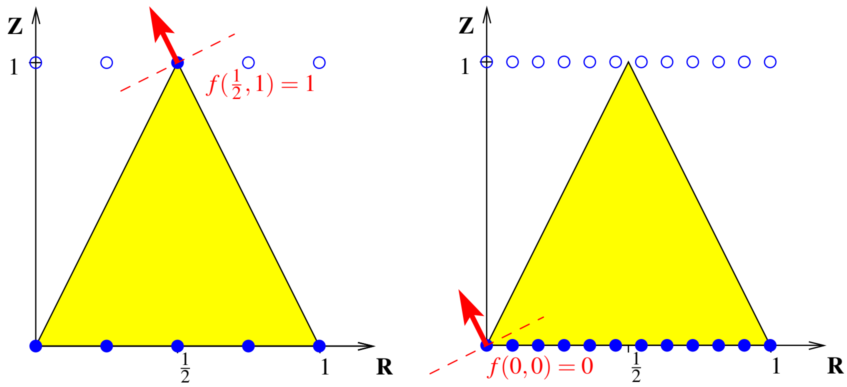

Fig. 1 A sequence of optimal solutions to grid problems with two limit points, for even m and for odd m

Theorem 8 (Grid Approximation) *Let d_1 be fixed. Let $P = \{\, (\mathbf{x},\mathbf{z}) \in \mathbf{R}^{d_1+d_2} : A\mathbf{x} + B\mathbf{z} \leq \mathbf{b} \,\}$, where $A \in \mathbf{Z}^{p\times d_1}$, $B \in \mathbf{Z}^{p\times d_2}$. Let $M \in \mathbf{R}$ be given such that $P \subseteq \{\, (\mathbf{x},\mathbf{z}) \in \mathbf{R}^{d_1+d_2} : |x_i| \leq M \text{ for } i = 1,\dots,d_1 \,\}$. There exists a polynomial-time algorithm to compute a number Δ such that for every $(\mathbf{x}^*,\mathbf{z}^*) \in P \cap (\mathbf{R}^{d_1} \times \mathbf{Z}^{d_2})$ and $\delta > 0$ the following property holds:*

> *Every lattice $\frac{1}{m}\mathbf{Z}^{d_1}$ for $m = k\Delta$ and $k \geq \frac{2}{\delta} d_1 M$ contains a lattice point $\lfloor \mathbf{x}^* \rceil_\delta$ such that $(\lfloor \mathbf{x}^* \rceil_\delta, \mathbf{z}^*) \in P \cap \left(\frac{1}{m}\mathbf{Z}^{d_1} \times \mathbf{Z}^{d_2}\right)$ and $\left\| \lfloor \mathbf{x}^* \rceil_\delta - \mathbf{x}^* \right\|_\infty \leq \delta$.*

The geometry of Theorem 8 is illustrated in Figure 2. The notation $\lfloor \mathbf{x}^* \rceil_\delta$ has been chosen to suggest that the coordinates of $\mathbf{x}^*$ have been "rounded" to obtain a nearby lattice point. The rounding method is provided by the next two lemmas; Theorem 8 follows directly from them.

Lemma 9 (Integral Scaling Lemma) *Let $P = \{\, (\mathbf{x},\mathbf{z}) \in \mathbf{R}^{d_1+d_2} : A\mathbf{x} + B\mathbf{z} \leq \mathbf{b} \,\}$, where $A \in \mathbf{Z}^{p\times d_1}$, $B \in \mathbf{Z}^{p\times d_2}$. For fixed d_1, there exists a polynomial time algorithm to compute a number $\Delta \in \mathbf{Z}_{>0}$ such that for every $\mathbf{z} \in \mathbf{Z}^{d_2}$ the polytope*

$$\Delta P_{\mathbf{z}} = \left\{\, \Delta \mathbf{x} : (\mathbf{x},\mathbf{z}) \in P \,\right\}$$

is integral, i.e., all vertices have integer coordinates. In particular, the number Δ has an encoding length that is bounded by a polynomial in the encoding length of P.

Proof Because the dimension d_1 is fixed, there exist only polynomially many simplex bases of the inequality system $A\mathbf{x} \leq \mathbf{b} - B\mathbf{z}$, and they can be enumerated in polynomial time. The determinant of each simplex basis can be computed in polynomial time. Then Δ can be chosen as the least common multiple of all these determinants. □

Lemma 10 *Let $Q \subset \mathbf{R}^d$ be an integral polytope. Let $M \in \mathbf{R}$ be such that $Q \subseteq \{\, \mathbf{x} \in \mathbf{R}^d : |x_i| \leq M \text{ for } i = 1,\dots,d \,\}$. Let $\mathbf{x}^* \in Q$ and let $\delta > 0$. Then every lattice $\frac{1}{k}\mathbf{Z}^d$ for $k \geq \frac{2}{\delta} dM$ contains a lattice point $\mathbf{x} \in Q \cap \frac{1}{k}\mathbf{Z}^d$ with $\|\mathbf{x} - \mathbf{x}^*\|_\infty \leq \delta$.*

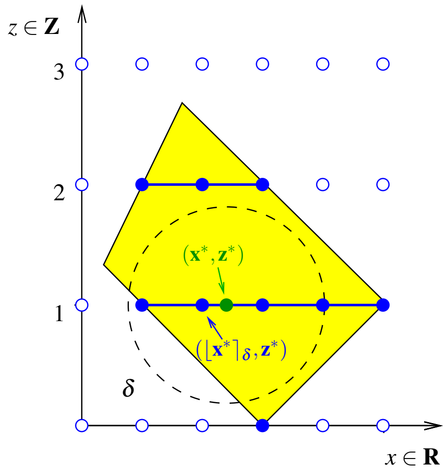

Fig. 2 The principle of grid approximation. Since we can refine the grid only in the direction of the continuous variables, we need to construct an approximating grid point $(\mathbf{x}, \mathbf{z}^*)$ in the same integral slice as the target point $(\mathbf{x}^*, \mathbf{z}^*)$.

Proof By Carathéodory's Theorem, there exist $d+1$ vertices $\mathbf{x}^0, \dots, \mathbf{x}^d \in \mathbf{Z}^d$ of Q and convex multipliers $\lambda_0, \dots, \lambda_d$ such that $\mathbf{x}^* = \sum_{i=0}^d \lambda_i \mathbf{x}^i$. Let $\lambda_i' := \frac{1}{k}\lfloor k\lambda_i \rfloor \geq 0$ for $i = 1, \dots, d$ and $\lambda_0' := 1 - \sum_{i=1}^d \lambda_i' \geq 0$. Moreover, we conclude $\lambda_i - \lambda_i' \leq \frac{1}{k}$ for $i = 1, \dots, d$ and $\lambda_0' - \lambda_0 = \sum_{i=1}^d (\lambda_i - \lambda_i') \leq d\frac{1}{k}$. Then $\mathbf{x} := \sum_{i=0}^d \lambda_i' \mathbf{x}^i \in Q \cap \frac{1}{k}\mathbf{Z}^d$, and we have

$$\|\mathbf{x} - \mathbf{x}^*\|_\infty \leq \sum_{i=0}^{d} |\lambda_i' - \lambda_i| \|\mathbf{x}^i\|_\infty \leq 2d\frac{1}{k}M \leq \delta,$$

which proves the lemma. □

4 Bounding techniques

Using the results of section 3 we are now able to approximate the mixed-integer optimal point by a point of a suitably fine lattice. The question arises how we can use the geometric distance of these two points to estimate the difference in objective function values. We prove Lemma 11 that provides us with a local Lipschitz constant for the polynomial to be maximized.

Lemma 11 (Local Lipschitz constant) *Let f be a polynomial in d variables with maximum total degree D. Let C denote the largest absolute value of a coefficient of f. Then there exists a Lipschitz constant L such that $|f(\mathbf{x}) - f(\mathbf{y})| \leq L\|\mathbf{x} - \mathbf{y}\|_\infty$ for all $|x_i|, |y_i| \leq M$. The constant L is $O(D^{d+1}CM^D)$.*

Proof Let $f(\mathbf{x}) = \sum_{\boldsymbol{\alpha} \in \mathscr{D}} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}}$, where $\mathscr{D} \subseteq \mathbf{Z}^d_{\geq 0}$ is the set of exponent vectors of monomials appearing in f. Let $r = |\mathscr{D}|$ be the number of monomials of f. Then we have

$$|f(\mathbf{x}) - f(\mathbf{y})| \leq \sum_{\boldsymbol{\alpha} \neq \mathbf{0}} |c_{\boldsymbol{\alpha}}|\, |\mathbf{x}^{\boldsymbol{\alpha}} - \mathbf{y}^{\boldsymbol{\alpha}}|.$$

We estimate all summands separately. Let $\boldsymbol{\alpha} \neq \mathbf{0}$ be an exponent vector with $n := \sum_{i=1}^{d} \alpha_i \leq D$. Let

$$\boldsymbol{\alpha} = \boldsymbol{\alpha}^0 \geq \boldsymbol{\alpha}^1 \geq \cdots \geq \boldsymbol{\alpha}^n = \mathbf{0}$$

be a decreasing chain of exponent vectors with $\boldsymbol{\alpha}^{i-1} - \boldsymbol{\alpha}^i = \mathbf{e}^{j_i}$ for $i = 1, \dots, n$. Let $\boldsymbol{\beta}^i := \boldsymbol{\alpha} - \boldsymbol{\alpha}^i$ for $i = 0, \dots, n$. Then $\mathbf{x}^{\boldsymbol{\alpha}} - \mathbf{y}^{\boldsymbol{\alpha}}$ can be expressed as the "telescope sum"

$$\begin{aligned}
\mathbf{x}^{\boldsymbol{\alpha}} - \mathbf{y}^{\boldsymbol{\alpha}} &= \mathbf{x}^{\boldsymbol{\alpha}^0}\mathbf{y}^{\boldsymbol{\beta}^0} - \mathbf{x}^{\boldsymbol{\alpha}^1}\mathbf{y}^{\boldsymbol{\beta}^1} + \mathbf{x}^{\boldsymbol{\alpha}^1}\mathbf{y}^{\boldsymbol{\beta}^1} - \mathbf{x}^{\boldsymbol{\alpha}^2}\mathbf{y}^{\boldsymbol{\beta}^2} + - \cdots - \mathbf{x}^{\boldsymbol{\alpha}^n}\mathbf{y}^{\boldsymbol{\beta}^n} \\
&= \sum_{i=1}^{n} \left(\mathbf{x}^{\boldsymbol{\alpha}^{i-1}}\mathbf{y}^{\boldsymbol{\beta}^{i-1}} - \mathbf{x}^{\boldsymbol{\alpha}^i}\mathbf{y}^{\boldsymbol{\beta}^i} \right) \\
&= \sum_{i=1}^{n} \left((x_{j_i} - y_{j_i}) \mathbf{x}^{\boldsymbol{\alpha}^i}\mathbf{y}^{\boldsymbol{\beta}^{i-1}} \right).
\end{aligned}$$

Since $\left|\mathbf{x}^{\boldsymbol{\alpha}^i}\mathbf{y}^{\boldsymbol{\beta}^{i-1}}\right| \leq M^{n-1}$ and $n \leq D$, we obtain

$$|\mathbf{x}^{\boldsymbol{\alpha}} - \mathbf{y}^{\boldsymbol{\alpha}}| \leq D \cdot \|\mathbf{x} - \mathbf{y}\|_\infty \cdot M^{n-1},$$

thus

$$|f(\mathbf{x}) - f(\mathbf{y})| \leq CrDM^{D-1} \|\mathbf{x} - \mathbf{y}\|_\infty.$$

Let $L := CrDM^{D-1}$. Now, since $r = O(D^d)$, we have $L = O(D^{d+1}CM^D)$. □

Moreover, in order to obtain an FPTAS, we need to put differences of function values in relation to the maximum function value. To do this, we need to deal with the special case of polynomials that are constant on the feasible region; here trivially every feasible solution is optimal. For non-constant polynomials, we can prove a lower bound on the maximum function value. The technique is to bound the difference of the minimum and the maximum function value on the mixed-integer set from below; if the polynomial is non-constant, this implies, for a non-negative polynomial, a lower bound on the maximum function value. We will need a simple fact about the roots of multivariate polynomials.

Lemma 12 *Let $f \in \mathbf{Q}[x_1, \dots, x_d]$ be a polynomial and let D be the largest power of any variable that appears in f. Then $f = 0$ if and only if f vanishes on the set $\{0, \dots, D\}^d$.*

Proof This is a simple consequence of the Fundamental Theorem of Algebra. See, for instance, [4, Chapter 1, §1, Exercise 6 b]. □

Lemma 13 *Let $f \in \mathbf{Q}[x_1, \dots, x_d]$ be a polynomial with maximum total degree D. Let $Q \subset \mathbf{R}^d$ be an integral polytope of dimension $d' \leq d$. Let $k \geq Dd'$. Then f is constant on Q if and only if f is constant on $Q \cap \frac{1}{k}\mathbf{Z}^d$.*

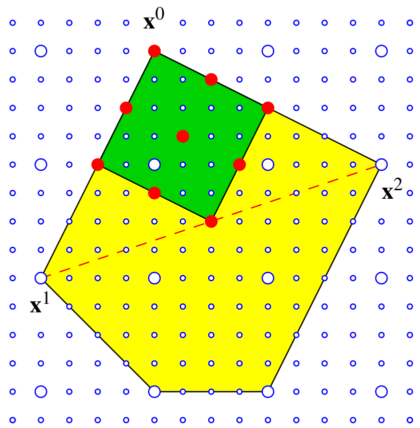

Fig. 3 The geometry of Lemma 13. For a polynomial with maximum total degree of 2, we construct a refinement $\frac{1}{k}\mathbf{Z}^d$ (small circles) of the standard lattice (large circles) such that $P \cap \frac{1}{k}\mathbf{Z}^d$ contains an affine image of the set $\{0,1,2\}^d$ (large dots).

Proof Let $\mathbf{x}^0 \in Q \cap \mathbf{Z}^d$ be an arbitrary vertex of Q. There exist vertices $\mathbf{x}^1, \dots, \mathbf{x}^{d'} \in Q \cap \mathbf{Z}^d$ such that the vectors $\mathbf{x}^1 - \mathbf{x}^0, \dots, \mathbf{x}^{d'} - \mathbf{x}^0 \in \mathbf{Z}^d$ are linearly independent. By convexity, Q contains the parallelepiped

$$S := \left\{ \mathbf{x}^0 + \textstyle\sum_{i=1}^{d'} \lambda_i (\mathbf{x}^i - \mathbf{x}^0) : \lambda_i \in [0, \frac{1}{d'}] \text{ for } i = 1, \dots, d' \right\}.$$

We consider the set

$$S_k = \tfrac{1}{k}\mathbf{Z}^d \cap S \supseteq \left\{ \mathbf{x}^0 + \textstyle\sum_{i=1}^{d'} \frac{n_i}{k} (\mathbf{x}^i - \mathbf{x}^0) : n_i \in \{0, 1, \dots, D\} \text{ for } i = 1, \dots, d' \right\};$$

see Figure 3. Now if there exists a $c \in \mathbf{R}$ with $f(\mathbf{x}) = c$ for all $\mathbf{x} \in Q \cap \frac{1}{k}\mathbf{Z}^d$, then all the points in S_k are roots of the polynomial $f - c$, which has only maximum total degree D. By Lemma 12 (after an affine transformation), $f - c$ is zero on the affine hull of S_k; hence f is constant on the polytope Q. □

Theorem 14 *Let $f \in \mathbf{Z}[x_1, \dots, x_{d_1}, z_1, \dots, z_{d_2}]$. Let P be a rational convex polytope, and let Δ be the number from Lemma 9. Let $m = k\Delta$ with $k \geq D d_1$, $k \in \mathbf{Z}$. Then f is constant on the feasible region $P \cap \left(\mathbf{R}^{d_1} \times \mathbf{Z}^{d_2}\right)$ if and only if f is constant on $P \cap \left(\frac{1}{m}\mathbf{Z}^{d_1} \times \mathbf{Z}^{d_2}\right)$. If f is not constant, then*

$$\left| f(\mathbf{x}_{\max}, \mathbf{z}_{\max}) - f(\mathbf{x}_{\min}, \mathbf{z}_{\min}) \right| \geq m^{-D}, \tag{11}$$

where $(\mathbf{x}_{\max}, \mathbf{z}_{\max})$ is an optimal solution to the maximization problem over the feasible region $P \cap \left(\mathbf{R}^{d_1} \times \mathbf{Z}^{d_2}\right)$ and $(\mathbf{x}_{\min}, \mathbf{z}_{\min})$ is an optimal solution to the minimization problem.

Proof Let f be constant on $P \cap \left(\frac{1}{m}\mathbf{Z}^{d_1} \times \mathbf{Z}^{d_2}\right)$. For fixed integer part $\mathbf{z} \in \mathbf{Z}^{d_2}$, we consider the polytope $\Delta P_{\mathbf{z}} = \left\{ \Delta \mathbf{x} : (\mathbf{x}, \mathbf{z}) \in P \right\}$, which is a slice of P scaled to become an integral polytope. By applying Lemma 13 with $k = (D+1)d$ on every polytope $\Delta P_{\mathbf{z}}$, we obtain that f is constant on every slice $P_{\mathbf{z}}$. Because f is also

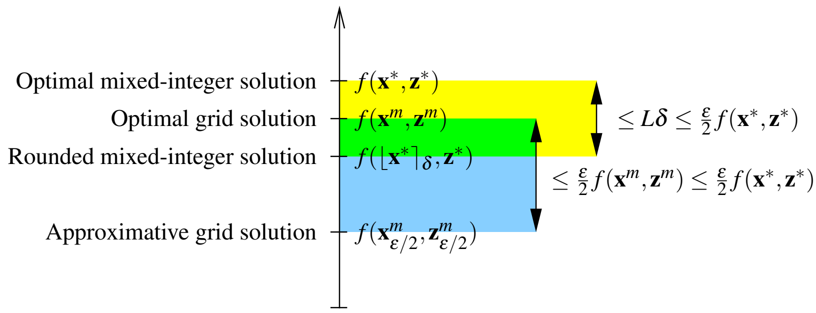

Fig. 4 Estimates in the proof of Theorem 1 (a)

constant on the set $P \cap \left(\frac{1}{m}\mathbf{Z}^{d_1} \times \mathbf{Z}^{d_2}\right)$, which contains a point of every non-empty slice $P_\mathbf{z}$, it follows that f is constant on P.

If f is not constant, there exist $(\mathbf{x}^1,\mathbf{z}^1)$, $(\mathbf{x}^2,\mathbf{z}^2) \in P \cap \left(\frac{1}{m}\mathbf{Z}^{d_1} \times \mathbf{Z}^{d_2}\right)$ with $f(\mathbf{x}^1,\mathbf{z}^1) \neq f(\mathbf{x}^2,\mathbf{z}^2)$. By the integrality of all coefficients of f, we obtain the estimate

$$|f(\mathbf{x}^1,\mathbf{z}^1) - f(\mathbf{x}^2,\mathbf{z}^2)| \geq m^{-D}.$$

Because $(\mathbf{x}^1,\mathbf{z}^1)$, $(\mathbf{x}^2,\mathbf{z}^2)$ are both feasible solutions to the maximization problem and the minimization problem, this implies (11). □

5 Proof of Theorem 1

Now we are in the position to prove the main result.

Proof (Proof of Theorem 1) Part (a). Let $(\mathbf{x}^*,\mathbf{z}^*)$ denote an optimal solution to the mixed-integer problem (1). Let $\varepsilon > 0$. We show that, in time polynomial in the input length, the maximum total degree, and $\frac{1}{\varepsilon}$, we can compute a point $(\mathbf{x},\mathbf{z})$ that satisfies (1b–1d) such that

$$|f(\mathbf{x},\mathbf{z}) - f(\mathbf{x}^*,\mathbf{z}^*)| \leq \varepsilon f(\mathbf{x}^*,\mathbf{z}^*). \tag{12}$$

We prove this by establishing several estimates, which are illustrated in Figure 4.

First we note that we can restrict ourselves to the case of polynomials with integer coefficients, simply by multiplying f with the least common multiple of all denominators of the coefficients. We next establish a lower bound on $f(\mathbf{x}^*,\mathbf{z}^*)$. To this end, let Δ be the integer from Lemma 9, which can be computed in polynomial time. By Theorem 14 with $m = Dd_1\Delta$, either f is constant on the feasible region, or

$$f(\mathbf{x}^*,\mathbf{z}^*) \geq (Dd_1\Delta)^{-D}, \tag{13}$$

where D is the maximum total degree of f. Now let

$$\delta := \frac{\varepsilon}{2(Dd_1\Delta)^D L(C,D,M)} \tag{14}$$

and let us choose the grid size

$$m := \Delta \left\lceil \frac{4}{\varepsilon}(Dd_1\Delta)^D L(C,D,M) d_1 M \right\rceil, \tag{15}$$

where $L(C,D,M)$ is the Lipschitz constant from Lemma 11. Then we have $m \geq \Delta \frac{2}{\delta} d_1 M$, so by Theorem 8, there is a point $\left(\lfloor \mathbf{x}^* \rceil_\delta, \mathbf{z}^*\right) \in P \cap \left(\frac{1}{m}\mathbf{Z}^{d_1} \times \mathbf{Z}^{d_2}\right)$ with $\left\| \lfloor \mathbf{x}^* \rceil_\delta - \mathbf{x}^* \right\|_\infty \leq \delta$. Let $(\mathbf{x}^m, \mathbf{z}^m)$ denote an optimal solution to the grid problem (7). Because $\left(\lfloor \mathbf{x}^* \rceil_\delta, \mathbf{z}^*\right)$ is a feasible solution to the grid problem (7), we have

$$f\left(\lfloor \mathbf{x}^* \rceil_\delta, \mathbf{z}^*\right) \leq f(\mathbf{x}^m, \mathbf{z}^m) \leq f(\mathbf{x}^*, \mathbf{z}^*). \tag{16}$$

Now we can estimate

$$\begin{aligned}
\left| f(\mathbf{x}^*, \mathbf{z}^*) - f(\mathbf{x}^m, \mathbf{z}^m) \right| &\leq \left| f(\mathbf{x}^*, \mathbf{z}^*) - f\left(\lfloor \mathbf{x}^* \rceil_\delta, \mathbf{z}^*\right) \right| \\
&\leq L(C,D,M) \left\| \mathbf{x}^* - \lfloor \mathbf{x}^* \rceil_\delta \right\|_\infty \\
&\leq L(C,D,M)\, \delta \\
&= \frac{\varepsilon}{2} (D d_1 \Delta)^{-D} \\
&\leq \frac{\varepsilon}{2} f(\mathbf{x}^*, \mathbf{z}^*),
\end{aligned} \tag{17}$$

where the last estimate is given by (13) in the case that f is not constant on the feasible region. On the other hand, if f is constant, the estimate (17) holds trivially.

By Corollary 6 we can compute a point $(\mathbf{x}^m_{\varepsilon/2}, \mathbf{z}^m_{\varepsilon/2}) \in P \cap \left(\frac{1}{m}\mathbf{Z}^{d_1} \times \mathbf{Z}^{d_2}\right)$ such that

$$(1 - \tfrac{\varepsilon}{2}) f(\mathbf{x}^m, \mathbf{z}^m) \leq f(\mathbf{x}^m_{\varepsilon/2}, \mathbf{z}^m_{\varepsilon/2}) \leq f(\mathbf{x}^m, \mathbf{z}^m) \tag{18}$$

in time polynomial in $\log m$, the encoding length of f and P, the maximum total degree D, and $1/\varepsilon$. Here $\log m$ is bounded by a polynomial in $\log M$, D and $\log C$, so we can compute $(\mathbf{x}^m_{\varepsilon/2}, \mathbf{z}^m_{\varepsilon/2})$ in time polynomial in the input size, the maximum total degree D, and $1/\varepsilon$. Now, using (18) and (17), we can estimate

$$\begin{aligned}
& f(\mathbf{x}^*, \mathbf{z}^*) - f(\mathbf{x}^m_{\varepsilon/2}, \mathbf{z}^m_{\varepsilon/2}) \\
&\quad \leq f(\mathbf{x}^*, \mathbf{z}^*) - (1 - \tfrac{\varepsilon}{2}) f(\mathbf{x}^m, \mathbf{z}^m) \\
&\quad = \tfrac{\varepsilon}{2} f(\mathbf{x}^*, \mathbf{z}^*) + (1 - \tfrac{\varepsilon}{2}) \big(f(\mathbf{x}^*, \mathbf{z}^*) - f(\mathbf{x}^m, \mathbf{z}^m) \big) \\
&\quad \leq \tfrac{\varepsilon}{2} f(\mathbf{x}^*, \mathbf{z}^*) + \tfrac{\varepsilon}{2} f(\mathbf{x}^*, \mathbf{z}^*) \\
&\quad = \varepsilon f(\mathbf{x}^*, \mathbf{z}^*).
\end{aligned}$$

Hence $f(\mathbf{x}^m_{\varepsilon/2}, \mathbf{z}^m_{\varepsilon/2}) \geq (1 - \varepsilon) f(\mathbf{x}^*, \mathbf{z}^*)$.

Part (b). Let the dimension $d \geq 2$ be fixed. We prove that there does not exist a PTAS for the maximization of arbitrary polynomials over mixed-integer sets of polytopes. We use the NP-complete problem AN1 on page 249 of [8]. This is to decide whether, given three positive integers a, b, c, there exists a positive integer $x < c$ such that $x^2 \equiv a \pmod{b}$. This problem is equivalent to asking whether the maximum of the quartic polynomial function $f(x,y) = -(x^2 - a - by)^2$ over the lattice points of the rectangle

$$P = \left\{ (x,y) : 1 \leq x \leq c-1,\ \frac{1-a}{b} \leq y \leq \frac{(c-1)^2 - a}{b} \right\}$$

is zero or not. If there existed a PTAS for the maximization of arbitrary polynomials over mixed-integer sets of polytopes, we could, for any fixed $0 < \varepsilon < 1$, compute in polynomial time a solution $(x_\varepsilon, y_\varepsilon) \in P \cap \mathbf{Z}^2$ with $\left| f(x_\varepsilon, y_\varepsilon) - f(x^*, y^*) \right| \leq \varepsilon \left| f(x^*, y^*) \right|$, where (x^*, y^*) denotes an optimal solution. Thus, we have $f(x_\varepsilon, y_\varepsilon) = 0$ if and only if $f(x^*, y^*) = 0$; this means we could solve the problem AN1 in polynomial time. $\square$

6 Extension to arbitrary polynomials

In this section we drop the requirement of the polynomial being positive over the feasible region. As we showed in Theorem 1, there does not exist a PTAS for the maximization of an arbitrary polynomial over polytopes in fixed dimension. We will instead show an approximation result like the one in [11], i.e., we compute a solution $(\mathbf{x}_\varepsilon, \mathbf{z}_\varepsilon)$ such that

$$\left| f(\mathbf{x}_\varepsilon, \mathbf{z}_\varepsilon) - f(\mathbf{x}_{\max}, \mathbf{z}_{\max}) \right| \leq \varepsilon \left| f(\mathbf{x}_{\max}, \mathbf{z}_{\max}) - f(\mathbf{x}_{\min}, \mathbf{z}_{\min}) \right|, \tag{19}$$

where $(\mathbf{x}_{\max}, \mathbf{z}_{\max})$ is an optimal solution to the maximization problem over the feasible region and $(\mathbf{x}_{\min}, \mathbf{z}_{\min})$ is an optimal solution to the minimization problem. Our algorithm has a running time that is polynomial in the input size, the maximum total degree of f, and $\frac{1}{\varepsilon}$. This means that while the result of [11] was a weak version of a PTAS (for fixed degree), our result is a weak version of an FPTAS (for fixed dimension).

The approximation algorithms for the integer case (Theorem 5) and the mixed-integer case (Theorem 1) only work for polynomial objective functions that are non-negative on the feasible region. In order to apply them to an arbitrary polynomial objective function f, we need to add a constant term to f that is large enough. As proposed in [6], we can use linear programming techniques to obtain a bound M on the variables and then estimate

$$f(\mathbf{x}) \geq -rCM^D =: L_0,$$

where C is the largest absolute value of a coefficient, r is the number of monomials of f, and D is the maximum total degree. However, the range $\left| f(\mathbf{x}_{\max}, \mathbf{z}_{\max}) - f(\mathbf{x}_{\min}, \mathbf{z}_{\min}) \right|$ can be exponentially small compared to L_0, so in order to obtain an approximation $(\mathbf{x}_\varepsilon, \mathbf{z}_\varepsilon)$ satisfying (19), we would need an $(1-\varepsilon')$-approximation to the problem of maximizing $g(\mathbf{x}, \mathbf{z}) := f(\mathbf{x}, \mathbf{z}) - L_0$ with an exponentially small value of ε'.

To address this difficulty, we will first apply an algorithm which will compute an approximation $[L_i, U_i]$ of the range $[f(\mathbf{x}_{\min}, \mathbf{z}_{\min}), f(\mathbf{x}_{\max}, \mathbf{z}_{\max})]$ with constant quality. To this end, we first prove a simple corollary of Theorem 1.

Corollary 15 (Computation of upper bounds for mixed-integer problems) *Let the dimension $d = d_1 + d_2$ be fixed. Let $P \subseteq \mathbf{R}^d$ be a rational convex polytope. Let $f \in \mathbf{Z}[x_1, \ldots, x_{d_1}, z_1, \ldots, z_{d_2}]$ be a polynomial function with integer coefficientsand maximum total degree D that is non-negative on $P \cap \left(\mathbf{R}^{d_1} \times \mathbf{Z}^{d_2}\right)$. Let $\delta > 0$. There exists an algorithm with running time polynomial in the input size, D, and $\frac{1}{\delta}$ for computing an upper bound u such that*

$$f(\mathbf{x}_{\max}, \mathbf{z}_{\max}) \leq u \leq (1+\delta) f(\mathbf{x}_{\max}, \mathbf{z}_{\max}), \tag{20}$$

where $(\mathbf{x}_{\max}, \mathbf{z}_{\max})$ is an optimal solution to the maximization problem of f over $P \cap \left(\mathbf{R}^{d_1} \times \mathbf{Z}^{d_2}\right)$.

Proof Let $\varepsilon = \frac{\delta}{1+\delta}$. By Theorem 1, we can, in time polynomial in the input size, D, and $\frac{1}{\varepsilon} = 1 + \frac{1}{\delta}$, compute a solution $(\mathbf{x}_\varepsilon, \mathbf{z}_\varepsilon)$ with

$$\left| f(\mathbf{x}_{\max}, \mathbf{z}_{\max}) - f(\mathbf{x}_\varepsilon, \mathbf{z}_\varepsilon) \right| \leq \varepsilon f(\mathbf{x}_{\max}, \mathbf{z}_{\max}). \tag{21}$$

Let $u := \frac{1}{1-\varepsilon} f(\mathbf{x}_\varepsilon, \mathbf{z}_\varepsilon) = (1+\delta) f(\mathbf{x}_\varepsilon, \mathbf{z}_\varepsilon)$. Then

$$f(\mathbf{x}_{\max}, \mathbf{z}_{\max}) \leq \frac{1}{1-\varepsilon} f(\mathbf{x}_\varepsilon, \mathbf{z}_\varepsilon) = u \tag{22}$$

and

$$\begin{aligned} (1+\delta) f(\mathbf{x}_{\max}, \mathbf{z}_{\max}) &\geq (1+\delta) f(\mathbf{x}_\varepsilon, \mathbf{z}_\varepsilon) \\ &= (1+\delta)(1-\varepsilon) u \\ &= (1+\delta)\left(1 - \frac{\delta}{1+\delta}\right) u = u. \end{aligned} \tag{23}$$

This proves the estimate (20). □

Algorithm 16 (Range approximation)
Input: Mixed-integer polynomial optimization problem (1), a number $0 < \delta < 1$.
Output: Sequences $\{L_i\}$, $\{U_i\}$ of lower and upper bounds of f over the feasible region $P \cap \left(\mathbf{R}^{d_1} \times \mathbf{Z}^{d_2}\right)$ such that

$$L_i \leq f(\mathbf{x}_{\min}, \mathbf{z}_{\min}) \leq f(\mathbf{x}_{\max}, \mathbf{z}_{\max}) \leq U_i \tag{24}$$

and

$$\lim_{i \to \infty} |U_i - L_i| = c\big(f(\mathbf{x}_{\max}, \mathbf{z}_{\max}) - f(\mathbf{x}_{\min}, \mathbf{z}_{\min})\big), \tag{25}$$

where c depends only on the choice of δ.

1. By solving $2d$ linear programs over P, we find lower and upper integer bounds for each of the variables $x_1, \ldots, x_{d_1}, z_1, \ldots, z_{d_2}$. Let M be the maximum of the absolute values of these $2d$ numbers. Thus $|x_i|, |z_i| \leq M$ for all i. Let C be the maximum of the absolute values of all coefficients, and r be the number of monomials of $f(x)$. Then

$$L_0 := -rCM^D \leq f(\mathbf{x}, \mathbf{z}) \leq rCM^D =: U_0,$$

as we can bound the absolute value of each monomial of $f(x)$ by CM^D.

2. Let $i := 0$.
3. Using the algorithm of Corollary 15, compute an upper bound u for the problem

$$\begin{aligned} \max \quad & g(\mathbf{x},\mathbf{z}) := f(\mathbf{x},\mathbf{z}) - L_i \\ \text{s.t.} \quad & (\mathbf{x},\mathbf{z}) \in P \cap \left(\mathbf{R}^{d_1} \times \mathbf{Z}^{d_2}\right) \end{aligned}$$

   that gives a $(1+\delta)$-approximation to the optimal value. Let $U_{i+1} := L_i + u$.
4. Likewise, compute an upper bound u for the problem

$$\begin{aligned} \max \quad & h(\mathbf{x},\mathbf{z}) := U_i - f(\mathbf{x},\mathbf{z}) \\ \text{s.t.} \quad & (\mathbf{x},\mathbf{z}) \in P \cap \left(\mathbf{R}^{d_1} \times \mathbf{Z}^{d_2}\right) \end{aligned}$$

   that gives a $(1+\delta)$-approximation to the optimal value. Let $L_{i+1} := U_i - u$.
5. $i := i+1$.
6. Go to 3.

Lemma 17 *Algorithm 16 is correct. For fixed* $0 < \delta < 1$, *it computes the bounds* L_n, U_n *satisfying* (24) *and* (25) *in time polynomial in the input size and* n.

Proof We have

$$U_i - L_{i+1} \leq (1+\delta)\big(U_i - f(\mathbf{x}_{\min},\mathbf{z}_{\min})\big) \tag{26}$$

and

$$U_{i+1} - L_i \leq (1+\delta)\big(f(\mathbf{x}_{\max},\mathbf{z}_{\max}) - L_i\big). \tag{27}$$

This implies

$$U_{i+1} - L_{i+1} \leq \delta(U_i - L_i) + (1+\delta)\big(f(\mathbf{x}_{\max},\mathbf{z}_{\max}) - f(\mathbf{x}_{\min},\mathbf{z}_{\min})\big).$$

Therefore

$$\begin{aligned} U_n - L_n &\leq \delta^n(U_0 - L_0) + (1+\delta)\left(\sum_{i=0}^{n-2}\delta^i\right)\big(f(\mathbf{x}_{\max},\mathbf{z}_{\max}) - f(\mathbf{x}_{\min},\mathbf{z}_{\min})\big) \\ &= \delta^n(U_0 - L_0) + (1+\delta)\frac{1-\delta^{n-1}}{1-\delta}\big(f(\mathbf{x}_{\max},\mathbf{z}_{\max}) - f(\mathbf{x}_{\min},\mathbf{z}_{\min})\big) \\ &\to \frac{1+\delta}{1-\delta}\big(f(\mathbf{x}_{\max},\mathbf{z}_{\max}) - f(\mathbf{x}_{\min},\mathbf{z}_{\min})\big) \quad (n \to \infty). \end{aligned}$$

The bound on the running time requires a careful analysis. Because in each step the result u (a rational number) of the bounding procedure (Corollary 15) becomes part of the input in the next iteration, the encoding length of the input could grow exponentially after only polynomially many steps. However, we will show that the encoding length only grows very slowly.

First we need to remark that the auxiliary objective functions g and h have integer coefficients except for the constant term, which may be rational. It turns out that the estimates in the proof of Theorem 1 (in particular, the local Lipschitz constant L and the lower bound on the optimal value) are independent from the constant term of the objective function. Therefore, the *same* approximating grid $\frac{1}{m}\mathbf{Z}^{d_1} \times \mathbf{Z}^{d_2}$ can be chosen in all iterations of Algorithm 16; the number m only depends on δ, the polytope P, the maximum total degree D, and the coefficients of f with the exception of the constant term.

The construction in the proof of Corollary 15 obtains the upper bound u by multiplying the approximation $f(\mathbf{x}_\varepsilon,\mathbf{z}_\varepsilon)$ by $(1+\delta)$. Therefore we have

$$\begin{aligned} U_{i+1} &= L_i + u \\ &= L_i + (1+\delta)\big(f(\mathbf{x}_\varepsilon,\mathbf{z}_\varepsilon) - L_i\big) \\ &= -\delta L_i + (1+\delta) f(\mathbf{x}_\varepsilon,\mathbf{z}_\varepsilon). \end{aligned} \tag{28}$$

Because the solution $(\mathbf{x}_\varepsilon,\mathbf{z}_\varepsilon)$ lies in the grid $\frac{1}{m}\mathbf{Z}^{d_1}\times\mathbf{Z}^{d_2}$, the value $f(\mathbf{x}_\varepsilon,\mathbf{z}_\varepsilon)$ is an integer multiple of m^{-D}. This implies that, because $L_0 \le f(\mathbf{x}_\varepsilon,\mathbf{z}_\varepsilon) \le U_0$, the encoding length of the rational number $f(\mathbf{x}_\varepsilon,\mathbf{z}_\varepsilon)$ is bounded by a polynomial in the input size of f and P. Therefore the encoding length U_{i+1} (and likewise L_{i+1}) only increases by an additive term that is bounded by a polynomial in the input size of f and P. ◻

We are now in the position to prove Theorem 2.

Proof (Proof of Theorem 2) Clearly we can restrict ourselves to polynomials with integer coefficients. Let $m = (D+1)d_1\Delta$, where Δ is the number from Theorem 8. We apply Algorithm 16 using $0<\delta<1$ arbitrary to compute bounds U_n and L_n for

$$n = \big\lceil -\log_\delta\big(2m^D(U_0-L_0)\big)\big\rceil.$$

Because n is bounded by a polynomial in the input size and the maximum total degree D, this can be done in polynomial time. Now, by the proof of Lemma 17, we have

$$\begin{aligned} U_n - L_n &\le \delta^n(U_0-L_0) + (1+\delta)\frac{1-\delta^{n-1}}{1-\delta}\big(f(\mathbf{x}_{\max},\mathbf{z}_{\max}) - f(\mathbf{x}_{\min},\mathbf{z}_{\min})\big) \\ &\le \frac{1}{2}m^{-D} + \frac{1+\delta}{1-\delta}\big(f(\mathbf{x}_{\max},\mathbf{z}_{\max}) - f(\mathbf{x}_{\min},\mathbf{z}_{\min})\big). \end{aligned} \tag{29}$$

If f is constant on $P\cap\big(\mathbf{R}^{d_1}\times\mathbf{Z}^{d_2}\big)$, it is constant on $P\cap\big(\frac{1}{m}\mathbf{Z}^{d_1}\times\mathbf{Z}^{d_2}\big)$, then $U_n - L_n \le \frac{1}{2}m^{-D}$. Otherwise, by Theorem 14, we have $U_n - L_n \ge f(\mathbf{x}_{\max},\mathbf{z}_{\max}) - f(\mathbf{x}_{\min},\mathbf{z}_{\min}) \ge m^{-D}$. This settles part (a).

For part (b), if f is constant on $P\cap\big(\mathbf{R}^{d_1}\times\mathbf{Z}^{d_2}\big)$, we return an arbitrary solution as an optimal solution. Otherwise, we can estimate further:

$$U_n - L_n \le \left(\frac{1}{2} + \frac{1+\delta}{1-\delta}\right)\big(f(\mathbf{x}_{\max},\mathbf{z}_{\max}) - f(\mathbf{x}_{\min},\mathbf{z}_{\min})\big). \tag{30}$$

Now we apply the algorithm of Theorem 1 to the maximization problem of the polynomial function $f' := f - L_n$, which is non-negative over the feasible region $P\cap\big(\mathbf{R}^{d_1}\times\mathbf{Z}^{d_2}\big)$. We compute a point $(\mathbf{x}_{\varepsilon'},\mathbf{z}_{\varepsilon'})$ where $\varepsilon' = \varepsilon\big(\frac{1}{2}+\frac{1+\delta}{1-\delta}\big)^{-1}$ such that

$$\big|f'(\mathbf{x}_{\varepsilon'},\mathbf{z}_{\varepsilon'}) - f'(\mathbf{x}_{\max},\mathbf{z}_{\max})\big| \le \varepsilon' f'(\mathbf{x}_{\max},\mathbf{z}_{\max}).$$

Then we obtain the estimate

$$\begin{aligned}\left|f(\mathbf{x}_{\varepsilon'},\mathbf{z}_{\varepsilon'})-f(\mathbf{x}_{\max},\mathbf{z}_{\max})\right| &\le \varepsilon'\bigl(f(\mathbf{x}_{\max},\mathbf{z}_{\max})-L_n\bigr)\\ &\le \varepsilon'\bigl(U_n-L_n\bigr)\\ &\le \varepsilon'\left(\frac{1}{2}+\frac{1+\delta}{1-\delta}\right)\bigl(f(\mathbf{x}_{\max},\mathbf{z}_{\max})-f(\mathbf{x}_{\min},\mathbf{z}_{\min})\bigr)\\ &= \varepsilon\bigl(f(\mathbf{x}_{\max},\mathbf{z}_{\max})-f(\mathbf{x}_{\min},\mathbf{z}_{\max})\bigr),\end{aligned}$$

which proves part (b). □

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