SIAM J. OPTIM.
Vol. 17, No. 3, pp. 806–821

ALL LINEAR AND INTEGER PROGRAMS ARE SLIM 3-WAY TRANSPORTATION PROGRAMS*

JESÚS A. DE LOERA† AND SHMUEL ONN‡

Abstract. We show that any rational convex polytope is polynomial-time representable as a 3-way line-sum transportation polytope of "slim" $(r, c, 3)$ format. This universality theorem has important consequences for linear and integer programming and for confidential statistical data disclosure. We provide a polynomial-time embedding of arbitrary linear programs and integer programs in such slim transportation programs and in bitransportation programs. Our construction resolves several standing problems on 3-way transportation polytopes. For example, it demonstrates that, unlike the case of 2-way contingency tables, the range of values an entry can attain in any slim 3-way contingency table with specified 2-margins can contain arbitrary gaps. Our smallest such example has format $(6, 4, 3)$. Our construction provides a powerful automatic tool for studying concrete questions about transportation polytopes and contingency tables. For example, it automatically provides new proofs for some classical results, including a well-known "real-feasible but integer-infeasible" $(6, 4, 3)$-transportation polytope of M. Vlach, and bitransportation programs where any feasible bitransportation must have an arbitrarily large prescribed denominator.

Key words. integer programming, linear programming, combinatorial optimization, convex polytopes, transportation problems, multicommodity flows, strongly polynomial time, contingency tables, multiway table, statistical table, data security, privacy, approximation algorithms, Markov basis, toric ideal, cofinetiality, disclosure

AMS subject classifications. 90C11, 52B55, 90B06, 68R05, 15A39, 62H17

DOI. 10.1137/040610623

1. Introduction.

Transportation polytopes, their integer points (called contingency tables by statisticians), and their projections have been used and studied extensively in the operations research and mathematical programming literature (see, e.g., [1, 2, 5, 17, 20, 23, 24, 29, 30] and references therein) and in the context of secure statistical data management by agencies such as the U.S. Census Bureau [28] (see, e.g., [3, 4, 9, 10, 13, 18, 22] and references therein).

We start right away with the statement of the main theorem of this article. Its proof will be the subject of section 3. Some of the many implications of the main theorem for linear and integer programming, combinatorial optimization, and confidential statistical data disclosure will be discussed in section 2. The consequences include the solution of several long-standing open questions stated by Vlach in 1986 [29]. Following a common convention we denote by $\mathbb{R}_{\geq 0}$ the nonnegative reals. In what follows, a 3-way transportation polytope is *slim* if one of its dimensions has depth three.

*Received by the editors June 27, 2004; accepted for publication (in revised form) April 12, 2006; published electronically October 4, 2006.

http://www.siam.org/journals/siopt/17-3/61062.html

†University of California at Davis, Davis, CA 95616 (deloera@math.ucdavis.edu, http://www.math.ucdavis.edu/∼deloera). The work of this author was supported in part by NSF grant 0309694, a 2003 UC-Davis Chancellor's fellow award, the Alexander von Humboldt foundation, and the American Institute of Mathematics.

‡Technion–Israel Institute of Technology, 32000 Haifa, Israel (onn@ie.technion.ac.il, http://ie.technion.ac.il/∼onn). The work of this author was supported in part by a grant from the Israel Science Foundation, the American Institute of Mathematics, the Technion President Fund, and the Jewish Communities of Germany Research Fund.

THEOREM 1.1. *Any rational polytope $P = \{y \in \mathbb{R}^n_{\geq 0} : Ay = b\}$ is polynomial-time representable as a slim 3-way transportation polytope:*

$$T = \left\{ x \in \mathbb{R}^{r\times c\times 3}_{\geq 0} \ : \ \sum_i x_{i,j,k} = w_{j,k}\ ,\ \sum_j x_{i,j,k} = v_{i,k}\ ,\ \sum_k x_{i,j,k} = u_{i,j} \right\}.$$

By saying that a polytope $P \subset \mathbb{R}^p$ is *representable* as a polytope $Q \subset \mathbb{R}^q$ we mean in the strong sense that there is an injection $\sigma : \{1, \ldots, p\} \longrightarrow \{1, \ldots, q\}$ such that the coordinate-erasing projection

$$\pi : \mathbb{R}^q \longrightarrow \mathbb{R}^p \ : \ x = (x_1, \ldots, x_q) \mapsto \pi(x) = (x_{\sigma(1)}, \ldots, x_{\sigma(p)})$$

provides a bijection between Q and P and between the sets of integer points $Q \cap \mathbb{Z}^q$ and $P \cap \mathbb{Z}^p$. In particular, if P is representable as Q, then P and Q are isomorphic in any reasonable sense: They are linearly equivalent, and hence all linear programming related problems over the two are polynomial-time equivalent; they are combinatorially equivalent and hence have the same facial structure; and they are integer equivalent, and therefore all integer programming and integer counting related problems over the two are polynomial-time equivalent as well. The polytope T in the theorem is a 3-way transportation polytope with specified *line-sums* $(u_{i,j}), (v_{i,k}), (w_{j,k})$ (2-*margins* in the statistical context to be elaborated upon below). The arrays in T are of size $(r, c, 3)$; that is, they have r rows, c columns, and "slim" depth 3, which is the best possible: 3-way line-sum transportation polytopes of depth ≤ 2 are equivalent to ordinary 2-way transportation polytopes which are not universal.

An appealing feature of Theorem 1.1 is that the defining system of T has only $\{0, 1\}$-valued coefficients and depends only on r and c. Thus, every rational polytope has a representation by one such system, where all information enters through the right-hand side $(u_{i,j}), (v_{i,k}), (w_{j,k})$.

We have also proved a second universality theorem about the following *bitransportation problems*: Given *supply* vectors $s^1, s^2 \in \mathbb{R}^r_{\geq 0}$, *demand* vectors $d^1, d^2 \in \mathbb{R}^c_{\geq 0}$, and *capacity* matrix $u \in \mathbb{R}^{r\times c}_{\geq 0}$, find a pair of nonnegative "transportations" $x^1, x^2 \in \mathbb{R}^{r\times c}_{\geq 0}$ satisfying supply and demand requirements $\sum_j x^k_{i,j} = s^k_i$, $\sum_i x^k_{i,j} = d^k_j$, $k = 1, 2$, and capacity constraints $x^1_{i,j} + x^2_{i,j} \leq u_{i,j}$. In other words, find $x^1, x^2 \geq 0$ such that x^k has row-sum s^k and column-sum d^k for $k = 1, 2$, and $x^1 + x^2 \leq u$.

THEOREM 1.2. *Any rational polytope $P = \{y \in \mathbb{R}^n_{\geq 0} : Ay = b\}$ is polynomial-time representable as a bitransportation polytope*

$$F = \left\{ (x^1, x^2) \in \mathbb{R}^{r\times c}_{\geq 0} \oplus \mathbb{R}^{r\times c}_{\geq 0} \ : \ x^1_{i,j} + x^2_{i,j} \leq u_{i,j}\ , \right.$$
$$\left. \sum_j x^k_{i,j} = s^k_i\ ,\ \sum_i x^k_{i,j} = d^k_j\ , \quad k = 1, 2 \right\} .$$

The proof is an easy adjustment of part of the proof of Theorem 1.1 (i.e., Theorem 3.3) and is presented in section 3.5. The theorem remains valid if we take all supplies to have the same value $s^k_i = U$, $i = 1, \ldots, r$, $k = 1, 2$; further, all capacities $u_{i,j}$ can be taken to be $\{0, U\}$-valued, giving a stronger statement.

The bitransportation problem gives at once a very simple two-commodity flow network as follows: start with the directed bipartite graph with vertex set $I \uplus J$,

$|I| = r$, $|J| = c$, and arc set $I \times J$ with capacities $u_{i,j}$, and augment it with two sources a_1, a_2 and two sinks b_1, b_2 and with arcs (a_k, i), $i \in I$, (j, b_k), $j \in J$, $k = 1, 2$ with capacities $u(a_k, i) := s_i^k$, $u(j, b_k) := d_j^k$. The feasible bitransportations are then precisely the two-commodity flows of maximal total value. This implies a result first obtained by A. Itai [19]: every linear program is polynomially equivalent to a two-commodity flow problem. It is worth noting that our transformation is in fact much simpler than Itai's. In particular, the above network is exceedingly special: every dipath has length three and is of the form (a_k, i, j, b_k) for some $k \in \{1, 2\}$, $i \in I$, and $j \in J$ and involves only one "interesting" arc ij. Further, each such arc ij carries flow of each commodity on precisely one path.

To demonstrate the concrete nature of our transformations, the procedures that convert any given data A, b to data to the representations of Theorems 1.1 and 1.2 have been implemented in a computer program which is available on-line (see [27]).

2. The consequences of the main results.

We now discuss some consequences of Theorems 1.1 and 1.2. A few of them were first presented in [7].

2.1. Universality of transportation polytopes: Solution of Vlach's problems.

As mentioned above, there is a large body of literature on the structure of various transportation polytopes. In particular, in the comprehensive paper [29], M. Vlach surveys some ten families of necessary conditions published over the years by several authors (including Schell, Haley, Smith, Morávek, and Vlach) on the line-sums $(u_{i,j}), (v_{i,j}), (w_{i,j})$ for a transportation polytope to be nonempty, and raises several concrete problems regarding these polytopes. Specifically, [29, Problems 4, 7, 9, 10] ask about the sufficiency of some of these conditions. Our results say that transportation polytopes (in fact already of slim, $(r, c, 3)$, arrays) are universal and include all polytopes. This indicated that the answer to each of Problems 4, 7, 9, and 10 has to be negative. Indeed we have already verified this.

Example 2.1 (Smith II conditions are not sufficient). Using our encoding, in particular, applying the algorithm of Theorem 3.2 to the infeasible polyhedron $P = \{(x, y) : x + y = 1, x + y = 2,\ x, y \geq 0\}$, with 2 as an upper bound on its entry values, we obtained concrete 2-margins (below). These 2-margins satisfy conditions (8.1)–(8.3) on page 72 of [29] while giving an infeasible system; thus the example solves open problem 7 in [29]. Note for reference that for the given matrices the top-left corners are the margin values $u_{1,1}, v_{1,1}, w_{1,1}$.

$$
\begin{pmatrix}
2 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 2 \\
0 & 2 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 2 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2
\end{pmatrix},
\quad
\begin{pmatrix}
2 & 2 & 2 \\
2 & 0 & 2 \\
2 & 2 & 2 \\
2 & 0 & 2 \\
2 & 0 & 2 \\
2 & 2 & 2 \\
2 & 0 & 2 \\
2 & 2 & 2 \\
2 & 2 & 2 \\
2 & 2 & 2 \\
2 & 2 & 2 \\
2 & 2 & 2
\end{pmatrix},
\quad
\begin{pmatrix}
2 & 2 & 0 \\
2 & 2 & 0 \\
2 & 2 & 0 \\
2 & 2 & 0 \\
7 & 0 & 1 \\
6 & 0 & 2 \\
3 & 0 & 5 \\
0 & 2 & 4 \\
0 & 2 & 4 \\
0 & 2 & 4 \\
0 & 2 & 4
\end{pmatrix}.
$$

Similarly, Problem 12 on page 76 of [29] asks whether all dimensions can occur as that of a suitable transportation polytope: the affirmative answer, given very recently in

[15], follows also at once from our universality result. Our construction also provides a powerful tool for studying concrete questions about transportation polytopes and their integer points, by allowing us to write down simple systems of equations that encode desired situations and lifting them up. Here is an example to this effect.

Example 2.2 (Vlach's rational-nonempty integer-empty transportation). Using our construction, we automatically recover the smallest known example, first discovered by Vlach [29], of a rational-nonempty integer-empty transportation polytope, as follows. We start with the polytope $P = \{y \geq 0 : 2y = 1\}$ in one variable, containing a (single) rational point but no integer point. Our construction represents it as a transportation polytope T of $(6, 4, 3)$-arrays with line-sums given by the three matrices below; by Theorem 1.1, T is integer equivalent to P and hence also contains a (single) rational point but no integer point.

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Returning to the Vlach problems, [29, Problem 13] asks for a characterization of those line-sums margins that guarantee an integer point in a 3-way transportation polytope T. In [18], Irving and Jerrum showed that deciding whether $T \cap \mathbb{Z}^{r\times c\times h} \neq \emptyset$ is NP-complete, and hence an efficient such characterization cannot exist unless $NP = coNP$. An immediate corollary of Theorem 1.1 strengthens this result to hold for slim arrays:

COROLLARY 2.3. *Deciding if a slim,* $(r, c, 3)$*, transportation polytope has an integer point is NP-complete.*

A comprehensive complexity classification of this decision problem under various assumptions on the array size and on the input, as well as of the related lattice point counting problem and other variants, appeared in [6].

The last Vlach problem [29, Problem 14] asks whether there is a *strongly polynomial-time* algorithm for deciding the (real) feasibility $T \neq \emptyset$ of a transportation polytope. Since the system defining T is $\{0, 1\}$-valued, the results of Tardos [26] provide an affirmative answer. However, the existence of a strongly polynomial-time algorithm for linear programming in general is open and of central importance; our construction embeds any linear program in an $(r, c, 3)$ transportation program in polynomial-time, but unfortunately this process is *not* strongly polynomial. Nonetheless, our construction may shed some light on the problem and may turn out useful in sharpening the boundary (if any) between strongly and weakly polynomial-time solvable linear programs.

2.2. Universality for approximations. The representation manifested by Theorem 1.1 allows us to represent an arbitrary integer programming problem $\min\{cy : y \in \mathbb{N}^n, Ay = b\}$ as a problem of finding minimum cost integer transportation,

$$\min\left\{\sum_{i,j,k} p_{i,j,k}x_{i,j,k} \ : \ x \in \mathbb{N}^{r\times c\times 3},\ \sum_i x_{i,j,k} = w_{j,k}\,, \right.$$
$$\left. \sum_j x_{i,j,k} = v_{i,k}\,,\ \sum_k x_{i,j,k} = u_{i,j} \right\}\ ,$$

by simply extending the cost vector c by zeros to a cost array p. In particular, the feasible (integer) solutions y to the original problem are in *cost-preserving* bijection with the feasible (integer) transportations x (that is, $cy = px$ for any corresponding pair). This shows that the representation preserves *approximations*, and that minimum cost transportation problems of slim format $(r, c, 3)$ are universal for approximation as well. In particular, any nonapproximability result—say, for the maximum clique problem [14]—lifts at once to the slim minimum cost transportation problem: just start with an integer programming formulation of the maximum clique problem with $\{0, 1\}$-valued right-hand-side vector b, and lift it up. We get the following hardness-of-approximation result.

COROLLARY 2.4. *Under the assumption $P \neq NP$, there is an $\epsilon > 0$ such that there is no polynomial-time $(rc)^\epsilon$-approximation algorithm for the minimum cost slim $(r, c, 3)$ line-sum transportation problem.*

We do not attempt here to provide the largest possible ϵ. Note, of course, that in particular, unless $P = NP$, there is no constant ratio approximation for the 3-way transportation problem (the problem is not in the class APX).

2.3. Confidential statistical data disclosure: Entry-range. Next, we briefly discuss some of the applications to statistical model theory: a comprehensive treatment can be found in [8]. A central goal of statistical data management by agencies such as the U.S. Census Bureau is to allow public access to information on their data base while protecting confidentiality of individuals whose data is in the base. A common practice [10], taken in particular by the Bureau [28], is to allow the release of some margins of tables in the base but not the individual entries themselves. The security of an entry is closely related to the range of values it can attain in any table with the fixed released collection of margins: if the range is "simple," then the entry may be exposed, whereas if it is "complex" the entry may be assumed secure.

In this subsection only, we use the following notation, which is common in statistical applications. A *d-table of size* $n = (n_1, \dots, n_d)$ is an array of nonnegative integers $x = (x_{i_1,\dots,i_d})$, $1 \le i_j \le n_j$. For any $0 \le k \le d$ and any k-subset $J \subseteq \{1, \dots, d\}$, the *k-margin* of x corresponding to J is the k-table $x^J := (x^J_{i_j : j \in J}) := (\sum_{i_j : j \notin J} x_{i_1,\dots,i_d})$ obtained by summing the entries over all indices *not in* J. For instance, the 2-margins of a 3-table $x = (x_{i_1,i_2,i_3})$ are its *line-sums* x^{12}, x^{13}, x^{23} such as $x^{13} = (x^{13}_{i_1,i_3}) = (\sum_{i_2} x_{i_1,i_2,i_3})$, and its 1-margins are its *plane-sums* x^1, x^2, x^3 such as $x^2 = (x^2_{i_2}) = (\sum_{i_1,i_3} x_{i_1,i_2,i_3})$.

A *statistical model* is a triple $\mathcal{M} = (d, \mathcal{J}, n)$, where $\mathcal{J}$ is a set of subsets of $\{1, \dots, d\}$ none containing the other and $n = (n_1, \dots, n_d)$ is a tuple of positive integers. The model dictates the collection of margins for d-tables of size n to be specified. Our results concern the models $(3, \{12, 13, 23\}, (r, c, 3))$, that is, slim, $(r, c, 3)$-tables, with all three of their 2-margins specified.

For any model $\mathcal{M} = (d, \mathcal{J}, n)$ and any specified collection of margins $u = (u^J : J \in \mathcal{J})$ under the model $\mathcal{M}$, the corresponding set of *contingency tables* with collection of margins u is

$$C(\mathcal{M}; u) := \{\, x \in \mathbb{N}^{n_1 \times \cdots \times n_d} \ : \ x^J = u^J \,,\ J \in \mathcal{J} \,\} \,.$$

Clearly, this set is precisely the set of integer points in the corresponding transportation polyhedron.

Finally, we define entry-ranges. Permuting coordinates, we may always consider the first entry $x_{\mathbf{1}}$, where $\mathbf{1} := (1, \dots, 1)$. The *entry-range* of a collection of margins

u under a model $\mathcal{M}$ is the set $R(\mathcal{M}; u) := \{x_{\mathbf{1}} : x \in C(\mathcal{M}; u)\} \subset \mathbb{N}$ of values $x_{\mathbf{1}}$ can attain in any table with these margins.

Often, the entry-range is an interval and hence "simple" and vulnerable, that is, for some $a, b \in \mathbb{N}$, $R(\mathcal{M}; u) = \{r \in \mathbb{N} : a \leq r \leq b\}$. For instance, as shown in [8], this indeed is the case for any 1-margin model $\mathcal{M} = (d, \{1, 2, \ldots, d\}, (n_1, \ldots, n_d))$ and any collection of margins $u = (u^1, \ldots, u^d)$ under $\mathcal{M}$.

In striking contrast with this situation and with recent attempts by statisticians to better understand entry behavior of slim 3-tables (cf. [3, 4, 10]), we have the following surprising consequence of Theorem 1.1, implying that entry-ranges of 2-margined slim 3-table models consist of all finite sets of nonnegative integers and hence are "complex" and presumably secure. For the proof, see [8].

Corollary 2.5 (universality of entry-range). *For any finite set $D \subset \mathbb{N}$ of nonnegative integers, there are $r, c,$ and 2-margins for $(r, c, 3)$-tables such that the set of values occurring in a fixed entry in all possible tables with these margins is precisely D.*

Example 2.6 (Gap in entry-range of 2-margined 3-tables). Applying our automatic universal generator [27] to the polytope $P = \{y \geq 0 : y_0 - 2y_1 = 0,\ y_1 + y_2 = 1\}$ in three variables, we obtain the following 2-margins for $(16, 11, 3)$-tables giving entry-range $D = \{0, 2\}$,

$$\begin{pmatrix}
2&0&0&0&0&0&2&0&0&2&0&0&0&0&0&0\\
0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&2\\
2&0&0&0&0&0&0&0&0&0&2&0&0&0&0&2\\
2&2&2&2&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&2&2&2&2&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&2&2&2&2&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&2&2&2&2\\
2&0&0&0&2&0&0&0&2&0&0&0&2&0&0&0\\
0&2&0&0&0&2&0&0&0&2&0&0&0&2&0&0\\
0&0&2&0&0&0&2&0&0&0&2&0&0&0&2&0\\
0&0&0&2&0&0&0&2&0&0&0&2&0&0&0&2
\end{pmatrix},$$

$$\begin{pmatrix}
2&2&2&2&2&2&2&2&2&2&2&2&2&2&2&2\\
4&0&0&0&0&2&2&0&0&2&2&0&0&0&0&4\\
2&2&2&2&2&2&2&2&2&2&2&2&2&2&2&2
\end{pmatrix},$$

$$\begin{pmatrix}
4&1&3&6&6&6&6&0&0&0&0\\
2&3&3&0&0&0&0&2&2&2&2\\
0&0&0&2&2&2&2&6&6&6&6
\end{pmatrix};$$

with a suitable "human" short cut it is possible to get it down to the following (possibly smallest) collection of margins for $(6, 4, 3)$-tables, giving again the entry-range $D = \{0, 2\}$ with a gap,

$$\begin{pmatrix}
2&1&2&0&2&0\\
1&0&2&0&0&2\\
1&0&0&2&2&0\\
0&1&0&2&0&2
\end{pmatrix}, \quad
\begin{pmatrix}
2&1&2&3&0&0\\
2&1&0&0&2&1\\
0&0&2&1&2&3
\end{pmatrix}, \quad
\begin{pmatrix}
2&3&2\\
2&1&2\\
2&1&2\\
2&1&2
\end{pmatrix}.$$

Further applications of Theorem 1.1 to statistical model theory are discussed in [8]; these include important consequences for *Markov bases* of 2-margined slim 3-way models. (Recall that a Markov basis is a set of moves that connects any pair of tables in the model that have the same set of margins, and is needed for the design of a random walk on the space of tables with fixed margins to address the problems of *sampling* and *estimating* various statistics on this space; see [8] for more details.)

2.4. Universality of the bitransportation problem. Our construction for Theorem 1.2 allows automatic generation of bitransportation programs with integer supplies, demands and capacities, where any feasible bitransportation must have an arbitrarily large prescribed denominator, in contrast with Hu's celebrated half-integrality theorem for the undirected case [16].

Example 2.7 (Bitransportations with arbitrarily large denominator). Fix any positive integer q. Start with the polytope $P = \{y \geq 0 : qy = 1\}$ in one variable containing the single point $y = \frac{1}{q}$. Our construction represents it as a bitransportation polytope F with integer supplies, demands and capacities, where y is embedded as the transportation $x^1_{1,1}$ of the first commodity from supply vertex $1 \in I$ to demand vertex $j \in J$. By Theorem 1.2, F contains a single bitransportation with $x^1_{1,1} = y = \frac{1}{q}$. For instance, for $q = 3$ we get the bitransportation problem with the data

$$u = \begin{pmatrix} 1&0&0&1&0&0&1&0\\ 0&1&0&1&0&0&0&1\\ 0&1&0&0&1&0&1&0\\ 0&0&1&0&1&0&0&1\\ 0&0&1&0&0&1&1&0\\ 1&0&0&0&0&1&0&1 \end{pmatrix}, \quad s^1 = \begin{pmatrix}1\\1\\1\\1\\1\\1\end{pmatrix}, \quad s^2 = \begin{pmatrix}1\\1\\1\\1\\1\\1\end{pmatrix},$$

$$d^1 = \begin{pmatrix}1&1&1&1&1&1&0&0\end{pmatrix}, \quad d^2 = \begin{pmatrix}0&0&0&1&1&1&2&1\end{pmatrix},$$

which has the following unique, $\{0, \frac{1}{3}, \frac{2}{3}\}$-valued, bitransportation solution:

$$x^1 = \frac{1}{3}\begin{pmatrix} 1&0&0&2&0&0&0&0\\ 0&2&0&1&0&0&0&0\\ 0&1&0&0&2&0&0&0\\ 0&0&2&0&1&0&0&0\\ 0&0&1&0&0&2&0&0\\ 2&0&0&0&0&1&0&0 \end{pmatrix}, \quad x^2 = \frac{1}{3}\begin{pmatrix} 0&0&0&1&0&0&2&0\\ 0&0&0&2&0&0&0&1\\ 0&0&0&0&1&0&2&0\\ 0&0&0&0&2&0&0&1\\ 0&0&0&0&0&1&2&0\\ 0&0&0&0&0&2&0&1 \end{pmatrix}.$$

By Theorem 1.2, any (say, feasibility) linear programming problem can be encoded as such a bitransportation problem (unbounded programs can also be treated by adding to the original system a single equality $\sum_{j=0}^{n} y_j = U$ with y_0 a new "slack" variable and U derived from the Cramer's rule bound of Theorem 10.3 [25]). Thus, any (hopefully combinatorial) algorithm for the bitransportation problem will give an algorithm for general linear programming. There has been much interest lately (A. Levin [21]) in combinatorial approximation algorithms for (fractional) multiflows, e.g., [11, 12]; these yield, via Theorem 1.2, approximation algorithms for general linear programming, which might prove a useful and fast solution strategy in practice. Details of this will appear elsewhere.

3. The three-stage construction. Our construction consists of three stages which are independent of each other as reflected in Lemma 3.1 and Theorems 3.2

and 3.3 below. Stage one, in section 3.1, is a simple preprocessing based on standard scaling ideas, in which a given polytope is represented as another whose defining system involves only small, $\{-1, 0, 1, 2\}$-valued coefficients, at the expense of increasing the number of variables. This enables us to make the entire construction run in time polynomial in the size of the input. However, for systems with small coefficients, such as in the examples above, this may result in unnecessary blow-up and can be skipped. Stage two, in section 3.2, represents any rational polytope as a 3-way transportation polytope with specified *plane-sums* and *forbidden-entries*. In the last stage, in section 3.3, any plane-sum transportation polytope with upper-bounds on the entries is represented as a slim 3-way line-sum transportation polytope. In section 3.4 these three stages are integrated to give Theorem 1.1, and a complexity estimate is provided to close the presentation. Theorem 1.2 is a result of an easy modification of Theorem 3.2, and it is the content of section 3.5.

3.1. Preprocessing: Coefficient reduction. Let $P = \{y \geq 0 : Ay = b\}$ where $A = (a_{i,j})$ is an integer matrix and b is an integer vector. We represent it as a polytope $Q = \{x \geq 0 : Cx = d\}$, in polynomial-time, with a $\{-1, 0, 1, 2\}$-valued matrix $C = (c_{i,j})$ of coefficients, as follows. Consider any variable y_j and let $k_j := \max\{\lfloor \log_2 |a_{i,j}| \rfloor : i = 1, \ldots, m\}$ be the maximum number of bits in the binary representation of the absolute value of any $a_{i,j}$. We introduce variables $x_{j,0}, \ldots, x_{j,k_j}$, and relate them by the equations $2x_{j,s} - x_{j,s+1} = 0$. The representing injection σ is defined by $\sigma(j) := (j, 0)$, embedding y_j as $x_{j,0}$. Consider any term $a_{i,j}\, y_j$ of the original system. Using the binary expansion $|a_{i,j}| = \sum_{s=0}^{k_j} t_s 2^s$ with all $t_s \in \{0, 1\}$, we rewrite this term as $\pm \sum_{s=0}^{k_j} t_s x_{j,s}$. To illustrate, consider a system consisting of the single equation $3y_1 - 5y_2 + 2y_3 = 7$. Then the new system is

$$
\begin{array}{ccccccccc}
2x_{1,0} & -x_{1,1} & & & & & & = & 0, \\
& & 2x_{2,0} & -x_{2,1} & & & & = & 0, \\
& & & 2x_{2,1} & -x_{2,2} & & & = & 0, \\
& & & & & 2x_{3,0} & -x_{3,1} & = & 0, \\
x_{1,0} & +x_{1,1} & -x_{2,0} & & -x_{2,2} & & +x_{3,1} & = & 7.
\end{array}
$$

It is easy to see that this procedure provides the sought representation, and we get the following.

LEMMA 3.1. *Any rational polytope $P = \{y \geq 0 : Ay = b\}$ is polynomial-time representable as a polytope $Q = \{x \geq 0 : Cx = d\}$ with $\{-1, 0, 1, 2\}$-valued defining matrix C.*

3.2. Representing polytopes as plane-sum entry-forbidden transportation polytopes. The next stage of construction we are about to explain will normally be applied to the output $Q = \{x \geq 0 : Cx = d\}$ of stage one, but we present the construction for a general polyhedron P since the construction holds in that generality. Let $P = \{y \geq 0 : Ay = b\}$, where $A = (a_{i,j})$ is an $m \times n$ integer matrix and b is an integer vector: we assume that P is bounded and hence a (possibly empty) polytope, with an integer upper bound U on the value of any coordinate y_j of any $y \in P$ (U can be derived efficiently from Cramer's rule as explained in Theorem 10.3 of [25]).

For each variable y_j, let r_j be the maximum of the sum of the positive coefficients of y_j over all equations and the sum of absolute values of the negative coefficients of y_j over all equations:

$$r_j := \max\left(\sum_k \{a_{k,j} : a_{k,j} > 0\}, \sum_k \{|a_{k,j}| : a_{k,j} < 0\}\right) .$$

Let $r := \sum_{j=1}^n r_j$, $R := \{1,\ldots,r\}$, $h := m+1$, and $H := \{1,\ldots,h\}$. We now describe how to construct vectors $u, v \in \mathbb{Z}^r$, $w \in \mathbb{Z}^h$, and a set $E \subset R \times R \times H$ of triples—the "enabled," non-"forbidden" entries—such that the polytope P is represented as the corresponding transportation polytope of $r \times r \times h$ arrays with plane-sums u, v, w and only entries indexed by E enabled:

$$T = \left\{ x \in \mathbb{R}^{r\times r\times h}_{\geq 0} \ : \ x_{i,j,k} = 0 \text{ for all } (i,j,k) \notin E\,, \quad\text{and}\right.$$
$$\left. \sum_{i,j} x_{i,j,k} = w_k\,,\ \sum_{i,k} x_{i,j,k} = v_j\,,\ \sum_{j,k} x_{i,j,k} = u_i \right\} .$$

We also indicate the injection $\sigma : \{1,\ldots,n\} \longrightarrow R \times R \times H$ giving the desired embedding of the coordinates y_j as the coordinates $x_{i,j,k}$ and the representation of P as T (see paragraph following Theorem 1.1).

Basically, each equation $k = 1,\ldots,m$ will be encoded in a "horizontal plane" $R \times R \times \{k\}$ (the last plane $R \times R \times \{h\}$ is included for consistency and its entries can be regarded as "slacks"); and each variable y_j, $j = 1,\ldots,n$, will be encoded in a "vertical box" $R_j \times R_j \times H$, where $R = \biguplus_{j=1}^n R_j$ is the natural partition of R with $|R_j| = r_j$, namely with $R_j := \{1 + \sum_{l<j} r_l, \ldots, \sum_{l\leq j} r_l\}$.

Now, all "vertical" plane-sums are set to the same value U, that is, $u_j := v_j := U$ for $j = 1,\ldots,r$. All entries not in the union $\biguplus_{j=1}^n R_j \times R_j \times H$ of the variable boxes will be forbidden. We now describe the enabled entries in the boxes; for simplicity we discuss the box $R_1 \times R_1 \times H$, the others being similar. We distinguish between the two cases $r_1 = 1$ and $r_1 \geq 2$. In the first case, $R_1 = \{1\}$; the box, which is just the single line $\{1\}\times\{1\}\times H$, will have exactly two enabled entries $(1,1,k^+), (1,1,k^-)$ for suitable k^+, k^- to be defined later. We set $\sigma(1) := (1,1,k^+)$, namely embed $y_1 = x_{1,1,k^+}$. We define the *complement* of the variable y_1 to be $\bar{y}_1 := U - y_1$ (and likewise for the other variables). The vertical sums u, v then force $\bar{y}_1 = U - y_1 = U - x_{1,1,k^+} = x_{1,1,k^-}$, so the complement of y_1 is also embedded. Next, consider the case $r_1 \geq 2$. For each $s = 1,\ldots,r_1$, the line $\{s\} \times \{s\} \times H$ (respectively, $\{s\} \times \{1 + (s \bmod r_1)\} \times H$) will contain one enabled entry $(s, s, k^+(s))$ (respectively, $(s, 1 + (s \bmod r_1), k^-(s))$. All other entries of $R_1 \times R_1 \times H$ will be forbidden. Again, we set $\sigma(1) := (1,1,k^+(1))$, namely embed $y_1 = x_{1,1,k^+(1)}$; it is then not hard to see that, again, the vertical sums u, v force $x_{s,s,k^+(s)} = x_{1,1,k^+(1)} = y_1$ and $x_{s,1+(s \bmod r_1),k^-(s)} = U - x_{1,1,k^+(1)} = \bar{y}_1$ for each $s = 1,\ldots,r_1$. Therefore, both y_1 and $\bar{y}_1$ are each embedded in r_1 distinct entries.

To clarify the above description it is helpful to visualize the $R \times R$ matrix $(x_{i,j,+})$ whose entries are the vertical line-sums $x_{i,j,+} := \sum_{k=1}^h x_{i,j,k}$. For instance, if we have three variables with $r_1 = 3, r_2 = 1, r_3 = 2$ then $R_1 = \{1,2,3\}$, $R_2 = \{4\}$, $R_3 = \{5,6\}$, and the line-sums matrix $x = (x_{i,j,+})$ is

$$\begin{pmatrix} x_{1,1,+} & x_{1,2,+} & 0 & 0 & 0 & 0 \\ 0 & x_{2,2,+} & x_{2,3+} & 0 & 0 & 0 \\ x_{3,1,+} & 0 & x_{3,3,+} & 0 & 0 & 0 \\ 0 & 0 & 0 & x_{4,4,+} & 0 & 0 \\ 0 & 0 & 0 & 0 & x_{5,5,+} & x_{5,6,+} \\ 0 & 0 & 0 & 0 & x_{6,5,+} & x_{6,6,+} \end{pmatrix} = \begin{pmatrix} y_1 & \bar{y}_1 & 0 & 0 & 0 & 0 \\ 0 & y_1 & \bar{y}_1 & 0 & 0 & 0 \\ \bar{y}_1 & 0 & y_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & U & 0 & 0 \\ 0 & 0 & 0 & 0 & y_3 & \bar{y}_3 \\ 0 & 0 & 0 & 0 & \bar{y}_3 & y_3 \end{pmatrix}.$$

We now encode the equations by defining the horizontal plane-sums w and the indices $k^+(s), k^-(s)$ mentioned above as follows. For $k = 1, \ldots, m$, consider the kth equation $\sum_j a_{k,j} y_j = b_k$. Define the index sets $J^+ := \{j : a_{k,j} > 0\}$ and $J^- := \{j : a_{k,j} < 0\}$, and set $w_k := b_k + U \cdot \sum_{j \in J^-} |a_{k,j}|$. The last coordinate of w is set for consistency with u, v to be $w_h = w_{m+1} := r \cdot U - \sum_{k=1}^m w_k$. Now, with $\bar{y}_j := U - y_j$ the complement of variable y_j as above, the kth equation can be rewritten as

$$\sum_{j \in J^+} a_{k,j} y_j + \sum_{j \in J^-} |a_{k,j}| \bar{y}_j = \sum_{j=1}^n a_{k,j} y_j + U \cdot \sum_{j \in J^-} |a_{k,j}| = b_k + U \cdot \sum_{j \in J^-} |a_{k,j}| = w_k.$$

We encode this equation by setting, for each $j \in J_+$, $k^+(s) = k$ for $|a_{k,j}|$ many different values of s (respectively, for each $j \in J_-$ we set $k^-(s) = k$ for enough values of s). By suitably setting $k^+(s) := k$ or $k^-(s) := k$, this has the effect of pulling enough copies of the variables y_j or $\bar{y}_j$ to the corresponding kth horizontal plane. Of course, once a variable is used at a certain horizontal level it cannot be used in others. By the choice of r_j there are sufficiently many copies of variables y_j $\bar{y}_j$, possibly with a few redundant copies which are absorbed in the last hyperplane by setting $k^+(s) := m+1$ or $k^-(s) := m+1$. For instance, if $m = 8$, the first variable y_1 has $r_1 = 3$ as above, its coefficient $a_{4,1} = 3$ in the fourth equation is positive, its coefficient $a_{7,1} = -2$ in the seventh equation is negative, and $a_{k,1} = 0$ for $k \neq 4, 7$, then we set $k^+(1) = k^+(2) = k^+(3) := 4$ (so $\sigma(1) := (1,1,4)$ embedding y_1 as $x_{1,1,4}$), $k^-(1) = k^-(2) := 7$, and $k^-(3) := h = 9$. This way, all equations are suitably encoded, and we obtain the following theorem.

THEOREM 3.2. *Any rational polytope $P = \{y \in \mathbb{R}^n_{\geq 0} : Ay = b\}$ is polynomial-time representable as a plane-sum entry-forbidden 3-way transportation polytope*

$$T = \left\{ x \in \mathbb{R}^{r \times r \times h}_{\geq 0} \ : \ x_{i,j,k} = 0 \ \text{for all} \ (i,j,k) \notin E\,, \quad \text{and} \right.$$
$$\left. \sum_{i,j} x_{i,j,k} = w_k\,,\ \sum_{i,k} x_{i,j,k} = v_j\,,\ \sum_{j,k} x_{i,j,k} = u_i \right\}.$$

Here E denotes the set of enabled, nonforbidden entries.

Proof. The proof follows from the construction outlined above and Lemma 3.1. ∎

3.3. Representing plane-sum entry-bounded as slim line-sum entry-free. Here we start with a transportation polytope of plane-sums and *upper-bounds*

$e_{i,j,k}$ on the entries,

$$P = \left\{ y \in \mathbb{R}_{\geq 0}^{l\times m\times n} : \sum_{i,j} y_{i,j,k} = c_k \,, \sum_{i,k} y_{i,j,k} = b_j \,, \sum_{j,k} y_{i,j,k} = a_i \,,\ y_{i,j,k} \leq e_{i,j,k} \right\}.$$

Clearly, this is a more general form than that of T appearing in Theorem 3.2 above; the forbidden entries can be encoded by setting a "forbidding" upper-bound $e_{i,j,k} := 0$ on all forbidden entries $(i,j,k) \notin E$ and an "enabling" upper-bound $e_{i,j,k} := U$ on all enabled entries $(i,j,k) \in E$. Thus, by Theorem 3.2, any rational polytope is representable also as such a plane-sum entry-bounded transportation polytope P. We now describe how to represent, in turn, such a P as a slim line-sum (unrestricted-entry) transportation polytope of the form of Theorem 1.1,

$$T \;=\; \left\{ x \in \mathbb{R}_{\geq 0}^{r\times c\times 3} \;:\; \sum_I x_{I,J,K} = w_{J,K} \,,\; \sum_J x_{I,J,K} = v_{I,K} \,,\; \sum_K x_{I,J,K} = u_{I,J} \right\} \;.$$

This stage of our construction was first presented in [6] while studying the complexity of deciding if T has an integer point; we include the details for completeness of the presentation. We give explicit formulas for $u_{I,J}, v_{I,K}, w_{J,K}$ in terms of a_i, b_j, c_k, and $e_{i,j,k}$ as follows. Put $r := l \cdot m$ and $c := n + l + m$. The first index I of each entry $x_{I,J,K}$ will be a pair $I = (i,j)$ in the r-set

$$\{(1,1), \ldots, (1,m), (2,1), \ldots, (2,m), \ldots, (l,1), \ldots, (l,m)\} \;.$$

The second index J of each entry $x_{I,J,K}$ will be a pair $J = (s,t)$ in the c-set

$$\{(1,1), \ldots, (1,n), (2,1), \ldots, (2,l), (3,1), \ldots, (3,m)\} \;.$$

The last index K will simply range in the 3-set $\{1,2,3\}$. We represent P as T via the injection σ given explicitly by $\sigma(i,j,k) := ((i,j),(1,k),1)$, embedding each variable $y_{i,j,k}$ as the entry $x_{(i,j),(1,k),1}$. Let U now denote the minimum between the two values $\max\{a_1, \ldots, a_l\}$ and $\max\{b_1, \ldots, b_m\}$. The 2-margins entries will be

$$u_{(i,j),(1,t)} = e_{i,j,t}, \quad u_{(i,j),(2,t)} = \begin{cases} U & \text{if } t = i, \\ 0 & \text{otherwise,} \end{cases} \quad u_{(i,j),(3,t)} = \begin{cases} U & \text{if } t = j, \\ 0 & \text{otherwise,} \end{cases}$$

$$v_{(i,j),t} = \begin{cases} U & \text{if } t = 1, \\ e_{i,j,+} & \text{if } t = 2, \\ U & \text{if } t = 3, \end{cases}$$

$$w_{(i,j),1} = \begin{cases} c_j & \text{if } i = 1, \\ m \cdot U - a_j & \text{if } i = 2, \\ 0 & \text{if } i = 3. \end{cases} \qquad w_{(i,j),2} = \begin{cases} e_{+,+,j} - c_j & \text{if } i = 1, \\ 0 & \text{if } i = 2, \\ b_j & \text{if } i = 3. \end{cases}$$

$$w_{(i,j),3} = \begin{cases} 0 & \text{if } i = 1, \\ a_j & \text{if } i = 2, \\ l \cdot U - b_j & \text{if } i = 3. \end{cases}$$

THEOREM 3.3. *Any rational plane-sum entry-bounded* 3*-way transportation polytope*

$$P = \left\{ y \in \mathbb{R}^{l\times m\times n}_{\geq 0} : \sum_{i,j} y_{i,j,k} = c_k \, , \sum_{i,k} y_{i,j,k} = b_j \, , \sum_{j,k} y_{i,j,k} = a_i \, , \; y_{i,j,k} \leq e_{i,j,k} \right\}$$

is strongly-polynomial-time representable as a line-sum slim transportation polytope

$$T = \left\{ x \in \mathbb{R}^{r\times c\times 3}_{\geq 0} \; : \; \sum_I x_{I,J,K} = w_{J,K} \, , \; \sum_J x_{I,J,K} = v_{I,K} \, , \; \sum_K x_{I,J,K} = u_{I,J} \right\} .$$

Proof. We outline the proof; complete details appeared in [6]. First, consider any $y = (y_{i,j,k}) \in P$; we claim the embedding via σ of $y_{i,j,k}$ in $x_{(i,j),(1,k),1}$ can be extended uniquely to $x = (x_{I,J,K}) \in T$. First, the entries $x_{I,(3,t),1}$, $x_{I,(2,t),2}$ and $x_{I,(1,t),3}$ for all $I = (i,j)$ and t are zero since so are the line-sums $w_{(3,t),1}$, $w_{(2,t),2}$ and $w_{(1,t),3}$. Next, consider the entries $x_{I,(2,t),1}$: since all entries $x_{I,(3,t),1}$ are zero, examining the line-sums $u_{I,(2,t)}$ and $v_{I,1} = U$, we find $x_{(i,j),(2,i),1} = U - \sum_{t=1}^{n} x_{(i,j),(1,t),1} = U - y_{i,j,+} \geq 0$ whereas for $t \neq i$ we get $x_{(i,j),(2,t),1} = 0$. This also gives the entries $x_{I,(2,t),3}$: we have $x_{(i,j),(2,i),3} = U - x_{(i,j),(2,i),1} = y_{i,j,+} \geq 0$ whereas for $t \neq i$ we have $x_{(i,j),(2,t),3} = 0$. Next, consider the entries $x_{I,(1,t),2}$: since all entries $x_{I,(1,t),3}$ are zero, examining the line-sums $u_{(i,j),(1,k)} = e_{i,j,k}$ we find $x_{(i,j),(1,k),2} = e_{i,j,k} - y_{i,j,k} \geq 0$ for all i, j, k. Next consider the entries $x_{I,(3,t),2}$: since all entries $x_{I,(2,t),2}$ are zero, examining the line-sums $u_{(i,j),(3,t)}$ and $v_{(i,j),2} = e_{i,j,+}$, we find $x_{(i,j),(3,j),2} = e_{i,j,+} - \sum_{k=1}^{l} x_{(i,j),(1,k),2} = y_{i,j,+} \geq 0$ whereas for $t \neq j$ we get $x_{(i,j),(3,t),2} = 0$. This also gives the entries $x_{I,(3,t),3}$: we have $x_{(i,j),(3,j),3} = U - x_{(i,j),(3,j),2} = U - y_{i,j,+} \geq 0$ whereas for $t \neq j$ we get $x_{(i,j),(3,t),3} = 0$. Using the relations established above, one can easily check that all line-sums are correct.

Conversely, given any $x = (x_{I,J,K}) \in T$, let $y = (y_{i,j,k})$ with $y_{i,j,k} := x_{(i,j),(1,k),1}$. Since x is nonnegative, so is y. Further, $e_{i,j,k} - y_{i,j,k} = x_{(i,j),(1,k),2} \geq 0$ for all i, j, k and hence y obeys the entry upper-bounds. Finally, using the relations established above $x_{(i,j),(3,t),2} = 0$ for $t \neq j$, $x_{(i,j),(2,t),3} = 0$ for $t \neq i$, and $x_{(i,j),(3,j),2} = x_{(i,j),(2,i),3} = y_{i,j,+}$, we obtain

$$\sum_{i,j} y_{i,j,k} \;=\; \sum_{i,j} x_{(i,j),(1,k),1} \;=\; w_{(1,k),1} \;=\; c_k \, , \qquad 1 \leq k \leq n \; ;$$

$$\sum_{i,k} y_{i,j,k} \;=\; \sum_{i} x_{(i,j),(3,j),2} \;=\; w_{(3,j),2} \;=\; b_j \, , \qquad 1 \leq j \leq m \; ;$$

$$\sum_{j,k} y_{i,j,k} \;=\; \sum_{j} x_{(i,j),(2,i),3} \;=\; w_{(2,i),3} \;=\; a_i \, , \qquad 1 \leq i \leq l \; .$$

This shows that y satisfies the plane-sums as well and hence is in P. Since integrality is also preserved in both directions, this completes the proof. ∎

3.4. The main theorem and a complexity estimate. Call a class $\mathcal{P}$ of rational polytopes *polynomial-time representable* in a class $\mathcal{Q}$ if there is a polynomial-time algorithm that represents any given $P \in \mathcal{P}$ as some $Q \in \mathcal{Q}$. The resulting binary relation on classes of rational polytopes is clearly transitive. Thus, the composition of Theorem 3.2 (which incorporates Lemma 3.1) and Theorem 3.3 gives at once Theorem 1.1 stated in the introduction. Working out the details of our three-stage construction, we can give the following estimate on the number of rows r and columns c in the resulting representing transportation polytope, in terms of the input. The computational complexity of the construction is also determined by this bound, but we do not dwell on the details here.

THEOREM 1.1 (with complexity estimate). *Any polytope* $P=\{y \in \mathbb{R}^n_{\geq 0} : Ay = b\}$ *with integer* $m \times n$ *matrix* $A = (a_{i,j})$ *and integer* b *is polynomial-time representable as a slim transportation polytope*

$$T = \left\{ x \in \mathbb{R}^{r \times c \times 3}_{\geq 0} \ : \ \sum_i x_{i,j,k} = w_{j,k}\,,\ \sum_j x_{i,j,k} = v_{i,k}\,,\ \sum_k x_{i,j,k} = u_{i,j} \right\}\ ,$$

with $r = O(m^2(n+L)^2)$ *rows and* $c = O(m(n+L))$ *columns, where*

$$L := \sum_{j=1}^{n} \max_{i=1}^{m} \lfloor \log_2 |a_{i,j}| \rfloor.$$

3.5. Proof of the universality of the bitransportation problem. We conclude with the modification of the proof of Theorem 3.3 that establishes Theorem 1.2.

THEOREM 1.2. *Any rational polytope* $P = \{y \in \mathbb{R}^n_{\geq 0} : Ay = b\}$ *is polynomial-time representable as a bipartite bitransportation polytope*

$$F = \left\{ (x^1, x^2) \in \mathbb{R}^{r \times c}_{\geq 0} \oplus \mathbb{R}^{r \times c}_{\geq 0} \ : \ x^1_{i,j} + x^2_{i,j} \leq u_{i,j}\,, \right.$$
$$\left. \sum_j x^k_{i,j} = s^k_i\,,\ \sum_i x^k_{i,j} = d^k_j\,, \quad k = 1, 2 \right\}\ .$$

Here r, c are the same values as presented in Theorem 1.1 above. Moreover, the statement remains valid with all supplies s^k_i having the same value U and all capacities $u_{i,j}$ being 0 or U for some suitable nonnegative integer U.

Proof. We do an easy adjustment of the proof of Theorem 3.3 above: We essentially need to describe the capacities, demands and supplies (for each of two commodities) for a bipartite network with $l \cdot m$ nodes for the first part and $n + l + m$ nodes in the second part, with $l \cdot m \cdot (n + l + m)$ arcs. Take the capacities of the arcs to be $u_{i,j}$ as defined in section 3.3; take the supplies to be $s^1_i := v_{i,1} = U$ and $s^2_i := v_{i,3} = U$ for all i, and take the demands to be $d^1_j := w_{j,1}$ and $d^2_j := w_{j,3}$ for all j. Note that by taking s^2_i and d^2_j to be $v_{i,3}$ and $w_{j,3}$ instead of $v_{i,2}$ and $w_{j,2}$ we can guarantee that all supplies have the same value U. Moreover, since the proof follows by the composition of Theorem 3.2 and Theorem 3.3, and the former makes use of forbidden entries only, rather than upper bounds, it is easy to see that we can take all upper bounds $e_{i,j,k}$ in the latter (and hence all $u_{i,j}$) to be either 0 or U, proving the stronger statement. More visually, the data can also be described in matrix form as follows:

$$
u = \begin{pmatrix}
e_{1,1,1} & e_{1,1,2} & \cdots & e_{1,1,n} & U & 0 & \cdots & 0 & U & 0 & \cdots & 0 \\
e_{1,2,1} & e_{1,2,2} & \cdots & e_{1,2,n} & U & 0 & \cdots & 0 & 0 & U & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
e_{1,m,1} & e_{1,m,2} & \cdots & e_{1,m,n} & U & 0 & \cdots & 0 & 0 & 0 & \cdots & U \\
 & & & & & & & & & & & \\
e_{2,1,1} & e_{2,1,2} & \cdots & e_{2,1,n} & 0 & U & \cdots & 0 & U & 0 & \cdots & 0 \\
e_{2,2,1} & e_{2,2,2} & \cdots & e_{2,2,n} & 0 & U & \cdots & 0 & 0 & U & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
e_{2,m,1} & e_{2,m,2} & \cdots & e_{2,m,n} & 0 & U & \cdots & 0 & 0 & 0 & \cdots & U \\
 & & & & & & & & & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & & & & & & & & & & & \\
e_{l,1,1} & e_{l,1,2} & \cdots & e_{l,1,n} & 0 & 0 & \cdots & U & U & 0 & \cdots & 0 \\
e_{l,2,1} & e_{l,2,2} & \cdots & e_{l,2,n} & 0 & 0 & \cdots & U & 0 & U & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
e_{l,m,1} & e_{l,m,2} & \cdots & e_{l,m,n} & 0 & 0 & \cdots & U & 0 & 0 & \cdots & U
\end{pmatrix},
$$

$$
s^1 = s^2 = \begin{pmatrix}
U \\ U \\ \vdots \\ U \\ \\ U \\ U \\ \vdots \\ U \\ \\ \vdots \\ \\ U \\ U \\ \vdots \\ U
\end{pmatrix},
$$

$$
\begin{aligned}
d^1 &= (\, c_1, \quad c_2, \quad \ldots, \quad c_n, \quad m \cdot U - a_1, \quad m \cdot U - a_2, \ldots, \quad m \cdot U - a_l, \quad 0, \quad 0, \quad \ldots, \quad 0 \,), \\
d^2 &= (\, 0, \quad 0, \quad \ldots, \quad 0, \quad a_1, \quad a_2, \quad \ldots, a_l, \quad l \cdot U - b_1, \quad l \cdot, U - b_2, \ldots, \quad l \cdot U - b_m \,). \qquad \square
\end{aligned}
$$

Acknowledgments. The authors are grateful to G. Rote, F. Santos, S. Hoşten, G. Ziegler, and the anonymous referees for their comments and suggestions.

REFERENCES

[1] M. Baiöu and M. L. Balinski, *The stable allocation (or ordinal transportation) problem*, Math. Oper. Res., 27 (2002), pp. 485–503.

[2] M. L. Balinski and F. J. Rispoli, *Signature classes of transportation polytopes*, Math. Program Ser. A, 60 (1993), pp. 127–144.

[3] L. H. Cox, *Bounds on entries in 3-dimensional contingency tables*, in Inference Control in Statistical Databases: From Theory to Practice, J. Domingo-Ferrer, ed., Lecture Notes in Comput. Sci. 2316, Springer, New York, 2002, pp. 21–33.

[4] L. H. Cox, *On properties of multi-dimensional statistical tables*, J. Stat. Plann. Inference, 117 (2003), pp. 251–273.

[5] M. Cryan, M. Dyer, H. Müller, and L. Stougie, *Random walks on the vertices of transportation polytopes with constant number of sources*, in Proceedings of the 14th Annual ACM-SIAM Symposium on Discrete Algorithms (Baltimore, MD), ACM, New York, 2003, pp. 330–339.

[6] J. De Loera and S. Onn, *The complexity of three-way statistical tables*, SIAM J. Comput., 33 (2004), pp. 819–836.

[7] J. De Loera and S. Onn, *All rational polytopes are transportation polytopes and all polytopal integer sets are contingency tables*, in Integer Programming and Combinatorial Optimization, Proceedings of the 10th International IPCO Conference, Lecture Notes in Comput. Sci. 3064, Springer, New York, 2004, pp. 338–351.

[8] J. De Loera and S. Onn, *Markov bases of three-way tables are arbitrarily complicated*, J. Symbolic Comput., 41 (2006), pp. 173–181.

[9] P. Diaconis and A. Gangolli, *Rectangular arrays with fixed margins* in Discrete Probability and Algorithms (Minneapolis, MN, 1993), D. Aldous, P. Diaconis, J. Spencer, and J. M. Steele, eds., IMA Vol. Math. Appl. 72, Springer, New York, 1995, pp. 15–41.

[10] G. T. Duncan, S. E. Fienberg, R. Krishnan, R. Padman, and S. F. Roehrig, *Disclosure limitation methods and information loss for tabular data*, in Confidentiality, Disclosure and Data Access: Theory and Practical Applications for Statistical Agencies, P. Doyle, J. I. Land, J. M. Theeuwes, and L. V. Zayatz, eds., North-Holland, Amsterdam, 2001, pp. 135–166.

[11] L. K. Fleischer, *Approximating fractional multicommodity flow independent of the number of commodities*, SIAM J. Discrete Math., 13 (2000), pp. 505–520.

[12] N. Garg and J. Könemann, *Faster and simpler algorithms for multicommodity flow and other fractional packing problems*, in Proceedings of the 39th Annual IEEE Symposium on Foundations of Computer Science (New York), IEEE, New York, 1998, pp. 300–309.

[13] D. Gusfield, *A graph theoretic approach to statistical data security*, SIAM J. Comput., 17 (1988), pp. 552–571.

[14] J. Håstad, *Clique is hard to approximate within* $n^{1-\epsilon}$, Acta Math., 182 (1999), pp. 105–142.

[15] S. Hoşten and S. Sullivant, *Gröbner bases and polyhedral geometry of reducible and cyclic models*, J. Combin. Theory Ser. A, 100 (2002), pp. 277–301.

[16] T. C. Hu, *Multi-commodity network flows*, Oper. Res., 11 (1963), pp. 344–360.

[17] F. K. Hwang, S. Onn, and U. G. Rothblum, *A polynomial time algorithm for shaped partition problems*, SIAM J. Optim., 10 (1999), pp. 70–81.

[18] R. Irving and M. R. Jerrum, *Three-dimensional statistical data security problems*, SIAM J. Comput., 23 (1994), pp. 170–184.

[19] A. Itai, *Two-commodity flow*, J. Assoc. Comput. Mach., 25 (1978), pp. 596–611.

[20] V. Klee and C. Witzgall, *Facets and vertices of transportation polytopes*, in Mathematics of the Decision Sciences, Part I (Stanford, CA, 1967), AMS, Providence, RI, 1968, pp. 257–282.

[21] A. Levin, *Personal communication*, 2004.

[22] C. R. Mehta and N. R. Patel, *A network algorithm for performing Fisher's exact test in* $r \times c$ *contingency tables*, J. Amer. Statist. Assoc., 78 (1983), pp. 427–434.

[23] S. Onn and U. G. Rothblum, *Convex combinatorial optimization*, Discrete Comput. Geom., 32 (2004), pp. 549–566.

[24] S. Onn and L. J. Schulman, *The vector partition problem for convex objective functions*, Math. Oper. Res., 26 (2001), pp. 583–590.

[25] A. Schrijver, *Theory of Linear and Integer Programming*, John Wiley & Sons, Ltd., Chichester, 1986.

[26] E. Tardos, *A strongly polynomial algorithm to solve combinatorial linear programs*, Oper. Res., 34 (1986), pp. 250–256.

[27] Universal Generator, http://www.math.ucdavis.edu/∼deloera, http://ie.technion.ac.il/∼onn.

[28] U.S. Bureau of the Census, *American FactFinder*, interactive information accessible at http://factfinder.census.gov/servlet/BasicFactsServlet.

[29] M. Vlach, *Conditions for the existence of solutions of the three-dimensional planar transportation problem*, Discrete Appl. Math., 13 (1986), pp. 61–78.

[30] V. A. Yemelichev, M. M. Kovalev, and M. K. Kravtsov, *Polytopes, Graphs and Optimisation*, Cambridge University Press, Cambridge, UK, 1984.