QUANTIZED PARAMETERIZED PARTICLE

Energy integral
\[ S = \frac{1}{2} \int dt \mathcal{L} \]

Minimized by parameterized geodesics \( x^{\mu}(t) \)
\[ \frac{\partial x^{\nu}}{\partial t} = \frac{\partial}{\partial t} x^{\nu}(t) = 0 \]

Equivalent 2nd order action
\[ S^2 = \int dt \left( p^{\mu} \frac{dx^{\mu}}{dt} - \mathcal{L} \right) \]

Poincare of canonical 2-form

Minimized by
\[ \{ x^{\mu}, x^{\nu} \} = g^{\mu\nu}(x) p_{\mu} p_{\nu} \]
\[ \{ p_{\mu}, p_{\nu} \} = -\frac{1}{2} \eta_{\mu\nu} p^2 \]

Represented on wavefunction \( \psi \) by states \( | \psi \rangle \) in a Hilbert space \( \mathcal{H} \)

Here states are wavefunctions
\[ \psi: M \to \mathbb{C} \] alias scalar fields (section of complex bundle over \( M \))

Inner product, inner metric (although could consider \( \mathcal{L} \) density of)
\[ \langle \phi | \psi \rangle = \int_M g^{\mu\nu} \phi^\nu \psi^\mu \] for simplicity assume \( M \) compact.
\[ i \mathcal{J} = (\det g) \frac{i}{2} \]

In Lorentzian signature
\[ \langle \psi | \phi \rangle = \int_M g^{-\mu\nu} \phi^\mu \psi^\nu \]

measuring probability (density) of finding particle at \( x^{\mu} \) via

Poisson brackets \( \{ p_{\mu}, x^{\nu} \} \) replaced by quantum commutators of operators
\[ [\hat{p}_{\mu}, \hat{x}^{\nu}] = -i \hbar [\mathcal{L}]^\mu_{\nu} \]

represented on wavefunctions by
\[ \{ \hat{p}_{\mu}, \psi(x) \} = x^{\mu} \psi(x) \] multiplication operator
\[ \{ \hat{p}_{\mu}, \psi(x) \} = -i \hbar \frac{\partial}{\partial x^\mu} \psi(x) \]

Will drop the \( ^\mu \)'s unless absolutely necessary.
In a similar vein, classical Hamiltonians become operators

\[ H = \pm \not{p} \phi + m \not{\phi} \rightarrow \hat{H} = \pm \not{p} \phi + m \not{\phi} \]

But this procedure is ambiguous like \([\phi, \phi^\dagger] \neq 0\).

Must use experiment to make choice (typical method is to impose symmetries).

Here we can use geometry (a diffeomorphism invariance) as one guiding principle:

Consider \( p_\mu \rightarrow -L_\mu \) (setting \( k = 1 \))

\[ \hat{H} \Psi = -\frac{\hbar^2}{2m} \not{p}_\mu g^{\mu\nu} \frac{\partial}{\partial x^\nu} \Psi \]

Notice that \( \frac{\partial}{\partial x^\nu} \) are components of a 1-form

\[ \Rightarrow g^{\mu\nu} \frac{\partial}{\partial x^\nu} \text{ is a vector} \]

But \( \frac{\partial}{\partial x^\nu} \) is NOT A SCALAR (or some vector)

Although \( \frac{\partial}{\partial x^\nu} = g^{\mu\nu} \frac{\partial}{\partial x^\mu} + \Gamma^{\mu}_{\rho\nu} \frac{\partial}{\partial x^\rho} \) is A SCALAR.

**FACT** (explicit proof using \( \phi = \exp(i \Theta) \) not hard)

\[ \left( \frac{\partial}{\partial x^\mu} \right)^2 \frac{\hbar^2}{2m} \left( \frac{\partial}{\partial x^\mu} \right)^2 g^{\mu\nu} \frac{\partial}{\partial x^\nu} \phi = \nabla^\mu \nabla_\mu \phi \]

Let \( g = \exp(\Theta g_\mu) \)

\[ \hat{H} = \pm \frac{\hbar^2}{2m} \not{p}_\mu g^{\mu\nu} (\not{p} \phi + m \phi) g^{\nu\lambda} \not{p}_\lambda \]

**Observe**

\[ -2 \langle \phi | H | \phi \rangle = \int g^{\mu\nu} g^{\rho\sigma} \left( \frac{\partial}{\partial x^\mu} g^{\rho\sigma} \left( \frac{\partial}{\partial x^\nu} g^{\rho\sigma} \right) \phi \right)^2 \]

\[ = -\int (\not{p} \phi) (\not{p} \phi) (\not{p} \phi) (\not{p} \phi) \phi \quad \text{(positive-definite)} \]

\[ = -2 \langle \phi \mid \hat{H} \mid \phi \rangle \quad \text{Hermitian} \]

**Summary** Quantization of particle moving in curved space \( M \)

\[ \rightarrow \text{scalars, } -2H = \Delta \quad \text{Laplacian} \]
We need to be able to integrate over worldline in coordinate invariant way.

Introduce worldline "top-form"/volume form/"density" $e$

In any choice of worldline coordinate $t$

$e = e(t) dt$

Under changes of coordinate $t = t(t')$

$e'(t') = e(t(t')) d(t(t')) = e(t(t')) (dt/dt') dt'$

Scalar transformation rule "orbit"
Now have world line 1-forms
\[
\begin{align*}
\theta^* &= p_\mu \gamma^\mu dt \\
eH &= e(\theta) H dt 
\end{align*}
\]

\[S[e,x^*,p] = \int_R \left( \theta^* - eH \right) = \int_R \left[ p_\mu \gamma^\mu dt - \frac{e}{2} p_\mu g^{\mu\nu} (\gamma^\nu) p_\nu \right] \]

Notice, $e$ is gauge multiplier for constraint

Claim $S$ is GAUGE INVARIANT

Obvious: by construction $S$ is worldline coordinate independent

Explicitly
\[
\begin{align*}
\delta e &= \frac{\delta S}{\delta e} = \frac{\delta}{\delta t} (S) \, \text{new} \\
\delta p_\mu &= \frac{\delta S}{\delta p_\mu} \, \text{worldline scalars} \\
\delta x^\mu &= \frac{\delta S}{\delta x^\mu} \, \text{worldline scalars}
\end{align*}
\]

Exercise, verify $\delta S = 0$ for these transformations

Note we can also add a term $\frac{m^2 e}{\lambda}$ to our action for $m^2$ constant $\pm 1$ is a 1-form

So study
\[S = \int_R \left[ p_\mu \gamma^\mu dt - \frac{e}{2} p_\mu g^{\mu\nu} (\gamma^\nu) p_\nu \right] \]

"Not difficult" to see this action is minimized on equations:

Integrate out $p_\mu$
\[
\frac{\delta S}{\delta p_\mu} = 0 \Rightarrow \gamma^\mu = \delta g^{\mu\nu} p_\nu \Rightarrow p_\mu = e g^{\mu\nu} \gamma^\nu
\]

This is an ALGEBRAIC FIELD EQUATION, so may substitute it back in action
\[p_\mu - \delta p_\mu + \frac{\delta}{\delta t} \frac{\delta S}{\delta x^\mu} = \frac{\delta}{\delta t} \frac{\delta S}{\delta e}
\]

\[S^{(e)} = \frac{1}{2} \int_R dt \left[ \frac{\dot{x}^n g^{n\nu} \dot{x}^\nu}{\delta} - m^2 e \right]
\]

Note $\dot{x}^n$ is a WORLDLINE 1-FORM $\rightarrow \frac{\dot{x}^n}{\delta}$ is a 1-FORM

Still worldline DIFFEOMORPHISM GAUGE INVARIANT

Now integrate out $e$
\[
\frac{\delta S}{\delta e} = 0 \Rightarrow -\frac{\dot{x}^n g^{n\nu} \dot{x}^\nu}{\delta} = m^2
\]
Back-substituting yields \( e = \sqrt{-\frac{i M_k}{m^2}} \).

Pullback of \( s = -i \) to worldline.

Clearly if the energy integral \( \int s \) was minimized on geodesics, so is the square root action.

Exercise (1) In the Hamiltonian approach, \( \theta \) is the Lagrange multiplier for the constraint:

\[ p^\mu \phi_{\mu} + m^2 = 0 \]

Analyze with when \( m^2 = 0, \neq 0 \) in Lorentzian/Riemannian signatures.

Notice with \( m^2 = 0 \) in Lorentzian/Riemannian signatures.

Rewrite this entire lecture replacing, wherever possible, the worldline \( \mathbb{R} \) with a world sheet \( \mathbb{R}^2 \), locally homogeneous to \( \mathbb{R}^2 \). Enter your treatise "string theory".

Notice gauge invariance \( \phi_{\mu} \rightarrow \phi_{\mu} + \delta \phi_{\mu} \) worldline diffeomorphisms.

Supersymmetry

Introduce classical Grassmann variables

\[ \theta^i_\mu = (\theta^i_\mu, \bar{\theta}^i_\mu) \] "anti-complex", \( i = 1, 2 \)

Really just want to have some involution labeled by "\( \bar{\cdot} \)".

Algebra

\[ \theta^i_\mu \bar{\theta}^j_\nu = -\theta^j_\nu \theta^i_\mu \] "anti-commuting"

Fermions

Structure common in mathematics

Differential forms

\[ dx^\mu \wedge dx^\nu = - dx^\nu \wedge dx^\mu \]

Notice

\[ \theta^i_\mu \bar{\theta}^i_\mu = 0 \] (no sum)

Also want independent copy of this algebra at each point on the worldline. Will write \( \theta^i_\mu (\vec{r}) \) with

\[ \theta^i_\mu (\vec{r}) \bar{\theta}^j_\nu (\vec{r}) = - \theta^j_\nu (\vec{r}) \theta^i_\mu (\vec{r}) \]

Grade algebra over \( \mathbb{C} \) by fermion number

\( \theta \wedge \bar{\theta} \rightarrow \theta \wedge \bar{\theta} \rightarrow \theta \wedge \bar{\theta} \rightarrow \theta \wedge \bar{\theta} \rightarrow \theta \wedge \bar{\theta} \rightarrow \theta \wedge \bar{\theta} \rightarrow \theta \wedge \bar{\theta} \)

Will write

\[ \theta^i_\mu + i \psi_{\mu} (\vec{r}) \theta^i_\mu \bar{\theta}^i_\mu \text{ etc...} \]
Consider new action

\[ S = \int \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + \frac{i}{2} \epsilon_{ij} \theta^i \frac{\partial \theta^j}{\partial x^k} + \frac{1}{16} \epsilon_{klm} \epsilon_{ij} \frac{\partial \theta^l}{\partial x^k} \frac{\partial \theta^m}{\partial x^j} \]