LECTURE #6  \( N=2 \) SUSY QM & DIFFERENTIAL FORMS

\[ S = \int dt \left\{ \frac{1}{2} \dot{x}^\mu \dot{x}_\mu + \frac{i}{2} \dot{\theta}^\mu \dot{\theta}_{\dot{\mu}} - \frac{1}{8} \bar{\theta} \gamma_\mu \theta \gamma^\mu \theta \right\} \]

SYMMETRIES

\[
\begin{align*}
S^x \mu &= \dot{x}^\mu \xi^i + \xi \dot{\xi}^\mu \\
S^\theta \mu &= \dot{\theta}^\mu \xi^i + \xi \dot{\theta}^\mu + \lambda \xi \theta \xi \\
&\uparrow \uparrow \uparrow \uparrow \uparrow \\
N=2 &\text{ SUSY} \quad \text{NMLine}\quad \text{running}\quad \text{symmetry} \\
&\text{R-Symmetry}
\end{align*}
\]

AIM: show quantum Noether charges for these symmetries are

\[ d, S = dx, \Delta_p, N \text{ acting on time} \]

Quantum Noether Charges

FIRST NEED SYMPLECTIC STRUCTURE, \( \to \) CAN BE OBTAINED FROM 1ST ORDER ACTION

\[
S^{(n)} = \int \left[ \Pi \mu \dot{x}^\mu + \frac{i}{2} \dot{\theta}^\mu \gamma^\mu \theta_{\dot{\mu}} - \left( \frac{i}{2} \Pi^\mu \gamma_\mu + \frac{1}{8} \bar{\theta} \gamma_\mu \theta \gamma^\mu \theta_{\dot{\mu}} \right) \right] \\
\]

WHERE \( \Pi \mu \gamma^\mu + \frac{i}{2} \theta_{\dot{\mu}} \gamma^\mu \theta = \left( \gamma^\mu + \frac{i}{2} \theta_{\dot{\mu}} \gamma^\mu \theta_{\dot{\mu}} \right) \gamma^a + \frac{1}{2} \theta_{\dot{\mu}} \theta^a_{\dot{\mu}} \gamma^a \theta^a_{\dot{\mu}} \gamma^a \)

"COVARIANT MOMENTUM" \( \equiv P \)

THE PAIRS \( (x^\mu, p_\mu) \), \( (\theta^\mu, \theta_{\dot{\mu}}) \) ARE "DARBOUX COORDINATES"

READ OFF SYMPLECTIC STRUCTURE/QUANTUM COMMUTATORS

\[
\begin{align*}
\{ p_\mu, x^\nu \} &= -i \delta_{\mu}^\nu \\
\{ \theta^\mu_{\dot{\mu}}, \theta^\nu_{\dot{\nu}} \} &= \delta_{\mu}^\nu \delta^a_{\dot{\mu}} \delta_{\dot{\nu}}^a
\end{align*}
\]

EXERCISE BUILD ANY HAMILTONIAN OF YOUR CHOICE INVOLVING AT LEAST 2 \( \theta^\mu \)S AND VERIFY THAT

\[ \theta_{\dot{\mu}} = -i \left[ \theta_{\dot{\mu}}, H \right] \quad \text{Heisenberg Comm} \]

MATCHES THE EXTREMUM OF

\[ S = \int \left( \frac{1}{2} \theta^\mu_{\dot{\mu}} \theta_{\dot{\mu}} - H \right) dt \]
TO BUILD CHARGES, STUDY BLINNARS BUILT FROM FERMIONS

\[
\begin{align*}
2N + d &\equiv N_{\mu} \varepsilon_{ijk} \quad \text{equivalently } N_{\mu} \varepsilon_{ijk} \\ M_{\mu} &\equiv \Phi_{\mu} \Phi_{\mu} \quad \text{symmetries } \sim g_{\mu}
\end{align*}
\]

THE OPERATOR \( N \) COUNTS FERMIONS WITH WEIGHT

\[
\{N, \theta^i_\sigma\} = \theta^i_\sigma \varepsilon_{ijk} \theta^j_\sigma, \theta^k_\sigma = \varepsilon^{\mu\nu} \theta^i_\sigma, \quad R\text{-SYMMETRY}
\]

\(\text{Le. } [N, (\theta^i_\sigma)] = (-1)^i (\theta^i_\sigma) = (+\theta^i_\sigma)\)

THE OPERATORS \(M_{\mu} \) GENERATE \( SO(4) \sim \text{LOCAL LORIENTZ} \)

\[
\begin{align*}
[M_{\mu}, M_{\nu}] &\equiv \theta^i_\mu \theta^j_\nu \varepsilon_{ijk} \varepsilon_{ijk} + 3 \text{ more} \\
&\equiv \theta^i_\mu \theta^j_\nu g_{\mu\nu} + 3 \text{ more} \\
&= M_{\mu\nu} g_{\mu\nu} + 3 \text{ more}
\end{align*}
\]

NOTATION:

\[
[\mathbf{X^k}, \mathbf{X^\mu}] \sim R^k_{\mu
}
\]

WORLDLINE TRANSLATIONS

\[
H = \frac{1}{2} (\pi^\mu + i M^\mu) \pi^\nu \pi^\nu + \frac{1}{8} R^k_{\mu
}
\]

QUANTUM HAMILTONIAN

\[
\frac{d}{dt} = i \frac{1}{\hbar} [\cdot, H] \quad \text{SCHRÖDINGER EQUATION}
\]

SUPERSYMMETRY

CLAIM \( Q^i \equiv \Phi^i_\mu g^{\mu\nu} T_\nu \) COMMUTES WITH \( H \)

MAKES SENSE \( \delta x^\mu = \frac{i}{\hbar} [x^\mu, Q^i] = -i \theta^i_\nu [x^\mu, T_\nu] = \partial^\mu \)

CHECK NON TRIVIAL COMPUTATION, NEED

\[
[\mathbf{x^\mu}, \mathbf{x^\nu}] \sim R^k_{\mu\nu}
\]
SUPERALGEBRA

$N$-grading $\downarrow \quad \mathbb{C}^*$

$Q^N \rightarrow$ odd-grading

($\#$ derivatives)

$Q$

suggests

$\{Q, Q^*\} = -2H$

To see why note

$\{Q, Q^*\} \sim \{0, \pi, \theta\}$

$\theta[0, 0] \rightarrow \mathbb{R}^*$

$\pi \times 0, \pi \rightarrow x, y, \pi^*$

$\pi \times \pi \rightarrow y^*, \pi^*$

Quantum term is trickiest

Actually these results will be guaranteed by geometry!

STATES

Need to represent the Heisenberg algebra:

$\{x^\mu, p_\nu\} = i\delta_\mu^\nu$

$\{\theta^\mu, \theta_\nu\} = \delta^\mu_\nu \delta^0_\nu$

The first is easy $\rightarrow$ wave functions $\psi(x) \rightarrow$ scalars!

$p_\mu = -i \frac{\partial}{\partial x^\mu}$

Consider

$\{x^\mu, \theta_\nu\} = \delta^\mu_\nu$ (writing out $\theta$ indices explicitly)

Build a Grassmann Fock Space

$\rightarrow$ treats $(\theta, \theta^*)$ like $(\alpha, \alpha^*)$ in harmonic oscillator

$\rightarrow$ a coherent state treatment gives an equivalent "wave-function" formulation

Define Fock vacuum $\mid 0\rangle$

$\delta_\mu^\nu \mid 0\rangle = 0$

i.e. here will find finite dim Grassmann Hilbert space $\rightarrow$ nothing will depend on this choice.
THE OPERATORS $\phi^m$ ARE CREATION OPERATORS, BUILD 2d STATES

\[ |0\rangle, \phi^m |0\rangle, \gamma^m |0\rangle, \ldots \]

\[ 1 + x + \frac{x(x-1)}{2} + \ldots = (1+i)^x \]

WE CAN VIEW $\phi^m$ AS CREATING AN ANTISYMMETRIC INDEX $\mu$. "DYNAMICS IN INDEX SPACE."

THE MOST GENERAL STATE IS

\[ |I\rangle = \sum \omega_{\mu_1...\mu_k} (x) \phi^{\mu_1} \cdots \phi^{\mu_k} |0\rangle \]

NOTE THE OPERATORS $\phi^m$ REMOVE INDICES
E.G. CALL

\[ |\omega_{\mu\nu}\rangle = \omega_{\mu\nu} \phi^\mu \phi^\nu |0\rangle \]

\[ \omega_{\mu\nu} = -\omega_{\nu\mu} \]

THEN

\[ \phi^m |\omega_{\mu\nu}\rangle = \omega_{\mu\nu} [\phi^m, \phi^\mu \phi^\nu] |0\rangle = 2 \omega_{\mu\nu} \delta^m \phi^\mu |0\rangle \]

\[ = \omega_{\mu\nu} \phi^\nu |0\rangle \] (UNFORTUNATELY OPEN INDEX $\mu$)

NOTICE $\theta^\mu = \frac{2}{d \delta m^\mu}$ ALC DERIVATIVE

REVOLUTIONARY NOTATION

\[ |0\rangle = 1 \]

\[ \phi^m = \partial \phi^m \]

\[ \gamma^m \phi^\nu = \partial \phi^m \wedge \partial \phi^\nu = -\partial \phi^\nu \wedge \partial \phi^m \]

\[ \theta^\mu = \frac{2}{d \delta \phi^m} \]

NOTICE

\[ \left( \frac{2}{d \delta \phi^m} \right) |1\rangle = 0 \Leftrightarrow \phi^m |0\rangle = 0 \]

NEW MOST GENERAL STATE IS

\[ |I\rangle = \sum \omega_{\mu_1...\mu_k} (x) d \phi^{\mu_1} \wedge \ldots \wedge d \phi^{\mu_k} \]

\[ I \in T(\text{AM}) \text{ DIFFERENTIAL FORM} \]

NOTE $\omega_{\mu\nu} + \omega_{\nu\mu}$ IS FORBIDDEN

BUT $\omega_{\mu} \phi^m + \omega_{\nu} \phi^m \wedge d \phi^k \wedge d \phi^l$ IS ALLOWED
QUANTUM NOETHER CHARGES & GEOMETRY

1. DEGREE

\[ N = \partial^\mu \delta_\mu = d_x \mu \frac{\partial}{\partial (d_x \mu)} \]

Eigenvalues are form degree

(\text{cf. } N = a^* a \text{ for Harmonic Oscillator})

Example

\[ N \omega dx^\mu dx^\nu \Rightarrow dx^\mu \frac{\partial}{\partial (dx^\mu)} \omega dx^\nu = \frac{2}{d_x \mu} \omega dx^\mu \]

\[ = 2 \omega dx^\mu \delta^\mu_\nu dx^\nu \]

\[ = 2 \omega dx^n dx^m dx^v \]

I.e. \( N \omega = \deg(\omega) \omega \) for forms of definite degree

2. EXTERIOR DERIVATIVE

\[ d \equiv iQ* = \partial^\mu \delta_\mu \]

\[ = dx^\mu \left( \frac{\partial}{\partial x^\mu} \right) \]

Related to \[ T^\mu_\nu \]

\[ d \text{ EXTERIOR DERIVATIVE} \rightarrow \text{metric independent} \]

\[ \text{NOTE} \quad [N, d] = \left[ dx \cdot \frac{\partial}{\partial dx} \right] = dx \cdot d = +1 \text{d} \]

\[ E \text{ik d ADDS 1 INDEX} \]

3. CODIFFERENTIAL

\[ \delta \equiv iQ = i \delta^\mu \delta_\mu \pi \nu = \frac{\partial}{\partial (dx^\mu)} \pi^\mu \nu \left( \frac{2}{d_x \mu} \right) \pi^\mu \nu \]

Consider

\[ \pi^\mu \nu \omega dx^\mu \pi^\mu \nu \Rightarrow \omega dx^\mu \]

\[ = \omega dx^\mu \pi^\mu \nu \rightarrow \pi^\mu \nu dx^\mu \]

Then

\[ \delta \omega dx^\mu \pi^\mu \nu dx^\mu = k \nabla^\mu \pi^\mu \nu dx^\mu \ldots dx^\mu \]

This is the standard codifferential/form divergence

It can also be written \( \Delta d_x \).

Exercise find an \( N=2 \) QM expression for

the operator \( \pi^\mu \nu \).

\[ \text{NOTICE} \quad [N, \delta] = -\delta \quad (\text{ex. check this}) \]

\( \delta \) contains

an index
FORM LAPLACIAN

CAN STEAL FROM GEOMETRY

\[ \Delta F = d \delta + \delta d = \{d, \delta\} \]

"SUPERSYMMETRY ALGEBRA \( \{Q, Q^*\} = -2H \)

THE FORM LAPLACIAN IS NOT BOCHNER LAPLACIAN

\[ \Delta F = \nabla^\mu \nabla_\mu + \frac{1}{4} R^{\# \#} \]

WHERE

\[ [\nabla_\mu, \nabla_\nu] = R^{\#}_{\mu\nu} \]

AND \# DENOTES THE USUAL ACTION OF RIEMANN

E.G.

\[ R^{\#}_{\mu\nu} \phi = \nabla^\rho \nabla_\rho \phi \]

in accordance with

\[ [\nabla_\mu, \nabla_\nu] \phi = R^{\#}_{\mu\nu} \phi \]

IN QM NOTATION \# DENOTES \( M_{\mu\nu} \) CONTRACTION

\[ R^{\#\#} = R^\rho_{\mu\nu} \phi \sum dx^\rho \partial^2 (\partial(x^\rho) \partial^2 (\phi(x))) \]

FINALLY IT IS NOW OBVIOUS THAT THE MODEL WITH

\[ -2H = \Delta F \]

IS SUPER-SYMMETRIC B/C

\[ [d, \Delta F] = d (d\delta + \delta d) - (d\delta + \delta d) d \]

\[ \delta \Omega \text{ (Q,H) commutator} \]

\[ \delta = 0 \]

NEXT TIME GAUGE THE SYMMETRIES \( \{d, \delta, \Delta F\} \)