Associated Bundles

Given any locally trivial "bundle of gadgets" $E \xrightarrow{\pi} M$ with standard fiber $F$ (some "gadget"), we've seen how to get a principal $G$ bundle, where $G = \text{Aut}(F)$ is the group of "gadget automorphisms" of $F$. Namely, we let

$$P_x = \{ \text{gadget isomorphisms } f: F \to E_x \}$$

be the space of "generalized frames" at $x$, and glue these together smoothly to get a bundle:

$$P = \bigcup_{x \in M} P_x$$

This is a principal bundle because each $P_x$ is isomorphic — as a right $G$-space, not as a group — to $G$ itself:

$$\exists g \in G \text{ st. } f' = f$$

Today, we'll see how to reconstruct the bundle $E$ from its principal bundle $P$ of generalized frames and its standard fiber $F$ — and also get lots of new bundles.
**Example:** Let $\Sigma$ be a surface, $F \Sigma$ its frame bundle, where a frame is any linear isomorphism

$$f : \mathbb{R}^2 \to T_x \Sigma.$$ 

Any two frames at $x$ are related by a unique elt. of

$$GL(2, \mathbb{R}) = \{ \text{linear automorphisms of } \mathbb{R}^2 \}.$$ 

So $F \Sigma$ is a principal $GL(2, \mathbb{R})$-bundle. Now consider the reverse process: suppose we’re given some principal $GL(2, \mathbb{R})$-bundle $\frac{P}{\xi}$ and we want to construct a 2d vector bundle $\frac{E}{\xi}$ that has $P$ as its “bundle of frames.”

Each $E_x$ should be a 2d vector space; if an elt. $f \in P_x$ is to be thought of as a “frame” at $x$, it should give a basis of $E_x$, as the image of the standard basis:

$$\mathbb{R}^2 \xrightarrow{f} \begin{array}{c}
\begin{array}{c}
\uparrow \\
\rightarrow
\end{array}
\end{array} \xrightarrow{f} \begin{array}{c}
\begin{array}{c}
\uparrow \\
\rightarrow
\end{array}
\end{array}$$

An elt. $v \in E_x$ can then be specified by a “frame” $f \in P_x$ together with an elt. $\tilde{v} \in \mathbb{R}^2$ — the “components of $v$ in the frame $f$.” But, the pair $(f, \tilde{v})$ is not unique! ...
If I change \( f \) to \( fg \), \( g \in GL(2,\mathbb{R}) \), then the same vector \( v \in \mathbb{R}^2 \) should now have components \( g^t v \in \mathbb{R}^2 \):

\[
\begin{array}{c}
\mathbb{R}^2 \\
\uparrow \\
\downarrow \\
\mathbb{R}^2
\end{array}
\xrightarrow{fg}
\begin{array}{c}
v \\
\uparrow \\
\downarrow \\
v
\end{array}
\]

So: we could say \( E_x \) consists of pairs \((f, \tilde{v}) \in P \times \mathbb{R}^2 \modulo \) identification of \((f, \tilde{v})\) with \((fg, \tilde{v})\) for all \( g \in GL(2,\mathbb{R}) \).

This example motivates the following definition:

First, let \( G \) be a Lie group, \( P \) a principal \( G \)-bundle over a manifold \( M \):

\[
P \xrightarrow{p} M
\]

and let \( V \) be a (left) \( G \)-space. Define an equivalence relation on \( P \times V \) by

\[
(p, v) \sim (p', v') \iff p' = pg, \quad v' = g^t v \quad \exists g \in G;
\]

in other words:

\[
(pg, v) \sim (p, g^t v) \quad \forall g \in G, \ p \in P, \ v \in V.
\]

Then let

\[
P \times_G V := \frac{P \times V}{\sim}
\]

be the space of equivalence classes, and denote the
class of \((p,v) \in P \times V\) by \([p,v] \in P \times_G V\).

**Def:** If \(P \overset{\pi}{\to} M\) is a principal \(G\)-bundle and \(V\) is a \(G\)-space, then the bundle

\[
P \times_G V \quad \xrightarrow{\pi} \quad [p,v] \quad \downarrow \quad \pi(p)
\]

is called the bundle **associated** to \(P\) with standard fiber \(V\).

(Note: this definition actually requires a few little theorems before it makes sense, e.g.:)

- \(P \times_G V\) is smooth mfld.
- the projection \([p,v] \mapsto \pi(p)\) is smooth.
- \(P \times_G V\) really is locally trivial — locally iso. to a trivial bundle — so the phrase "with standard fiber" has its standard meaning.

Most of this is easy, but we won't worry about it.)

Back to our example, we really do get

\[
F \Sigma \times_{\text{GL}(2,\mathbb{R})} \mathbb{R}^2 \cong T \Sigma
\]

So "taking the associated bundle" seems to be the inverse of "taking the bundle of frames". This is true more generally...
Thm: Suppose $E \xrightarrow{\pi} M$ is a "bundle of gadgets" with standard fiber $V$ and let $P \xrightarrow{\pi} M$ be the bundle of generalized frames $f: V \to E_x$ — a principal $G$ bundle, $G = \text{Aut}(V)$. Then there is a canonical isomorphism of bundles of gadgets:

$$P \times_G V \cong E$$

Pf: Exercise! (You can ignore technical details such as smoothness conditions — just construct an explicit "gadget-bundle isomorphism".)

Examples of Associated Bundles

Once we have a principal $G$-bundle, we can construct tons of new bundles — one for every $G$-space $V$ we can imagine. Better yet, if $V$ has the structure of some gadget and the action of $G$ is as gadget automorphisms, the associated bundle will be a bundle of gadgets.

Examples: (for $G$ and a principal $G$-bundle $P \xrightarrow{\pi} M$)

1) If $G$ acts as linear transformations of the vector space $V$ (i.e. $G$ has a representation on $V$) then we get a vector bundle $P \times_G V$.

(e.g. $G = \text{GL}(n, \mathbb{R})$, $P = FM$, $V = \mathbb{R}^n \implies P \times_G V \cong TM$)
2) If $V$ is an inner product space and $G$ has a rep. on $V$ that preserves the inner product, then we get a **bundle of inner product spaces** $P \times_\theta V$.

(e.g. $G = O(n)$, $P = FM$ = the orthonormal frame bundle, $V = \mathbb{R}^n$ $\mapsto$ $P \times_\theta V = (TM, g)$ = tangent bundle with metric)

3) If $V = G'$ is a group, and $G \leq G'$, then $G$ acts on $G'$ by multiplication

$$L : G \times G' \longrightarrow G'$$

$$(g, g') \mapsto g g' = : L_g g'$$

and the associated bundle $P \times_\theta G'$ is a **principal $G'$-bundle** — Not a bundle of groups! Why? Because left multiplication by $g \in G$ is not an automorphism of $G'$ as a group. But it is an automorphism of $G'$ as a right $G'$-space.

**Exercise**: Show that $P \times_\theta G'$ really is a principal $G'$ bundle. If $G = G'$, show $P \times_\theta G \cong P$ as principal bundles.
4) Again taking $G \leq G'$, $G$ also acts on $G'$ by conjugation:
\[
AD : G \times G' \longrightarrow G' \\
(g, g') \longmapsto gg'g^{-1} = AD(g)g'
\]

For each $g$, $AD(g) : G' \longrightarrow G'$ is a group automorphism, so the associated bundle is a bundle of groups, not just torsors. To distinguish it from example 3, we denote the associated bundle
\[
P \times_{AD_g} G'.
\]

**Example 4** is very important, especially in the case $G = G'$.

**Thm:** Let $P$ be a principal $G$-bundle, $P \times_{AD} G$ the bundle of groups associated to $P$ via the action of $G$ on itself by conjugation. Then there is a group isomorphism:

\[
\begin{array}{c}
\text{Gauge transformations} \\
P \xrightarrow{\alpha} P
\end{array}
\]

\[
\begin{array}{c}
\text{(w. composition of principal} \\
\text{bundle maps)}
\end{array}
\]

\[
\begin{array}{c}
\text{Sections of associated} \\
\text{group-bundle} \\
P \times_{AD} G
\end{array}
\]

\[
\begin{array}{c}
\text{(w. pointwise multiplication of} \\
\text{sections)}
\end{array}
\]

**Pf:** Homework! (This is the version for nontrivial bundles of our earlier observation (lecture 2) that gauge transformations may be viewed either as bundle automorphisms or a sections of a certain group bundle.)