Recall that a connection on a principal $G$-bundle $\mathcal{P}$ is a smoothly varying family of linear maps (one for each $p \in \mathcal{P}$):

$$A_p : T_{\pi(p)}M \rightarrow T_p \mathcal{P}$$

s.t.

1) $d\pi \circ A_p = 1_{T_{\pi(p)}M}$

2) $A_p = dR_g \circ A_p$

We saw that doing parallel translation along $\gamma : [0,1] \rightarrow M$:

\[ \begin{array}{ccc}
    \gamma & \longrightarrow & P \\
    \gamma \circ x & \longrightarrow & P_x \\
    \gamma \circ y & \longrightarrow & P_y
\end{array} \]

meant solving the ODE

$$\tilde{\gamma}'(t) = A_{\tilde{\gamma}'(t)}(\gamma'(t)) \quad (*)$$

for $\tilde{\gamma} : [0,1] \rightarrow \mathcal{P}$. We defined the holonomy of $\gamma$ to be

the isomorphism

$$\text{hol}(\gamma) : P_{\gamma(0)} \longrightarrow P_{\gamma(1)}$$

$$p \longmapsto \tilde{\gamma}_p(1)$$

where $\tilde{\gamma}_p$ is the unique solution of $(*)$ satisfying the initial condition $\tilde{\gamma}_p(0) = p \in P_{\gamma(0)}$. 

D. Wise
Using parallel translation in a principal $G$-bundle $P$, we immediately get:

1) Parallel translation in any associated bundle

$$E = P \times e V = \frac{P \times V}{(pg,v) - (p,gv)}$$

If $\tilde{\gamma}_p(t)$ is the horizontal lift of $\gamma$ starting at $p \in P$, then parallel translation of the point $[p,v] \in E_x$ along $\gamma$ is given by

$$[\tilde{\gamma}_p(t), v] \in E_{\tilde{\gamma}(t)}$$

Note this is well defined, since parallel translation of $[pg, g^{-1}v] = [p,v]$ along $\gamma$ is given by

$$[\tilde{\gamma}_p(t), g^{-1}v] = [\tilde{\gamma}_p(t)g, g^{-1}v] = [\tilde{\gamma}_p(t), v]$$

since we showed last time that horizontal lifts are related by $\tilde{\gamma}_p(t) = \tilde{\gamma}_p(t)g$.

2) Covariant derivatives of sections of any associated vector bundle:

If $E = P \times e V$ is a vector bundle (i.e. $V$ is a vector space on which $G$ acts linearly), then given $w \in T_x M$, choose $\gamma$ s.t. $\gamma(0) = x, \gamma'(0) = w$.

For a section $\sigma : M \to E$, define the covariant derivative

$$(D_w \sigma)(x) = \lim_{t \to 0} \frac{\sigma(\tilde{\gamma}(t)) - [\tilde{\gamma}_p(t), v]}{t} \in E_x$$

of $\sigma$ at $x$ in direction $w$.

For $w$ a vector field, $(D_w \sigma)(x)$ gives a new section:

$$\downarrow \quad D_w \sigma$$

the covariant derivative of $\sigma$ w.r.t. $w$. 
To make these ideas more concrete, let's see what connections and parallel translation amount to in a local trivialization. For notational simplicity, we just assume $P$ is trivial:

$$P = M \times G \quad \text{(x,g)}$$

$$\downarrow \quad \downarrow$$

$$M \quad x$$

By definition, a connection $A$ gives a family of maps

$$A_{(x,g)} : T_{(x,g)}M \longrightarrow T_{(x,g)}(M \times G)$$

$$\downarrow \quad \downarrow$$

$$T_xM \quad T_xM \times T_g G$$

satisfying:

1) $d\pi \circ A_{(x,g)} = 1_{T_xM}$.

Since $d\pi : T_xM \times T_g G \longrightarrow T_xM$ is given by $(v, x) \mapsto v$, this equation just says $A_{(x,g)}(v) = (v, \tilde{A}_{(x,g)}(v))$ for some linear map

$$\tilde{A}_{(x,g)} : T_xM \longrightarrow T_g G$$

2) $A_{(x,g)}^\prime = dR_{g} \circ A_{(x,g)}$.

This is equivalent (Exercise!) to $\tilde{A}_{(x,g)} = dR_{g} \circ \tilde{A}_{(x,g)}$ where $R_{g}$ here denotes the right action of $G$ on itself. I.e.,

$$T_xM \xrightarrow{\tilde{A}_{(x,g)}} T_g G \xrightarrow{dR_{g}} T_{g}G \quad \text{commutes}.$$

So, all of the $\tilde{A}_{(x,g)}$ are determined by

$$A_x := \tilde{A}_{(x,1)} : T_xM \longrightarrow \mathfrak{g}$$

where $\mathfrak{g} = T_{e}G$ is the Lie algebra of $G$. 
So, on a trivial principal $G$-bundle (or in a local trivialization), a connection amounts to a smooth map

$$A : TM \longrightarrow \mathfrak{g}$$

that is linear on each $T_xM$. In other words, it's a Lie algebra-valued 1-form on $M$.

**Exercise:** Show that on the trivial principal $G$-bundle $P$, the ODE for parallel translation reduces to

$$\tilde{\gamma}_0'(t) = dR_{\tilde{\gamma}_0(t)} \circ A_{\tilde{\gamma}_0(t)}(\gamma'(t))$$

where the horizontal lift of $\gamma$ to $P = M \times G$ is given by

$$\tilde{\gamma} : [0, 1] \longrightarrow P$$

$$t \longmapsto (\gamma(t), \tilde{\gamma}_0(t))$$

and we take $\gamma_0 : [0, 1] \longrightarrow G$ s.t. $\gamma_0(0) = 1 \in G$.

(Note that $dR_{\tilde{\gamma}_0(t)} \circ A_{\tilde{\gamma}_0(t)} : T_{\tilde{\gamma}_0(t)}M \longrightarrow \mathfrak{g} \longrightarrow T_{\tilde{\gamma}_0(t)}G$ and the claim is: $\gamma' : \text{velocity at point on } M \longrightarrow \tilde{\gamma}'_0 : \text{velocity of path in } G$.)
Next, if \( P = M \times G \), then any associated bundle can be trivialized in a canonical way:

\[
E = P \times_\sigma V \xrightarrow{\sim} M \times V
\]

\[
[(x, g), v] \mapsto (x, g_0 v)
\]

(Exercise: check that this is a bundle isomorphism) Under this identification, parallel translation in \( E \) is given by

\[
[(\gamma(t), g_0(t)), v] = (\gamma(t), g_0(t)v) \in M \times V.
\]

I.e. parallel translation of \( v \in V \) along \( \gamma \) is given by

\[
[0, 1] \longrightarrow V
\]

\[
t \mapsto \gamma_0(t)v
\]

where \( \gamma_0(t) \in G \) acts on \( v \in V \) via the action that defines the associated bundle.

If \( E \) is a vector bundle, and \( \sigma \) is a section, we can use the trivialization \( E = M \times V \) to think of \( \sigma \) as a function \( \sigma : M \rightarrow V \), and define

\[
D_{\gamma'(0)} \sigma := \lim_{t \to 0} \frac{\sigma(\gamma(t)) - \gamma_0(t) \sigma(\gamma(0))}{t}
\]

\[
= \lim_{t \to 0} \left( \frac{\sigma(\gamma(t)) - \sigma(\gamma(0))}{t} - \frac{(\gamma_0(t) - 1) \sigma(\gamma(0))}{t} \right)
\]

\[
= \gamma'(0) \left[ \sigma \right] \quad - \quad \gamma_0'(0) \sigma(\gamma(0))
\]

\( \gamma' \) directional derivative of \( \gamma : M \rightarrow V \)

\( \gamma_0' \) elt. of \( G \) elt. of \( V \)

action of \( \gamma \) on \( V \)

elt. of \( \gamma \) elt. of \( V \)

is derivative of action of \( G \) on \( V \) defining the associated bundle \( E \).
We can simplify this further, since we know \( \tilde{y}_0 \) satisfies the IVP:

\[
\tilde{y}_0'(t) = dR_{\tilde{y}_0(t)} \circ A_{\tilde{y}(t)}(y'(t))
\]
\[
\tilde{y}_0(0) = 1 \in G.
\]

Using this, we get

\[
D_{\tilde{y}(0)} \sigma = \tilde{y}'(0) \sigma - A_{\tilde{y}(0)}(y'(0)).
\]

Working in local coordinates \( x^\mu \), if we take \( y'(0) = \frac{\partial}{\partial x^\mu} \) and define \( D_\mu := D_{x^\mu} \) and \( A_\mu(x) = A_x(\partial_{x^\mu}) \), we get

\[
D_\mu \sigma = \partial_\mu \sigma - A_\mu \sigma,
\]

as the formula for the covariant derivative in local coordinates. (Note: sometimes people call \( A = A \) instead, to get a "+" sign in the above formula — this is just a convention.)