The 2D Dijkgraaf-Witten model & EF Theory.

If $\mathcal{M}$ is a compact oriented 2-manifold with triangulation $\Delta_n$ and $\mathcal{G}$ is a compact Lie group, we expect that 2d BF theory gives

$$\tilde{Z}(\mathcal{M}) = \text{measure of } \frac{A_c(\Delta_n)}{\mathcal{G}(\Delta_n)}$$

—the measure of the "moduli stack of flat $\mathcal{G}$-bundles over $\mathcal{M}$."

Until we figure out what a measure on a groupoid should be like, we can wimp out & try

$$\bar{Z}(\mathcal{M}) = \text{measure of } \frac{\bar{A}_c(\Delta_n)}{\mathcal{G}(\Delta_n)}$$

—hoping that reducible connections don't count for much.

There's still the question: which measure?

Thm: There's a natural symplectic structure on the open dense subset of $\frac{\bar{A}_c(\Delta_n)}{\mathcal{G}(\Delta_n)}$ consisting of irreducible connections — a 2k-dimensional manifold. Thus

$$\omega^k = \omega \wedge \cdots \wedge \omega$$

$k$ times

is a volume form on this open dense subset.
Proof Sketch: We only get our nondegenerate closed 2-form \( \omega \) thanks to the fact that \( M \) is 2-dimensional. \( \omega \) should eat two tangent vectors

\[ u, v \in T_x \frac{A_n}{g(A_n)} \]

\& give a number \( \omega(u, v) \), whenever \( x = [A_0] \) for some reducible flat connection. I'll just give a formula for

\[ \omega(SA, SA') \]

where \( SA, SA' \in T_{A_0} \mathcal{A}_G(P) \)

where \( P \) is some principal \( G \)-bundle \( P \rightarrow M \). Note:

\[ SA, SA' \in \Omega^1(M, \text{Ad } P) \]

so we can define

\[ \omega(SA, SA') = \int_M \text{tr}(SA \wedge SA') \]

where \( \text{tr}(SA \wedge SA') \) is defined by wedge product of 1-form parts, trace of matrix product of \( g \)-valued parts (thinking locally of \( \text{Ad}(P) \)-val. form as locally a \( g \)-val differential form and thinking of \( g \) as a Lie algebra of matrices.) This uses the fact that \( M \) is 2-dimensional!

Need to check \( \omega \) is nondegenerate.
Given this, we could try to calculate $\tilde{Z}(M)$ as

$$\int \psi^k \quad \text{over \ irreducible \ reps.}\]$$

Or: we could copy what worked in Dijkgraaf-Witten model, instead of just generalizing the end result.

### 2d DW

- $G$: finite group
- $A = C[G] = \{ \psi: G \to \mathbb{C} \}$

![Diagram](triangle)

$m: A \otimes A \to A$

given by

$$\delta_g \ast \delta_h = \delta_{gh}$$

or

$$(\psi \ast \varphi)(g) = \sum_{h,k \in G \text{ st. } hk=g} \psi(h) \varphi(h')$$

$$= \sum_{h \in G} \psi(h) \varphi(h^{-1}g)$$

### 2d EF

- $G$: compact Lie group
- $A = L^2(G, \mu)$

![Diagram](triangle)

$m: A \otimes A \to A$

is convolution:

$$(\psi \ast \varphi)(g) = \int \psi(h) \varphi(h^{-1}g) \, d\mu(h)$$

(Note: $\int |\psi|^2 < \infty$ & $\int |\varphi|^2 < \infty$

$\Rightarrow \int |\psi \ast \varphi|^2 < \infty$.)
\[ \begin{array}{c}
\begin{array}{c}
\Diamond \\
\downarrow \\
\rightarrow
\end{array} =
\begin{array}{c}
\Box \\
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\rightarrow
\end{array} \quad \text{A is associative}
\end{array}
\]

\[ \begin{array}{c}
\begin{array}{c}
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\rightarrow
\end{array} \quad \text{comes from:}
\begin{array}{c}
\text{A is semisimple -}
\end{array}
\begin{array}{c}
g(a,b) = \text{tr}(L_a L_b)
\end{array}
\begin{array}{c}
is nondegenerate, where
\end{array}
\begin{array}{c}
L_a: A \rightarrow A
\end{array}
\begin{array}{c}
x \mapsto ax
\end{array}
\end{array}
\]

\[ \begin{array}{c}
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\Box \\
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\end{array} \quad \text{A is associative}
\end{array}
\]

\[ \text{Alas,} \]

\[ g(a,b) = \text{tr}(L_a L_b) \]

\[ \text{is ill-defined: A is infinite dimensional & \text{tr}(L_a L_b) doesn't always converge - e.g. if } a = b = 1 \]

\[ \text{we get } \dim(A) = \infty. \]

But all is not lost! We can "smooth out"

\[ m: L^2(G) \otimes L^2(G) \rightarrow L^2(G) \]

\[ \psi \otimes \phi \quad \mapsto \quad \psi \ast \phi \]

to ensure that \( \text{tr}(L_a L_b) \) converges. There's a Laplacian \( \nabla^2 \) on a compact Lie group - coming from invariant metric & thus for \( \alpha > 0 \) an operator

\[ e^{-\alpha H} : L^2(G) \rightarrow L^2(G) \]

where \( H = -\nabla^2 \) has all nonnegative eigenvalues, one for each irrep of \( G \):

\[ L^2(G) = \bigoplus_{\rho \in \text{Irep}(G)} \rho \otimes \rho^* \quad \text{(Peter-Weyl Thm)} \]
Using this, we define for each \( a > 0 \),

\[
\mathcal{M}_a : L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})
\]

\[
\psi \otimes \phi \rightarrow e^{-\mathcal{H}_a} (\psi \ast \phi)
\]

This makes

\[
\text{tr} (L \psi \phi)
\]

converge, where

\[
L \psi : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})
\]

\[
\phi \rightarrow e^{-\mathcal{H}_a} \psi \ast \phi
\]

We think of this \( \mathcal{M}_a \) as related to a triangle:

\[
\begin{array}{c}
\downarrow \phi \\
\downarrow \phi \\
\downarrow \phi \\
\end{array}
\]

where \( a \) is the area of the triangle. Luckily:

\[
\begin{array}{c}
\downarrow a_1 \\
\downarrow a_2 \\
\end{array}
\quad \begin{array}{c}
\downarrow a_3 \\
\downarrow a_4 \\
\end{array}
\]

holds if the areas match:

\[
a_1 + a_2 = a_3 + a_4
\]

So: we get not a topological quantum field theory, but an "area-logical field theory" — where we get well defined operators...
from triangulated manifolds, cobordisms equipped with total area for each component:

\[ A_1 \quad A_2 \]

So we get:

\[ Z : \text{2Cob}_{\text{with areas!}} \to \text{Vect} \]

Witten showed that this approach gives \( Z(M) \in C \) for a closed manifold with area. This agrees with the "symplectic structure on moduli space" approach in the limit as \( \text{area} \to 0 \), when the limit exists. (modulo some fudge factor involving Euler characteristic of \( M \).)