1.) \( !n \) is the number of elements of \( S_n \) that are not in any stabilizer subgroup. Since \( (S_n)_{\{i\}} \cong S_{n-1} \), we have \( |(S_n)_{\{i\}}| = (n - 1)! \). Similarly, \( |(S_n)_{\{i,j\}}| = (n - 2)! \) for \( i \neq j \), and so on. Also, there are \( \binom{n}{1} \) stabilizers of one element, \( \binom{n}{2} \) stabilizers of two elements, and so on. Thus, by the inclusion-exclusion principle:

\[
!n = |S_n| - \sum_i |(S_n)_{\{i\}}| + \sum_{i<j} |(S_n)_{\{i,j\}}| - \cdots + (-1)^n |(S_n)_{\{1,2,\ldots,n\}}|,
\]

or

\[
!n = n! - \binom{n}{1}(n - 1)! + \binom{n}{2}(n - 2)! - \cdots + (-1)^n (n - n)!,
\]

or more concisely,

\[
!n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n - k)!
\]

2.) This can be simplified using the definition of \( \binom{n}{k} \):

\[
!n = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} (n - k)!
\]

\[
= n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}
\]

3.) The probability of the coat-swapping being a derangement is

\[
\frac{!n}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!}
\]
Hence,
\[
\lim_{n \to \infty} \frac{!n}{n!} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = e^{-1}.
\]

4.) Theorem: For \( n > 0 \), \( !n \) is the closest integer to \( n!e^{-1} \).

Proof: Since we know \( !n \) is an integer, we only need to show that the absolute difference between \( !n \) and \( n!e^{-1} \) is less than a half.

\[
\left| \frac{n!}{e} - !n \right| = \left| n! \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} - n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} \right|
\]

\[
= n! \left| \sum_{k=n+1}^{\infty} \frac{(-1)^k}{k!} \right|
\]

\[
< n! \left| \frac{1}{(n+1)!} \right|
\]

\[
= \frac{1}{n+1}
\]

Here we have used the fact that an alternating series whose terms are decreasing in magnitude is strictly dominated by its first term. This proves the theorem for \( n \geq 2 \). Since the case \( n = 1 \) is easily verified (\( !1 = 0 \), and \( 1!/e \approx 0.37 \)), we are done. ♠

5.) To permute the elements of a set, we decide which elements will remain fixed under the permutation and then we derange the rest. That is, we split the set into two (possibly empty) parts, putting the vacuous structure “being a finite set” on the first part, and deranging the second part. Thus,

\[ P \cong E^Z D. \]

6.) Decategorifying, we get the following equation between generating functions:

\[ |P|(z) = e^z|D|(z). \]

If we rearrange this and use the fact that \( |P|(z) \) is just a geometric series, we get:

\[ |D|(z) = e^{-z}|P|(z) = \frac{e^{-z}}{1-z} \]
7.)

\[(1 - z) \frac{d}{dz} |D|(z) = (1 - z) \frac{d}{dz} \frac{e^{-z}}{1 - z} \]

\[= (1 - z) \left( \frac{e^{-z}}{(1 - z)^2} - \frac{e^{-z}}{1 - z} \right) = |D|(z) - e^{-z}.\]

8.) We now apply the result of part 7 to the power series representation of $|D|$, given by

\[|D|(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n.\]

For the left hand side we get:

\[(1 - z) \frac{d}{dz} |D|(z) = (1 - z) \sum_{n=0}^{\infty} \frac{1}{n!} z^n.\]

\[= (1 - z) \sum_{n=1}^{\infty} \frac{1}{(n-1)!} z^{n-1} = \sum_{n=1}^{\infty} \frac{1}{n!} (n+1) z^n - \sum_{n=1}^{\infty} \frac{1}{(n-1)!} z^n\]

\[= \frac{1}{0!} z^0 + \sum_{n=1}^{\infty} \left[ \frac{1}{n!} (n+1) - \frac{1}{(n-1)!} \right] z^n\]

And for the right hand side:

\[|D|(z) - e^{-z} = \sum_{n=1}^{\infty} \left[ \frac{1}{n!} - \frac{(-1)^n}{n!} \right] z^n\]

or, since the first term is zero,

\[|D|(z) - e^{-z} = \sum_{n=1}^{\infty} \left[ \frac{1}{n!} - \frac{(-1)^n}{n!} \right] z^n.\]
Equating corresponding coefficients in our two power series representations of \(|D|(z)\) we see that for \(n \geq 1\),

\[
\frac{!(n+1)}{n!} - \frac{n}{(n-1)!} = \frac{!n}{n!} - \frac{(-1)^n}{n!}.
\]

Multiplying this by \(n!\), we get the desired result:

\[(n + 1) - n!n = !n - (-1)^n\]

or,

\[(n + 1) = (n + 1)!n + (-1)^{n+1}.
\]

So far, we have only shown this is true for \(n \geq 1\), but it is easily verified to hold also when \(n = 0\).

9.) I’m not sure the first 6 will be sufficient to convince me that this stuff is really cool, so I’ll do the first 7:

<table>
<thead>
<tr>
<th>(n)</th>
<th>(n! = 1)</th>
<th>(1/e = )</th>
<th>(1/e = \approx)</th>
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<td>1</td>
<td>.367...</td>
</tr>
<tr>
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<td>1!</td>
<td>.735...</td>
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<td>(2(0) + 1 = 1)</td>
<td>2!</td>
<td>2.207...</td>
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<td>3!</td>
<td>8.829...</td>
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<td>(4(2) + 1 = 9)</td>
<td>4!</td>
<td>44.15...</td>
</tr>
<tr>
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<td>5!</td>
<td>264.87...</td>
</tr>
<tr>
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<td>(6(44) + 1 = 265)</td>
<td>6!</td>
<td>1854.11...</td>
</tr>
<tr>
<td>7</td>
<td>(7(265) - 1 = 1854)</td>
<td>7!</td>
<td>1854.11...</td>
</tr>
</tbody>
</table>

Hey, that’s really cool!