\[
[p, q^2] = [p, q]q + q[p, q] = -2iq
\]

14 October 2003

JB forgot his notebook today...

The \textit{Weyl algebra} is the associative alg. over \(\mathbb{C}\) generated by \(p, q\) satisfying

\[pq - qp = -i\]

We'll do calculations in this algebra (or some larger algebra containing things like \(e^{iH}\) where \(H\) is an elt. of Weyl algebra). Starting with:

\[
[p, q] = -i
\]

\[
[p, q^2] = [p, q]q + q[p, q] = -2iq
\]

\[
[p, q^n] = -inq^{n-1}
\]

or for any polynomial \(F,\)

\[
[p, F(q)] = -iF'(q)
\]

\textbf{Def: for short}

\[
[p, \cdot] = -i \frac{\partial}{\partial q}
\]

Next:

\[
[q, p^2] = [q, p]p + p[q, p] = 2ip
\]

\[
[q, p^n] = inp^{n-1}
\]

or for any poly. \(F\)

\[
[q, F(p)] = iF'(p)
\]

or:

\[
[q, \cdot] = i \frac{\partial}{\partial p}
\]
If \( H = \frac{1}{2} p^2 + V(q) \) (\( V \) a poly. or perhaps an analytic fn. (power series))

then we have

\[
\frac{d}{dt} O(t) = [iH, O(t)]
\]

where

\[
O(t) = e^{it\, H} \, O \, e^{-it\, H}
\]

In particular, for the observable \( q \),

\[
\frac{d}{dt} q(t) = \frac{d}{dt} (e^{it\, H} q \, e^{-it\, H})
\]

\[
= e^{it\, H} (iHq - q \, iH) e^{-it\, H}
\]

\[
= e^{it\, H} [iH, q] e^{-it\, H}
\]

Now

\[
[iH, q] = i \left[ \frac{p^2}{2} + V(q), q \right]
\]

\[
= \frac{i}{2} \left[ p^2, q \right]
\]

\[
= \frac{i}{2} \cdot -2i\, p
\]

\[
= p
\]

So

\[
\frac{d}{dt} q(t) = e^{it\, H} \, p \, e^{-it\, H} = p(t)
\]

i.e.

\[
\dot{q}(t) = p(t) \quad \text{"velocity = momentum" (when mass = 1)}
\]

Next, try

\[
[iH, p] = i \left[ \frac{p^2}{2} + V(q), p \right]
\]

\[
= i \left[ V(q), p \right]
\]

\[
= i \cdot i \, V'(q)
\]

\[
= -V'(q) \quad \text{force at time 0.}
\]

So

\[
\frac{d}{dt} p(t) = e^{it\, H} (-V(q)) e^{-it\, H} = -V'(e^{it\, H} q \, e^{it\, H}) = -V'(q(t))
\]

\( \text{If } V \text{ is a poly, do this term by term.} \)
In short
\[ \dot{p}(t) = -V'(q(t)) \quad \text{"time derivative of momentum is force"} \]

These give
\[ \ddot{q}(t) = -V'(q) \quad \text{"F = a"} \]

or if we but m back in & define \( F(t) = -V'(q(t)) \) - the force operator - \textit{except} and the \( \ddot{a}(t) = \ddot{q}(t) \) - the acceleration operator, we get
\[ F(t) = ma(t) \]

(For some reason, nobody talks about where \( F = ma \) goes in QM. TB has spent lots of time on spr: trying to convince people this is how it works.)

For the harmonic oscillator:
\[ H = \frac{1}{2} (p^2 + q^2) \]

& then
\[ \ddot{q}(t) = p(t) \quad \dot{p}(t) = -q(t) \]
\[ \& \quad \ddot{q}(t) = -q(t) \]

These have solutions
\[ q(t) = (\cos t)q + (\sin t)p \]
\[ p(t) = -(\sin t)q + (\cos t)p \]

(John is trying to give the impression that QM is "just like" CM except done in a Weyl algebra instead of a Poisson algebra.)

To go further, we want to look at representations of the Weyl algebra as linear operators, i.e. some vector space \( V \) & linear operators \( p,q : V \to V \) s.t.
\[ pq - qp = -i I_V \]
You could hope for a finite-dim $V$, e.g. $V = \mathbb{C}^n$ so then $p, q \in M_n(\mathbb{C})$ (Matrix mechanics) In fact there are no fin. dim. reps except for the zero-dim rep $n=0$.

Proof: Suppose $p, q \in M_n(\mathbb{C})$, then

$$0 = tr(pq - qp) = tr(-iI) = -i \dim(V) = -ni$$

So we conclude $n = 0$ (Obv's ok if we kept $k$ in, then the other alternative would be $k = 0$).

Next: try to find some normed vector space $V$ s.t. $p, q : V \to V$ are bounded lin. ops. on $V$. In fact, there are no reps like this except the zero dimensional one.

Proof:

$$n \|q^n\| = \| -i q^{n-1} \| = \|[p, q^n]\| = \|pq^n - q^n p\|$$

$$\leq \|pq^n\| + \|q^n p\|$$

$$\leq \|pq\| \cdot \|q^{n-1}\| + \|qp\| \cdot \|q^{n-1}\|$$

So

$$n \leq \|pq\| + \|qp\| \quad \forall n$$

a contradiction (since $p \& q$ are bounded)

unless $\|q^{n-1}\| = 0$, in which case $\|q^n\| = \|q\| \cdot \|q^{n-1}\| = 0$

so $q^n = 0$ so $\|[p, q^n]\| = 0$

so $\|q^{n-1}\| = 0$ so $q^{n-1} = 0$ so $\ldots$ so $q = 0$.

So $1_V = 0$ so $\dim V = 0$. 
So: we can try to describe $p$ & $q$ as unbounded operators on a Hilbert space. The classic example is called the Schrödinger representation of the Weyl algebra.

Idea: to let $p$ & $q$ be operators on some space of funs $\Psi: \mathbb{R} \to \mathbb{C}$:

\[
(p\Psi)(x) = -i \Psi'(x) \quad \text{or} \quad p = -i \frac{\partial}{\partial x},
\]
\[
(q\Psi)(x) = x \Psi(x) \quad \text{or} \quad q = M_x \quad \text{mult. by } x.
\]

\[
p(q\Psi)(x) - q(p\Psi)(x) = -i \frac{\partial}{\partial x} (x \Psi)(x) + i x \frac{d\Psi}{dx}(x) =
\]
\[
- i \Psi(x) - i x \frac{d\Psi}{dx}(x) + i x \frac{d\Psi}{dx}(x) = -i \Psi(x)
\]

The idea: the fun $\Psi$ is a "wavefunction" that describes the probability of finding the particle in some $x \in \mathbb{R}$: the probability is

\[
\int_S |\Psi(x)|^2 \, dx
\]

(if $\Psi$ is normalized so that $\int_{\mathbb{R}} (|\Psi(x)|^2 = 1)$

(The Plan:)

To write $p$ & $q$ as $\infty \times \infty$ matrices, as Heisenberg did, we'll take Schrödinger rep & pick a basis of functions on the real line — a basis of eigenvectors of

\[
H = \frac{1}{2}(p^2 + q^2) = \frac{1}{2}(-\frac{d^2}{dx^2} + x^2)
\]

the order "mult. by $x^2$".

What kind of funs on the real line? One choice: Schwartz functions $\mathcal{S}(\mathbb{R}) = \{ \Psi: \mathbb{R} \to \mathbb{C} : \int |x^n \frac{d^m}{dx^m} \Psi(x)| < C \}$

$\forall n, m, C$ s.t.
"All derivatives of $\Psi$ exist & vanish faster than the reciprocal of any polynomial."

Note: Schwartz functions satisfy both of the nice properties in analysis:
- Smoothness
- Fast decay

These are dual notions. The Fourier transform makes smooth functions decay fast & vice versa.

$$\mathcal{S}(\mathbb{R}) = \{ \Psi : q^m p^n \Psi \text{ is bounded } \forall n, m \}$$
$$= \{ \Psi : p^m q^n \Psi \text{ is bounded } \forall n, m \}.$$

We will find linearly independent eigenfunctions of $H$, say $\Psi_i$, s.t. finite linear combs are dense in the natural topology on $\mathcal{S}(\mathbb{R})$.

(These are in practice Hermite polynomials times gaussians.)

16 October 2003

The Schwartz space

Let $\mathcal{S}(\mathbb{R}) = \{ \Psi : \mathbb{R} \rightarrow \mathbb{C} : x^m \frac{d^n \Psi}{dx^n} \text{ is bounded } \forall n, m \}$

Given a sequence or net $\Psi_\alpha \in \mathcal{S}(\mathbb{R})$ we say $\Psi_\alpha \rightarrow \Psi \in \mathcal{S}(\mathbb{R})$ if

$$\sup \left| x^m \frac{d^n}{dx^n} (\Psi_\alpha - \Psi) \right| \rightarrow 0 \quad \forall n, m$$

making $\mathcal{S}(\mathbb{R})$ into a topological vector space.

We have cts. lim. ops

$$p, q : \mathcal{S}(\mathbb{R}) \rightarrow \mathcal{S}(\mathbb{R})$$

given by

$$(p \Psi)(x) = -i \Psi'(x)$$
$$(q \Psi)(x) = x \Psi(x)$$

(Easy Exercise: check that these are continuous)
8 also

\[ H : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R}) \]

given by

\[ H = \frac{1}{2} (p^2 + q^2) \]

Q: What are the eigenvectors of H?

(Note: p & q don't have eigenvectors in \( \mathcal{S}(\mathbb{R}) \), since \( e^{ikx} \& \mathcal{S}(x-x) \) aren't in \( \mathcal{S}(\mathbb{R}) \).)

Here's one eigenvector of H:

\[ \psi_0(x) = e^{-x^2/2} \]

Let's check how this works:

\[ \frac{d \psi_0}{dx} = -xe^{-x^2/2} \]

\[ \frac{d^2 \psi_0}{dx^2} = -(1+x^2)e^{-x^2/2} \]

\[ p^2 \psi_0 = (1-x^2) \psi_0 \]

\[ q^2 \psi_0 = x^2 \psi_0 \]

\[ \Rightarrow H \psi_0 = \frac{1}{2} (p^2 + q^2) \psi_0 = \frac{1}{2} \psi_0 . \]

It turns out \( \frac{1}{2} \) is the lowest eigenvalue of H. In general, if

\[ H \psi = E \psi \]

we say \( \psi \) describes a state where the particle has energy \( E \).

So \( \psi_0 = e^{-x^2/2} \) is the state of least energy, i.e., the ground state.
of the harmonic oscillator. Note that the ground state energy is $\frac{1}{2}$, not (as in the classical harmonic oscillator) 0!

Due to the uncertainty principle we can't get both $p$ & $q$ to be 0 in QM. Now how do we get more eigenvectors?

We'll now define cts. lin. ops $a, a^*$:

$$a, a^*: \mathcal{S}(\mathbb{R}) \rightarrow \mathcal{S}(\mathbb{R})$$

s.t. if

$$H \psi = E \psi$$

then

$$H a^* \psi = (E+1) a^* \psi$$

(so $a^*$ raises the energy by 1)

&

$$H a \psi = (E-1) a \psi$$

(a lowers the energy by 1)

We call $a$ the **annihilation operator** (because $a \psi_0 = 0$ — $a$ annihilates $\psi_0$)

or **lowering operator**; we call

$a^*$ the **creation op.** or **raising op.**

These ladder operators are

$$a = q + ip$$

$$a^* = q - ip$$

(people usually divide these by $\sqrt{2}$; we'll do that later.)
Recall:
\[ [iH, p] = -q \quad [iH, q] = p \]

so
\[ [iH, a] = [iH, q + ip] \]
\[ = [iH, q] + [iH, ip] \]
\[ = p - iq \]
\[ = -ia \]

\[ \Rightarrow [H, a] = -a \]

So if \( H\psi = E\psi \) then
\[ Ha^*\psi = (a^*H + [H, a^*])\psi \]
\[ = (a^*H + a^*)\psi \]
\[ = a^*E + a^* \]
\[ = (E + 1) a^*\psi \]

\&
\[ Ha\psi = (aH + [H, a])\psi \]
\[ = (aE - a)\psi \]
\[ = (E - 1)a\psi \]

So we can take \( \psi_0 \) and hit it with \( a^* \) \( n \) times & get
\[ \psi_n = a^{*^n}\psi_0 \]

so
\[ H\psi_n = (n + \frac{1}{2})\psi_n \]
These are in fact all of the eigenvectors of $H$, so we have:

$$a \Psi_0 = (q + ip) \Psi_0$$
$$= (x + \frac{d}{dx}) e^{-x^2/2}$$
$$= 0 , \text{ as expected.}$$

If $n \geq 1$,

$$a \Psi_n = a a^n \Psi_0$$

To do this, we would like to commute the $a$ through all of the $a^*$'s. We need the commutator:

$$[a,a^*] = [q + ip, q - ip]$$
$$= [q,q] + i[p,q] - i[q,p] + [p,p]$$
$$= 2i [p,q]$$
$$= 2i (-i)$$
$$= 2 \quad \text{This 2 is why people usually divide $a$ and $a^*$ by $\sqrt{2}$.}$$

So:

$$a \Psi_n = aa^* a^{n-1} \Psi_0$$

$$= (a^*a + [a,a^*]) a^{n-1} \Psi_0$$

$$= (a^*a + 2) a^{n-1} \Psi_0 = \ldots \text{ & etc., pushing $a$ to the right.}$$
An easier way:

$[a, \cdot] \text{ is a derivation, so}$

$[a, a^*] = 2 \ implies$  

$[a, a^{*n}] = 2na^{*n-1}$

Then

$a \psi_n = a a^{*n} \psi_0 = (a^{*n} a + [a, a^{*n}]) \psi_0$

$= (0 + 2na^{n-1}) \psi_0$

$= 2n \psi_{n-1}$

Note: $a$ & $a^*$ are not inverses of each other.

There's a big difference between creating a particle and then annihilating a particle, and the other way round.

Usually, people get rid of these "2's" by setting

$$a = \frac{q + ip}{\sqrt{2}} \quad a^* = \frac{q - ip}{\sqrt{2}}$$

Then

$[a, a^*] = 1$

but we still have

$[H, a] = -a$

$[H, a^*] = a^*$

Now if we set

$\psi_n = a^{*n} \psi_0$
we get

\[ H \psi_n = (n + \frac{1}{2}) \psi_n \]

\[ a^+ \psi_n = \psi_{n+1} \]

\[ a \psi_n = n \psi_{n-1} \]

\[ \psi_n \] is like a box

with n indistinguishable balls

\[ a^+ \psi_n = \psi_{n+1} \] says there's only one way to

add a ball to the box.

\[ a \psi_n = n \psi \] says there are n ways to
take a ball out.

\[ [a, a^+] = 1 \text{ or } \]

\[ a a^* = a^* a + 1 \]

"There's one more way to put in a ball and
then take one out than to take one out and
then put one in.

What are the functions \( \psi_n \) like?

\[ \psi_0 = e^{-x^2/2} \] \( \frac{1}{\sqrt{2}} \psi_0 \) \( \psi_0 \) the gaussian of

\[ \sigma \text{ std. deviation} 1. \]

\[ \psi_1 = a^+ \psi_0 = \frac{a - i p}{\sqrt{2}} \psi_0 \]

\[ = \frac{1}{\sqrt{2}} \left( x - \frac{d}{dx} \right) e^{-x^2/2} \]

\[ = \frac{1}{\sqrt{2}} 2x e^{-x^2/2} \]
\[ \Psi_2 = \delta^* \Psi_1 \]
\[ = \frac{1}{\sqrt{2}} (x - \frac{d}{dx}) \Psi_1 e^{-x^2/2} \]
\[ = (x^2 - 1^2 + x^2) e^{-x^2/2} \]
\[ = (2x^2 - 1) e^{-x^2/2} \]

Certainly
\[ \Psi_n(x) = H_n(x) e^{-x^2/2} \]

where \( H_n \) is some polynomial of degree \( n \) - the \( n \)th Hermite polynomial (up to fudge factors).

It's easy to see \( H_n(x) \) has degree \( n \), and
\( H_n \) is also even/odd depending on parity of \( n \).

This comes from the fact that multi by \( x \) and diff by \( x \) both turn odd \( H_n \)s into even \( H_n \)s and even \( H_n \)s into odd \( H_n \)s, so \( x - \frac{d}{dx} \) does too.

It's also true (but not trivial) that \( H_n \) has \( n \) real roots.

Thus: The \( H_n \)s form a (topological) basis of \( \mathcal{S}(\mathbb{R}) \)

i.e. they're linearly indep. & the finite lin. combs
\[ \sum_{n=0}^{N} c_n \Psi_n \]
are dense in \( \mathcal{S}(\mathbb{R}) \)

"Pf:" They're obviously lin. indep. & finite lin. combs are
the same as fns \( P(x)e^{-x^2/2} \) where \( P \) is any poly.
So we need: polynomials times \( e^{-x^2/2} \) are
dense in \( \mathcal{S}(\mathbb{R}) \). This follows from the "L^2 Stone-Weierstrass Thm" & some extra stuff.
Prop: \( S(\mathbb{R}) \) are dense in \( L^2(\mathbb{R}) \).

So, either \( E_n = E_m \) or \( \langle Y_n, Y_m \rangle = 0 \).

\[ \langle H Y_n, Y_m \rangle = \langle Y_n, H Y_m \rangle = \begin{cases} 1, & \text{if } n = m \\ 0, & \text{if } n \neq m \end{cases} \]

So, \( \langle H \psi, \psi \rangle = \langle \psi, H \psi \rangle \) since \( H = \frac{1}{2}(p^2 + x^2) \).

\[ \langle \phi, \psi \rangle = \int_{\mathbb{R}} \overline{\phi} \psi \, dx \]

Pf.

Note \( \langle \phi, \psi \rangle = \int_{\mathbb{R}} \overline{\phi} \psi \, dx \) if \( \phi, \psi \) are orthogonal in \( L^2(\mathbb{R}) \):

\[ \int_{\mathbb{R}} \overline{\phi} \psi \, dx = 0 \]

Vigoda \( \langle \phi, \psi \rangle = \int_{\mathbb{R}} \overline{\phi} \psi \, dx \) if \( \phi, \psi \) are orthogonal in \( L^2(\mathbb{R}) \):

\[ \int_{\mathbb{R}} \overline{\phi} \psi \, dx = 0 \]
Thus: If we normalize $\psi_n$ to get

$$|n\rangle = \frac{\psi_n}{\|\psi_n\|}$$

these form an orthonormal basis of $L^2(\mathbb{R})$

Pf: They're orthonormal, since the $\psi_n$ are orthogonal.

They form a basis since the finite lin. combs. of the $\psi_n$ are dense in $L^2(\mathbb{R})$ & thus by previous prop. in $L^2(\mathbb{R})$. 

The Weyl Algebra: the assoc. alg. gen. by $p,q$ s.t.

$$[p, q] = -i$$

is also generated by

$$a = \frac{q + ip}{\sqrt{2}}$$

$$a^* = \frac{q - ip}{\sqrt{2}}$$

with $[a, a^*] = 1$, since

$$\frac{a + a^*}{\sqrt{2}} = q$$

$$\frac{a - a^*}{\sqrt{2}i} = p$$

We considered the Schrödinger rep of

the Weyl algebra, where

$$p,q : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$$

$$\mathcal{S}(\mathbb{R}) = \left\{ \psi : \mathbb{R} \to \mathbb{C} : |x^r \frac{d^s}{dx^s} \psi| \text{ bounded} \right\}$$

Note the similarity with

$$z = x + iy$$

$$\frac{z - \bar{z}}{2i} = y$$

$$\frac{z + \bar{z}}{2} = x$$

is phase space for classical harm. oscillator.

Now we're doing a quantum version of the same thing:

$a, a^*$ are like $z \& \bar{z}$

$\Rightarrow$ noncommutative complex analysis