Stuff Types

**Review:**

We say a functor $F : C \to D$ is

- **essentially surjective** if $\forall d \in D \ \exists \tilde{d} \in C$ s.t. $F(\tilde{d}) = d$.

- **full** if $\forall c, c' \in C$,
  
  $$F : \text{hom}(c, c') \to \text{hom}(F(c), F(c'))$$

  is onto

- **faithful** if $\forall c, c' \in C$,
  
  $$F : \text{hom}(c, c') \to \text{hom}(F(c), F(c'))$$

  is one-to-one.

We say $F$:

- **forgets nothing** if it's faithful, full, & essentially surjective

- **forgets properties** if it's faithful & full

- **forgets structure** if it's faithful

- **forgets stuff always**

Here "forgets _______" really means "forgets at most _______," so each of the 4 conditions implies the next.

**Def:** A **stuff type** is a "groupoid over FinSet₀," i.e. a groupoid $X$ and a functor

$$X \downarrow \FinSet₀$$
We say "over" because the finite set $F(x)$ is like the "shadow" of $x \in X$; objects in $X$ are finite sets equipped with extra stuff. ($F$ forgets stuff.)

**Def:** A stuff type is a **structure type** if $F$ forgets structure (i.e. is faithful).

**Warning:** This looks completely different from our old definition of a structure type as

$$F : \text{FinSet}_0 \rightarrow \text{Set}$$

—but we'll see they're equivalent.

**Def:** A stuff type is a **property type** if $F$ forgets properties (i.e. is faithful & full)

**Def:** A stuff type is a **vacuous property type** if $F$ forgets nothing.

**Examples:**

1)

$$E^2 = [\text{2-colored finite sets}]_0$$

$$\xrightarrow{\text{F}}$$

--- forget the coloring

$$E = \text{FinSet}_0$$

(Recall: If $F$ is a str-type & $Z_0$ is a groupoid, $F(Z_0)$ is the groupoid of "$F$-structured finite sets with elements labelled by objects of $Z_0"$

$E^2$ = "being a finite set" so $E^2 = \"\text{finite sets w/ elts labelled by elts of } Z (\text{the 2-elt set})\"$ = "2-colored finite sets."
Is $F$:

- faithful? \hspace{1cm} YES \hspace{1cm} - a color preserving bijection determines a unique bijection
- full? \hspace{1cm} NO \hspace{1cm} - not all bijections preserve coloring
- ess. surjective? \hspace{1cm} YES \hspace{1cm} - any finite set can be 2-colored

So $F$ forgets structure — it's a structure type!

Let's do a similar example: some groupoids, but different functor:

2)

$E^2 = [\text{2-colored finite sets}]$

$\xrightarrow{F} \quad \text{forget the white dots}$

$E = \text{FinSet}_0$

Is $F$:

- faithful? \hspace{1cm} NO \hspace{1cm} - what morphisms do to white dots gets forgotten
- full? \hspace{1cm} YES \hspace{1cm} - any bijection of black dots extends to a color preserving bijection
- ess. surjective? \hspace{1cm} YES \hspace{1cm} - any finite set is (\equiv to) the set of black dots in a 2-colored set,

So $F$ forgets stuff — it's a structure type.

stuff type that's not a structure type,
3) \( \text{COSH}(1) = [\text{even sets}] \)

- the obvious inclusion

\[ E = \text{FinSet}_0 \]

(Recall: \( \text{COSH}(2) = "\text{being an even set}" \))

Is \( F \):

- faithful? YES  - what morphisms do to dots is not forgotten
- full? YES  - every bijection of even sets is a bijection of finite sets.
- ess. surjective? NO  - e.g. the 5-elt set is not even

So \( F \) forgets properties - it's a "property type."

4) Let \( \mathcal{N} \) be a skeleton of the category \( \text{FinSet} \) -

a subcategory of \( \text{FinSet} \) that has one object from each isomorphism class & all morphisms between them.

\[ \mathcal{N} \] can have objects:

- 0 = \( \emptyset \)
- 1 = \{0\}
- 2 = \{0,1\}
- 3 = \{0,1,2\} etc.

-von Neumann's favorite n-element sets.

Let \( \mathcal{N}_0 = \) the groupoid with 0,1,2,... as objects

& bijections as morphisms ( the "underlying groupoid" of \( \mathcal{N} \)).
Consider:

\[
\begin{array}{ccc}
N_0 & \to & \text{the obvious inclusion} \\
\downarrow & & \downarrow \\
\text{FinSet}_0 & & \\
\end{array}
\]

Is \( F \):
- faithful? \(\text{YES}\)
- full? \(\text{YES}\)
- ess. surjective? \(\text{YES}\)

\[
\begin{array}{ccc}
0 & 1 & 2 & 3 \\
\downarrow & & & \\
0 & 1 & 2 & 3 \\
\end{array}
\]

\[
\begin{array}{c}
\text{"same hom sets"} \\
\end{array}
\]

\[
\begin{array}{c}
\text{every } n\text{-elt set is } \cong \text{ to } \\
\text{"the" } n\text{-elt set.}
\end{array}
\]

So \( F \) forgets nothing. This happens whenever \( F \) is the inclusion of a skeleton.

8 April 2004

\textbf{STUFF TYPES & THE HARMONIC OSCILLATOR}

We've seen that structure types are a categorified version of states of the quantum harmonic oscillator — the same holds for stuff types, but it works even better!

Now we'll:
1) Define a generating function \( 1F/1\) for a stuff type

\[
\begin{array}{ccc}
X & \to & \\
\downarrow F & & \downarrow \\
\text{FinSet}_0 & & \\
\end{array}
\]

(a groupoid over \( \text{FinSet}_0 \))
Sometimes we'll call \( F \) just "F" for short. The generating function should reduce to our previous definition when \( F \) is a structure type.

2) Define how to apply a stuff type \( F \) to a groupoid \( Z_0 \) & get a groupoid \( F(Z_0) \) - the groupoid of "finite sets equipped with \( F \)-stuff & with elts, labelled by objects of \( Z_0 \)," or "\( F \)-stuffed \( Z_0 \)-colored finite sets," for short.

3) Check that:

\[
|F(Z_0)| = |F|(1|Z_0|)
\]

4) Given two stuff types \( F \) & \( G \), define a stuff type \( F \circ G \) such that

\[
F \circ G(Z_0) \simeq F(G(Z_0))
\]

Recall: we saw that even if \( F \) & \( G \) are structure types, \( F \circ G \) may not be a structure type (e.g. \( \mathbb{E}^2 \)) - it will be a stuff type.

5) Given two stuff types \( F \) & \( G \), define the "inner product" \( \langle F, G \rangle \), a groupoid with

\[
|\langle F, G \rangle| = |F| \cdot |G|
\]

where the right-hand side is the inner product on
the Hilbert space (Fock representation) for the quantum harmonic oscillator: given \( f, g \in \mathbb{C}[z] \), this had

\[
f(z) = \sum_{n \in \mathbb{N}} a_n z^n \quad \Rightarrow \quad \langle f, g \rangle = \sum_{n \in \mathbb{N}} n! \bar{a}_n \bar{b}_n
\]

\[g(z) = \sum_{n \in \mathbb{N}} b_n z^n\]

We saw this in the homework on "categorified coherent states," where we showed that \( \langle 1, 1 \rangle = 1 \) & \( \langle a^* f, g \rangle = \langle f, a g \rangle \) determine the inner product

\[
\langle z^n, z^m \rangle = n! \delta_{n,m}
\]

The idea:

\[
\langle z^n, z^n \rangle = \langle a^* z^{n-1}, z^n \rangle
\]

\[= \langle z^{n-1}, a z^n \rangle\]

\[= \langle z^{n-1}, \frac{d}{dz} z^n \rangle\]

\[= n \langle z^{n-1}, z^{n-1} \rangle = \cdots = n! \langle 1, 1 \rangle = n!\]

It's a minor miracle that this inner product comes naturally from category theory.

6) Define "stuff operators," which map stuff types to stuff types. E.g.:

\[A = "\text{annihilation operator}" \quad \text{with} \quad |AF| = a |F|\]

\[A^* = "\text{creation operator}" \quad \text{with} \quad |A^* F| = a^* |F|\]
7) Categorify the theory of Feynman diagrams.

The Generating Function of a Stuff Type

Suppose \( F \) is a stuff type. Define \( |F| \in \mathbb{C}[z] \) by

\[
|F|(z) = \sum_{n \in \mathbb{N}} |X_n| z^n
\]

where \( |X_n| \) is the groupoid cardinality (if it converges!) of \( X_n \), the "groupoid of \( F \)-stuffed \( n \)-elt. sets," i.e. \( n \)-elt. sets equipped with \( F \)-stuff.

More precisely: note

\[
\text{FinSet}_0 \cong \sum_{n \in \mathbb{N}} [n\text{-elt sets}]_0
\]

- a sum (aka. disjoint union or coproduct) of groupoids whose objects are \( n \)-elt. sets and whose morphisms are bijections. Therefore

\[
X \cong \sum_{n \in \mathbb{N}} X_n
\]

where the objects of \( X_n \) are those \( x \in X \) for which \( F(x) \) is an \( n \)-elt. set, and the morphisms are all morphisms in \( X \) between these.
So if $X$ is the groupoid of "F-stuffed finite sets," then $X_n$ is the groupoid of "F-stuffed n-elt. sets," & we get

$$
\begin{array}{c}
X_n \\
\downarrow \quad F_n \\
[n\text{-elt sets}]_n
\end{array}
$$

where $F_n = F|_{X_n}$.

Note: unlike our old definition for structure types, there's no $\frac{1}{n!}$ in this definition of the generating function - but it'll give the same result when $F$ is a structure type!

**Examples:**

1) Categorified Coherent Stacks, revisited.

Let $F = "$being a $C$-colored finite set,$"$ where $C$ is any groupoid. If $C$ is just the set $k$ (viewed as a groupoid), this is "being a $k$-colored finite set" & we've seen that its generating function is $e^{kz} = \sum_{n=0}^{\infty} \frac{k^n z^n}{n!}$ since there are $k^n$ ways to $k$-color an $n$-elt set. This is a structure type, but we've
also mentioned "being a $\frac{1}{2}$-colored finite set" - where $C$ is $\mathbb{Z}/2$ thought of as a groupoid with one object. This will be a stuff type.

More precisely, we have:

\[
X = \text{groupoid of } C\text{-colored finite sets} \\
F : X \rightarrow \text{FinSet}_0
\]

Here $X$ has objects finite sets with elements labelled by objects of $C$, & morphisms bijections with "strands" labelled by morphisms of $C$.

\[
\begin{array}{c}
C_1 & \xrightarrow{f_1} & C_2 \\
\downarrow{f_2} & & \downarrow{f_3} \\
C_2' & \xleftarrow{f_2'} & C_1'
\end{array}
\]

\[f_i : C_i \rightarrow C_{0(i)}\]
where $\sigma$ is a bijection

Applying $F$ to this morphism we get

\[
\begin{array}{c}
\times \\
\downarrow{f_1}
\end{array}
\]

$F$ forgets stuff because in general it's not faithful - 2 different morphisms in $X$ can have the same
underlying bijection. If $C$ has only identity morphisms (i.e., it's secretly a set), then $F$ will be faithful, hence a structure type.

What's the generating function $|F|$?

We have:

$$X \sim E^C = \sum_{n \in \mathbb{N}} \frac{C^n}{n!}$$

$$F \downarrow$$

$$F_{\mathbb{N} Set} \sim E = \sum_{n \in \mathbb{N}} \frac{1}{n!}$$

& thus

$$X_n = \frac{C^n}{n!} = \mathbb{A}[C\text{-colored } n\text{-elt sets}]_0$$

$$F_n \downarrow$$

$$[n\text{-elt sets}]_0 = \frac{1}{n!}$$

So

$$|F|(z) = \sum_{n \in \mathbb{N}} |X_n| z^n$$

$$= \sum_{n \in \mathbb{N}} \left| \frac{C^n}{n!} \right| z^n$$

$$= \sum_{n \in \mathbb{N}} \frac{|C|^n}{n!} z^n = e^{|C|z}$$

This is a coherent state of the harmonic oscillator with:

- expected position $|C|/\sqrt{2}$
- expected momentum 0
We can't yet get coherent states with nonzero expected momentum, or negative expected position, since we don't know gadgets with complex or negative cardinality.