Feynman Diagrams

We've seen that we can draw an object of:

- the stuff type \( \Psi \) as
  
  ![Diagram of \( \Psi \)]

- the stuff operator \( T \) as:
  
  ![Diagram of \( T \)]

- the stuff type \( T\Psi \) as:
  
  ![Diagram of \( T\Psi \)]

\[\text{the isomorphism } \alpha \text{ in def. of "weak pullback"}\]
- The inner product $\langle \psi, T\psi \rangle$ as

![Diagram of $\psi$, $t$, and $\phi$]

This should remind you of Dirac's Bra-Ket notation:

$\langle \psi | T | \psi \rangle := \langle \psi, T \psi \rangle$

Given two stuff operators $S$ & $T$ we can draw an object of $ST$ as:

![Diagram of $S$ and $T$]

We've looked at two stuff operators in particular:

- $\Phi^n$ here a typical object (for $n=6$) looks like:
  ![Feynman diagram with one 6-valent vertex with totally ordered incidences]

- $\frac{\Phi^n}{n!}$ has typical object ($n=6$):
  ![Feynman diagram with one 6-valent vertex.]
These pictures are pictures of extra stuff that can be put on a pair of finite sets!

Here are some objects in \( \Phi^4 \Phi^4 \Phi^4 \):

Now... let's study the time evolution in the perturbed harmonic oscillator. Let's see how to calculate

\[ \langle z^k, e^{-i t H} z^l \rangle \]

where

\[ H = H_0 + V \]

\[ V = \frac{\Phi^m}{m!} \]

Recall:

\[ \langle z^k, e^{-i t H} z^l \rangle = \sum_{n=0}^{\infty} \int \langle z^k, e^{-i(t-t_n)H_0} V \cdots V e^{-i(t_n)H_0} z^l \rangle dt_1 \cdots dt_n \]

\( 0 \leq t_n \leq t \)
Calculating $\langle z^k, e^{-i(t-t_0)H_0} V e^{-i(t_{m-1}-t_0)H_0} V \ldots V e^{-i(t, H_0) z^k} \rangle$

is almost like calculating

$\langle z^k, V V \ldots V z^k \rangle \equiv \langle z^k, \frac{z^m}{m!} \ldots \frac{z^n}{m!} z^k \rangle$ except to keep track of the $e^{-i(t_{p-1}-t_p)H_0}$ terms we label each edge by a phase: we label any edge from the $p$th vertex to the $q$th vertex ($p \leq q$) by $e^{-i(t_{p-1}-t_p)}$, e.g.

![Diagram](image)

(or $e^{-it_p}$ if the edge comes from bottom to the $p$th vertex or $e^{-i(t_{p-1})}$ if it goes from the $q$th vertex to the top). In short: label each edge with $e^{-iT}$ where $T$ is the amount of time that passes along that edge. These phases are called propagators.
We calculate
\[ \langle z^k, \ e^{-i(t-t_k)H_0} \ldots \ e^{-i(t-t_1)H_0} z^k \rangle \]
by summing over Feynman diagrams, each weighted by the product of all these phases (\& divided by the size of their symmetry group, as usual in groupoid cardinality).

Why? Short answer: each "quantum" has energy 1, so applying \( e^{-i(t_p-t_{p-1})H_0} \) to a state with \( E \) of the quanta \((z^E)\) multiplies it by \( e^{-i(t_p-t_{p-1})E} \).

Here we're getting that effect by attaching a phase to each edge (corresponding to a quantum).

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Let's see why our Feynman diagram recipe for calculating transition amplitudes like
\[ \langle z^k, \ e^{-i(t-t_k)H_0} \ldots \ e^{-i(t_{p-1}t_{p-1})H_0} z^k \rangle \]
agrees with a "direct" computation. We'll look at some examples coming from the homework, where we did
\[ \langle z^2, \frac{\Phi^3}{3!} \frac{\Phi^3}{3!} z^2 \rangle \]

But now consider
\[ \langle z^2, \ e^{-i(t-t_2)H_0} \frac{\Phi^3}{3!} \ e^{-i(t_1-t_2)H_0} \frac{\Phi^3}{3!} \ e^{-i(t_1-t_2)H_0} z^2 \rangle \]
This will be a sum of many terms, since $\phi^3 = (a + a^*)^3$, and each term corresponds to one or more Feynman diagrams. Let's look at a couple.

Example 1:

$$\langle z^2, e^{-i(t-t_2)H_0} \frac{a^* a}{3!} e^{-i(t-t_2)H_0} \frac{a^* a}{3!} e^{-i t_1 H_0} z^2 \rangle$$

corresponding to:

![Diagram]

Let's calculate this transition amplitude directly & using Feynman diagrams:

$$\langle z^2, e^{-i(t-t_2)H_0} e^{-i t_1 H_0} e^{-i(t-t_2)H_0} \frac{a^* a}{3!} e^{-i(t-t_2)H_0} \frac{a^* a}{3!} e^{-i t_1 H_0} z^2 \rangle$$

The phases work out right: product of phases over all edges is $e^{-2i(t-t_2)} e^{-i(t-t_2)} e^{-i t_1}$, since there were 2 quanta from time 0 to $t_1$. 

time $t_1$ -- $\frac{a^* a}{3!}$

time $t_2$ -- $\frac{a^* a}{3!}$

time $0$ -- $z^2$
Example 2:

\[ \langle z^2, e^{-i(t-t_0)\mathcal{H}_0} \left( \frac{a^3}{3!} e^{-i(t_1-t_0)\mathcal{H}_0} \right) \frac{a^3}{3!} e^{-i(t_1,\mathcal{H}_0) z^2} \rangle \]

corresponds to many Feynman diagrams, including:

\[ \begin{align*}
\langle z^2, & \rangle \\
& e^{-2i(t-t_0)} e^{-5i(t_1-t_0)} e^{-2i t_0} \frac{5}{3!} 3! 2! \\
& e^{-i(t-t_0)\mathcal{H}_0} \\
& e^{-5i(t_1-t_0)-2i t_0} \frac{5}{3!} 3! 2! \\
& e^{-i(t-t_0)\mathcal{H}_0} \\
& e^{-5i(t_1-t_0)-2i t_0} \frac{5}{3!} 3! 2! \\
\end{align*} \]
Even in this example the product of phase labelings on edges matches our "direct" calculation:

$$e^{-2it}e^{3it_1}e^{-3it_2} = e^{-2i(t-t_2)}e^{-5i(t_1-t_2)}e^{-2it_1}.$$ 

Why does it work? 

Break each edge by horizontal lines at times $0, t_1, t_2, t$, and factor its phase as above!