Asymptotic Behavior of Marginally Trapped Tubes

Catherine Williams

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Preliminaries — general relativity

General relativity says that spacetime is described by a Lorentzian 4-manifold $(\mathcal{M}, g)$ satisfying the Einstein field equations

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = 8\pi T_{\alpha\beta}$$

where

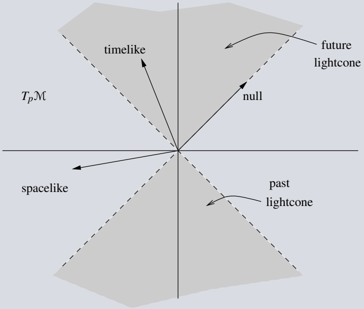
- $R_{\alpha\beta}$ is the Ricci curvature of g,
- R is the scalar curvature of g, and
- $T_{\alpha\beta}$ is the stress-energy tensor describing all matter and energy in the spacetime.

Preliminaries — causal notions

Since g has signature $(-,+,+,+)$, we can partition tangent vectors of $\mathcal{M}$ into three types: for $X \in T_p\mathcal{M}$,

$$g(X,X) < 0 \iff X \text{ is } \textit{timelike}$$
$$g(X,X) = 0 \iff X \text{ is } \textit{null} \text{ (or } \textit{lightlike}\text{)}$$
$$g(X,X) > 0 \iff X \text{ is } \textit{spacelike};$$

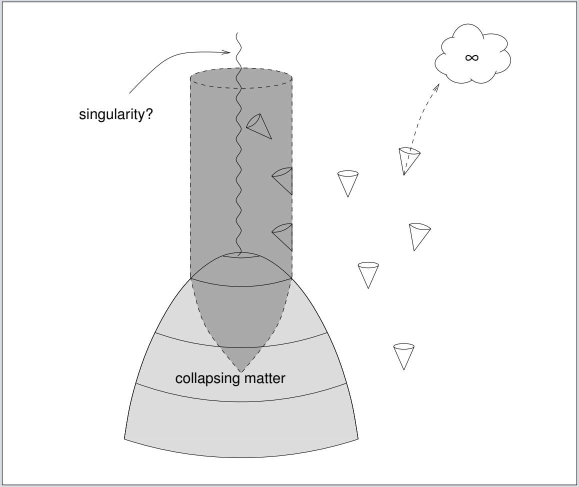
if X is either timelike or null, it is called *causal*;
if X is either null or spacelike, we will call it *achronal*.



We will assume that $\mathcal{M}$ is *time orientable* — causal vectors may be partitioned into two sets, the future- and past-directed lightcones.

Causal characterizations also extend to differentiable curves and submanifolds.

Preliminaries — black holes (heuristically)

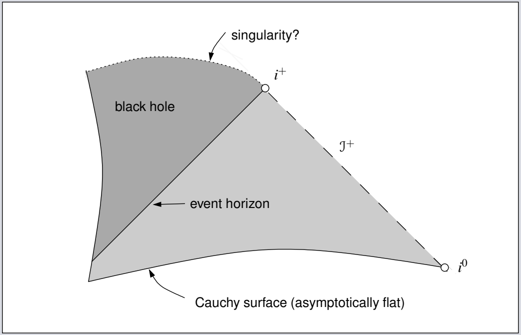


Preliminaries — black holes (mathematically)

To make this idea rigorous, one must locate future null infinity, $\mathcal{I}^+$.

One traditionally does this by making a conformal compactification of the spacetime $(\mathcal{M}, g)$ and identifying $\mathcal{I}^+$ as a null part of its conformal boundary.

The black hole region is then $\mathcal{M} \setminus J^-(\mathcal{I}^+)$.



Note that one must have the entire spacetime at hand in order to find the black hole.

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Given any spacelike 2-surface S and a future null vector field ℓ^α orthogonal to it, the *expansion* of S in the direction ℓ is

$$\theta_{(\ell)} = \mathrm{div}_S\, \ell$$

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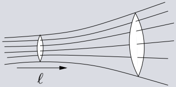
The expansion $\theta_{(\ell)}$ measures the infinitesimal change in surface area of S in the direction ℓ.

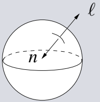


Preliminaries — trapped surfaces

A spacelike 2-surface S has exactly two orthogonal future null directions, given by vector fields ℓ and n, say.

If ℓ points "out" and n points "in," set

$$\begin{aligned} \theta_+ &= \theta_{(\ell)} \\ \theta_- &= \theta_{(n)}. \end{aligned}$$



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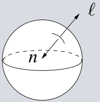

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If $\theta_- < 0$ and $\theta_+ < 0$, the surface S is said to be *trapped*.

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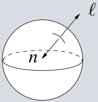

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Singularity Theorem (Penrose, 1965)

Let $(\mathcal{M}, g)$ be a connected, physically reasonable spacetime. If $\mathcal{M}$ contains a closed trapped surface S, then $\mathcal{M}$ is singular.

In recent years a quasi-local model for surfaces of black holes has been proposed:

Definition

A *marginally trapped tube* (MTT) $\mathcal{A}$ is a hypersurface foliated by closed marginally trapped (spacelike) 2-surfaces.

Related terminology:

- a *dynamical horizon* is an MTT which is spacelike;
- an *isolated horizon* is (essentially) an MTT which is null;
- a *timelike membrane* is an MTT which is timelike.

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Dynamical and isolated horizons appear to be well-suited to model the surfaces of dynamical and equilibrium black holes, respectively [Ashtekar & Krishnan, others].

(Timelike membranes have no apparent physical meaning.)

Preliminaries — spherical symmetry

A spherically symmetric spacetime is one which admits an $SO(3)$-action by isometries. One can work with the 1+1-dimensional Lorentzian quotient manifold

$$\mathcal{Q} = \mathcal{M}/SO(3)$$

instead of $\mathcal{M}$ without loss of information.

Conformally embedding $\mathcal{Q}$ into a bounded subset of $\mathbb{M}^{1+1}$, we obtain a representation of the spacetime called a *Penrose diagram*, off of which essentially all causal and asymptotic information may be read.

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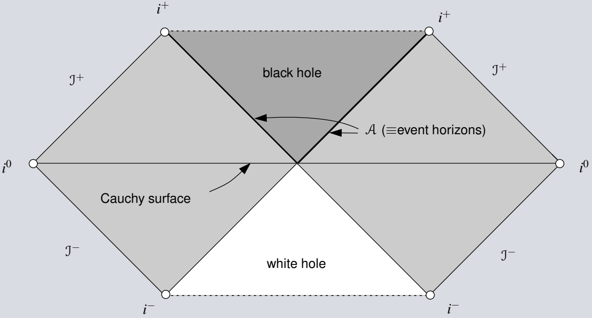
Remark. All known explicit (exact) examples of MTTs are spherically symmetric, and all existing theorems concerning their asymptotic behavior assume spherical symmetry.

Examples — Schwarzschild & Reissner-Nordström

The marginally trapped tubes in a Schwarzschild spacetime of mass M are isolated horizons which coincide with the black hole event horizons.

$$g = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr + r^2 d\sigma^2, \qquad T = 0.$$



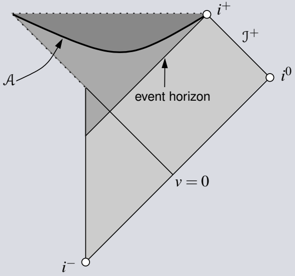
The same is true in a Reissner-Nordström (electrovac) spacetime of mass M and charge e.

Vaidya spacetimes with nonconstant, nondecreasing mass functions $M(v)$ provide the simplest examples of dynamical horizons.

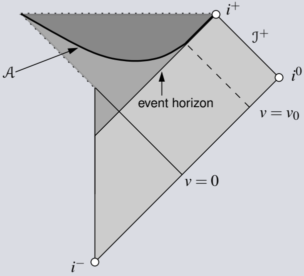
$$g = -\left(1 - \frac{2M(v)}{r}\right) dv^2 + 2dvdr + r^2 d\sigma^2,$$

$$T = \frac{\dot{M}(v)}{r^2} dv^2,$$

where and $M(v)$ is any smooth function of v.



$M(v) \nearrow M_0$ as $v \to \infty$



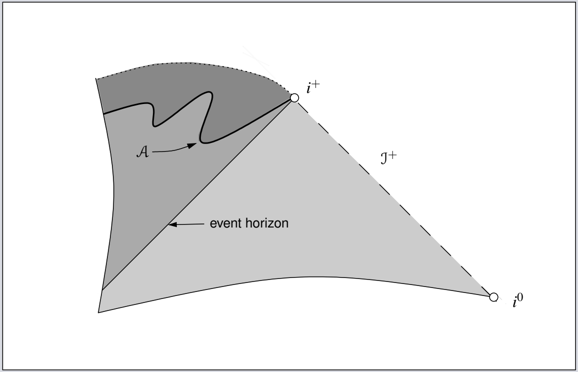
$M(v) \equiv M_0$ for $v \geq v_0$

Examples — evolutionary setting

For certain matter models, it is known that the maximal development of spherically symmetric asymptotically flat initial data contains an MTT which is asymptotic to the event horizon (i.e. terminates at i^+):

- ⋆ massless scalar fields [Christodoulou 1993]
- ⋆ Einstein-Maxwell scalar fields [Dafermos 2005, Dafermos & Rodnianski 2005]
- ⋆ Einstein-Vlasov (collisionless matter) [Dafermos & Rendall 2007]

Additionally, in the first two cases the MTT is known to be achronal near i^+.



Possible Asymptotic Behavior — good

What about the general case, for arbitrary (spherically symmetric) matter?

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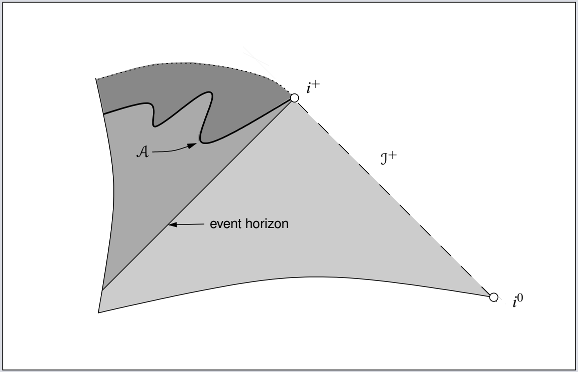
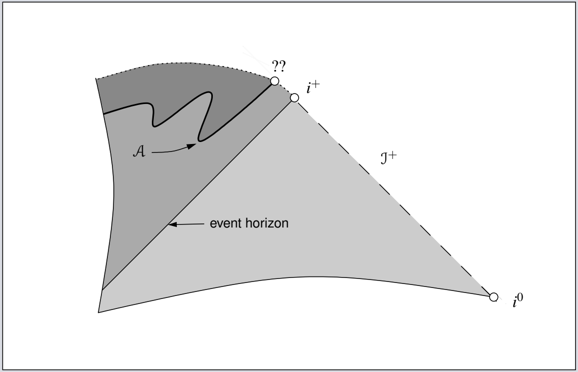
Must a marginally trapped tube be asymptotic to the event horizon?



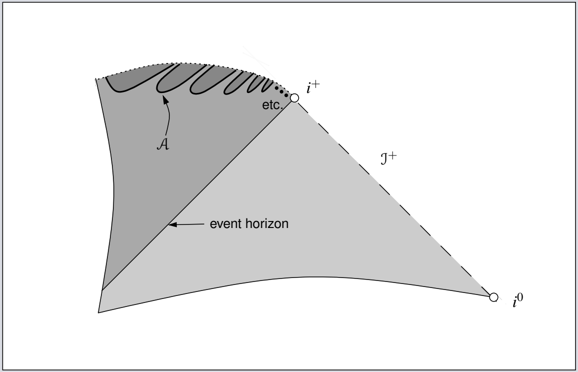
Possible Asymptotic Behavior — bad

Or could it terminate far inside the black hole?



Possible Asymptotic Behavior — ugly

One can imagine all sorts of bad behavior.



General black hole spacetimes — statement of the problem

Problem 1

Consider the class of all spherically symmetric black hole spacetimes satisfying the dominant energy condition. Are there general conditions which can be imposed on some or all of the metric and stress-energy tensor components which are sufficient to guarantee that the spacetime will contain a marginally trapped tube which is asymptotic to the event horizon?

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Conformally embedding $\mathcal{Q} = \mathcal{M}/SO(3) \hookrightarrow (\mathbb{R}^2, -du\,dv)$, its metric takes the form $-\Omega^2 du\,dv$, where $\Omega = \Omega(u,v) > 0$ is smooth on $\mathcal{Q}$. The metric g on $\mathcal{M}$ is

$$g = -\Omega^2 du\,dv + r^2 d\sigma^2,$$

where the radial function $r = r(u,v) \geq 0$ is smooth on $\mathcal{Q}$ and positive away from the center of symmetry, and $d\sigma^2$ is the round metric on S^2.

General black hole spacetimes — spherical symmetry revisited

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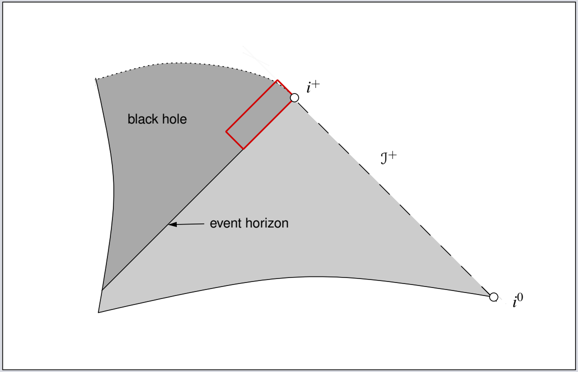
The Einstein equations on $\mathcal{M}$ yield a system of equations on $\mathcal{Q}$:

$$\begin{aligned}
\partial_u(\Omega^{-2}\partial_u r) &= -r\Omega^{-2} T_{uu} \\
\partial_v(\Omega^{-2}\partial_v r) &= -r\Omega^{-2} T_{vv} \\
\partial_u m &= 2r^2\Omega^{-2}\left(T_{uv}\partial_u r - T_{uu}\partial_v r\right) \\
\partial_v m &= 2r^2\Omega^{-2}\left(T_{uv}\partial_v r - T_{vv}\partial_u r\right),
\end{aligned}$$

where T_{uu}, T_{uv}, and T_{vv} are component functions of $T_{\alpha\beta}$ on M and m is the Hawking mass,

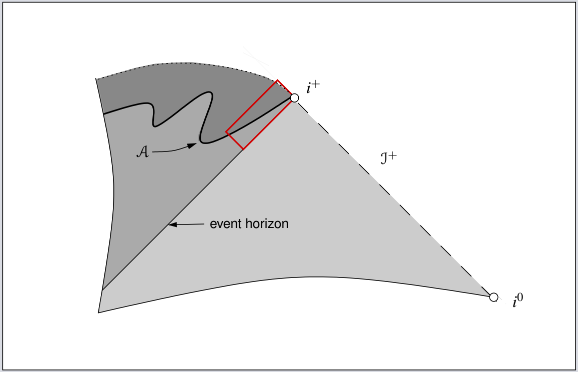
$$m = m(u,v) = \frac{r}{2}\left(1 - |\nabla r|^2\right) = \frac{r}{2}\left(1 + 4\Omega^{-2}\partial_u r\,\partial_v r\right).$$

In order to approach the problem, focus on a characteristic rectangle near timelike infinity, i^+:



General black hole spacetimes — approaching the problem

In order to approach the problem, focus on a characteristic rectangle near timelike infinity, i^+; whether $\mathcal{A}$ is asymptotic to the event horizon or not is determined by its behavior in this rectangle.



We begin with a characteristic rectangle

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and characteristic initial hypersurfaces

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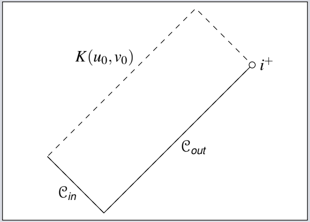

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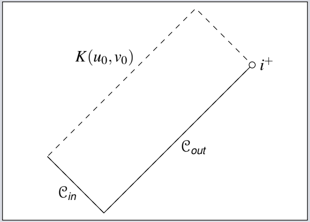

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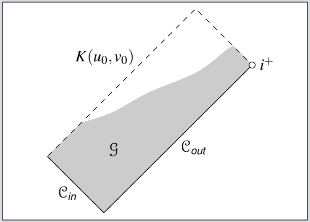

Given a specific matter model, we would prescribe initial data along $\mathcal{C}_{in} \cup \mathcal{C}_{out}$ and use the coupled Einstein-matter field equations to obtain the maximal future development of the data in K.

Here we instead assume simply that there exists a globally hyperbolic spacetime $\mathcal{G} \subset K$ containing $\mathcal{C}_{in} \cup \mathcal{C}_{out}$.

Each point $(u, v) \in \mathcal{G}$ represents a 2-sphere of radius $r = r(u, v)$ in $\mathcal{M}$.
The two future null normal directions are ∂_u and ∂_v — let u be the ingoing direction and v the outgoing direction. Then:

$$
\begin{array}{lllll}
 & \theta_+ & = & \theta_{(\partial_v)} & = & 2(\partial_v r) r^{-1} \\
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Therefore we define the *regular region* as

$$\mathcal{R} = \{(u,v) \in \mathcal{G} : \partial_v r > 0 \text{ and } \partial_u r < 0\},$$

the *trapped region* as

$$\mathcal{T} = \{(u,v) \in \mathcal{G} : \partial_v r < 0 \text{ and } \partial_u r < 0\},$$

and the *marginally trapped tube* as

$$\mathcal{A} = \{(u,v) \in \mathcal{G} : \partial_v r = 0 \text{ and } \partial_u r < 0\}.$$

Note that $\mathcal{A}$ is a hypersurface of $\mathcal{G}$ as long as 0 is a regular value of $\partial_v r$.

General black hole spacetimes — assumptions

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* $T_{uu} \geq 0,\ T_{uv} \geq 0,$ and $T_{vv} \geq 0$ everywhere in $\mathcal{G}$.

This is what the dominant energy condition on $\mathcal{M}$ boils down to on $\mathcal{Q}$.

(Upstairs, the dominant energy condition requires that $-T^{\alpha}_{\ \beta}\xi^{\beta}$ be future-directed causal for all future-directed timelike ξ^{α} — i.e. mass-energy can never be observed to be flowing faster than the speed of light.)

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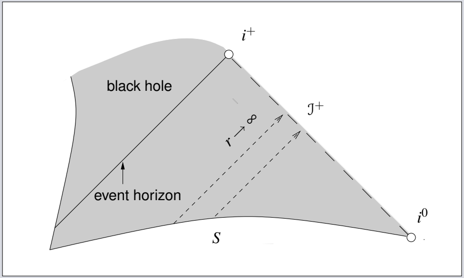
$*$ A certain “extension principle” holds. Paraphrased, it says that a first singularity evolving from the regular region in a spherically symmetric spacetime must emanate from the center of symmetry.

This is known to hold for self-gravitating Higgs fields and self-gravitating collisionless matter and expected to hold for other physically reasonable matter models [Dafermos 2005, Dafermos & Rendall 2005].

General black hole spacetimes — assumptions

$$* \qquad \sup_{\mathcal{C}_{out}} r = r_+ < \infty$$



General black hole spacetimes — assumptions

$$* \quad \partial_u r < 0 \text{ along } \mathcal{C}_{out}$$

Let us call a surface for which $\partial_u r \geq 0$ a *weakly anti-trapped surface*. Then this assumption is just that no weakly anti-trapped surfaces are present initially. It is motivated primarily by the following:

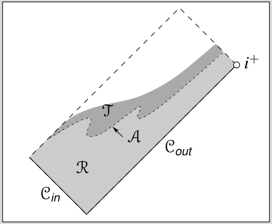
Proposition (Christodoulou)

If $\partial_u r < 0$ along $\mathcal{C}_{out}$, then $\mathcal{G} = \mathcal{R} \cup \mathcal{T} \cup \mathcal{A}$ — that is, weakly anti-trapped surfaces cannot evolve if none are present initially.

General black hole spacetimes — assumptions

Proposition (Christodoulou)

If $(u,v) \in \mathcal{T} \cup \mathcal{A}$, then $(u,v^*) \in \mathcal{T} \cup \mathcal{A}$ for all $v^* > v$. Similarly, if $(u,v) \in \mathcal{T}$, then $(u,v^*) \in \mathcal{T}$ for all $v^* > v$.



The trapped region $\mathcal{T}$ must lie inside the black hole, so $\partial_v r \geq 0$ along $\mathcal{C}_{out}$ if the latter is to lie along the event horizon. But if $\partial_v r = 0$ at a single point along $\mathcal{C}_{out}$, then by the above proposition, $\mathcal{A}$ must coincide with $\mathcal{C}_{out}$ to the future of that point. Therefore, in order to avoid the trivial case,

$$* \qquad \partial_v r > 0 \text{ along } \mathcal{C}_{out}$$

Theorem 1

Given a spacetime $(\mathcal{G}, \Omega, r)$ as described, suppose there exist positive constants c_0, c_1, c_2, ε, v', and δ such that

$$c_0 < \frac{1}{4r_+^2}, \quad \varepsilon < \frac{1}{4r_+^2} - c_0, \quad \text{and} \quad v_* \geq v_0,$$

and for

$$\mathcal{W} = \mathcal{W}(\delta) = \{(u,v) : r(u,v) \geq r_+ - \delta\},$$

the following conditions hold:

A′ $T_{uv}\,\Omega^{-2} \leq c_0$ in $\mathcal{W} \cap \mathcal{R}$

B1 $T_{uu}/(\partial_u r)^2 \leq c_1$ in $\mathcal{W} \cap \mathcal{R}$

B2 $\partial_v(T_{uv}\Omega^{-2})(u,\cdot) \in L^1([v_0,\infty))$ for all $u \in [0,u_0]$, and

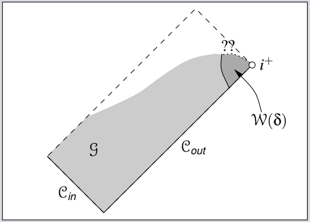
$\int_{v_*}^{v} \partial_v(T_{uv}\Omega^{-2})(u,\bar{v})\,d\bar{v} < \varepsilon$ for all $(u,v) \in \mathcal{W} \cap \mathcal{R}$ with $v \geq v_*$

C $\kappa := \partial_v r \left(1 - \frac{2m}{r}\right)^{-1} \geq c_2$ along $\mathcal{C}_{out} \cap \mathcal{W}$.

Then $\mathcal{G}$ contains a marginally trapped tube $\mathcal{A}$ which is asymptotic to the event horizon. Moreover, near i^+, $\mathcal{A}$ is connected and achronal with no ingoing null segments.

Theorem 1 — remarks

Remarks. $\mathcal{W}$ is essentially a δ-radius ball about the point i^+:



Also, the expression $T_{uv}\Omega^{-2}$ (seen in conditions **A**′ and **B2**) takes a particularly simple form in several matter models:

- ⋆ perfect fluid of pressure P and energy density ρ: $\frac{1}{4}(\rho - P)$
- ⋆ self-gravitating Higgs field ϕ with potential $V(\phi)$: $\frac{1}{2}V(\phi)$
- ⋆ Einstein-Maxwell massless scalar field of charge e: $\frac{1}{4}e^2 r^{-4}$

Theorem 1 — sketch of proof

Lemma 1

If $\mathcal{A}$ is nonempty and $T_{uv}\,\Omega^{-2} < \frac{1}{4r_+^2}$ (condition **A**) holds in $\mathcal{A}$, then each of its connected components is achronal with no ingoing null segments.

Remark. Condition $\mathbf{A}'$ is slightly stronger, requiring $T_{uv}\,\Omega^{-2} \leq c_0 < \frac{1}{4r_+^2}$.

Theorem 1 — sketch of proof

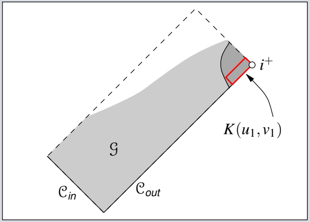
Lemma 1

If $\mathcal{A}$ is nonempty and $T_{uv}\Omega^{-2} < \frac{1}{4r_+^2}$ (condition **A**) holds in $\mathcal{A}$, then each of its connected components is achronal with no ingoing null segments.

Remark. Condition **A**$'$ is slightly stronger, requiring $T_{uv}\Omega^{-2} \le c_0 < \frac{1}{4r_+^2}$.

Lemma 2

Suppose condition **A** is satisfied in $\mathcal{W} \cap \mathcal{A}$. If $\mathcal{G}$ does not contain a marginally trapped tube which is asymptotic to the event horizon, then $\mathcal{W} \cap \mathcal{R}$ contains a rectangle $K(u_1, v_1)$ for some $u_1 \in (0, u_0]$, $v_1 \in [v_0, \infty)$.



Theorem 1 — sketch of proof (cont'd)

From the Einstein equations, we deduce the relation

$$\partial_v \log(-\partial_u r) = 2\kappa r^{-2}\alpha,$$

where

$$\begin{aligned}
\kappa &= \partial_v r \left(1 - \frac{2m}{r}\right)^{-1}, \\
\alpha &= m - 2r^3\,\Omega^{-2} T_{uv}.
\end{aligned}$$

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Inside of $K(u_1, v_1)$, we have

Conditions **B1** and **C** $\longrightarrow$ $\kappa \geq \kappa_0 > 0$

Condition **A**$'$ $\longrightarrow$ $\exists$ a small ingoing segment $[0,U] \times \{V\}$ on which $\alpha > \alpha_0 > 0$

Condition **B2** $\longrightarrow$ $\alpha > \alpha_1 > 0$ on $K(U,V) = [0,U] \times [V,\infty)$.

Thus in $K(U,V)$,

$$\partial_v \log(-\partial_u r) > 2\kappa_0 r_+^{-2}\alpha_1 > 0.$$

Theorem 1 — sketch of proof (cont'd)

In $K(U,V)$, we have
$$\partial_v \log(-\partial_u r) > C > 0.$$

Theorem 1 — sketch of proof (cont'd)

In $K(U,V)$, we have

$$\partial_v \log(-\partial_u r) > C > 0.$$

Integrating along an outgoing ray $\{u\} \times [V,v]$ yields

$$-\partial_u r(u,v) > -\partial_u r(u,V)\, e^{C(v-V)},$$

and since $[0,U]$ is compact and $\partial_u r < 0$, $\exists b > 0$ such that

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Finally, integrating along an ingoing null ray $[0,u] \times \{v\}$ gives

$$r(u,v) < r(0,v) - be^{C(v-V)}u.$$

But for any $u > 0$, the RHS tends to $-\infty$ as $v \to \infty$, while the LHS is positive — contradiction.

□

Theorem 1 — application?

Problem 2

Can the hypotheses of Theorem 1 be shown to hold for any particular matter model for which the asymptotic result was not already known?

Application to Higgs fields

A self-gravitating Higgs field with non-zero potential consists of a scalar function ϕ on the spacetime and a potential function $V(\phi)$ such that

$$\Box_g \phi = V'(\phi). \qquad (*)$$

The stress-energy tensor then takes the form

$$T_{\alpha\beta} = \nabla_\alpha \phi \nabla_\beta \phi - \left(\frac{1}{2} \nabla^\gamma \phi \nabla_\gamma \phi + V(\phi) \right) g_{\alpha\beta}. \qquad (**)$$

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In spherically symmetry, $\phi = \phi(u,v)$ and the evolution equation $(*)$ becomes

$$V'(\phi) = -4\Omega^{-2}(\partial^2_{uv}\phi + \partial_u \phi\,(\partial_v \log r) + \partial_v \phi\,(\partial_u \log r))$$

and in double-null coordinates, $(**)$ yields

$$\begin{aligned} T_{uu} &= (\partial_u \phi)^2, \\ T_{vv} &= (\partial_v \phi)^2, \text{ and} \\ T_{uv} &= \tfrac{1}{2}\Omega^2 V(\phi). \end{aligned}$$

Note that the dominant energy condition is satisfied if and only if $V(\phi) \geq 0$.
The extension principle is known to hold for self-gravitating Higgs fields precisely when V is bounded below [Dafermos 2005].

Application to Higgs fields – explicit decay

Theorem 2

Assume we have suitable initial data for the spherically symmetric Einstein-Higgs system for which $V \geq 0$. Fix a constant $p > \frac{1}{2}$ and a function $\eta(v) > 0$ such that $\eta(v)$ decreases monotonically to 0 as v tends to infinity. If V'' is bounded, and if along $\mathcal{C}_{out}$ the initial data satisfy

$$\partial_v r < \eta(v),$$

$$|\partial_v \phi| = O(v^{-p}),$$

$$|V'(\phi)| = O(v^{-p}),$$

$$\kappa \text{ is bounded above and away from } 0,$$

and

$$\liminf_{v \to \infty} V(\phi) < \frac{1}{4r_+^2},$$

then the result of Theorem 1 holds for the maximal development of these initial data.

Application to Higgs fields – smallness and monotonicity

Theorem 3

Assume we have suitable initial data for the spherically symmetric Einstein-Higgs system for which $V \geq 0$. If V'' is bounded and nonnegative, and along $\mathcal{C}_{out}$

$$\partial_v r, |\partial_v \phi|, \text{ and } V(\phi) \text{ are sufficiently small,}$$

$$\kappa \text{ is bounded above and away from } 0,$$

$$V'(\phi) < C|\partial_v \phi|,$$

$$\partial_v \phi,\ \partial_u \phi < 0,$$

$$\text{either } V'(\phi) \leq 0 \text{ or } |\inf_{\mathcal{C}_{out}} \phi| < \infty,$$

& a technical inequality relating $V'(\phi), \partial_v \log r,$ and $\partial_v \phi$ is satisfied,

then the result of Theorem 1 holds for the maximal development of these initial data.

Higgs field spacetimes — "physically reasonable" problem

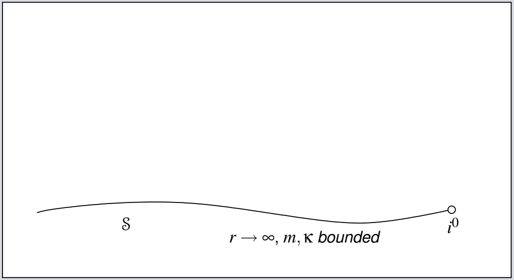
Problem 3

Consider spacelike spherically symmetric initial data for the Einstein-Higgs field equations with at least one asymptotically flat end whose future development contains a black hole. Are there conditions which can be imposed on the initial data which are sufficient to guarantee that the spacetime will contain a marginally trapped tube asymptotic to the event horizon?

Theorem 4 — assumptions

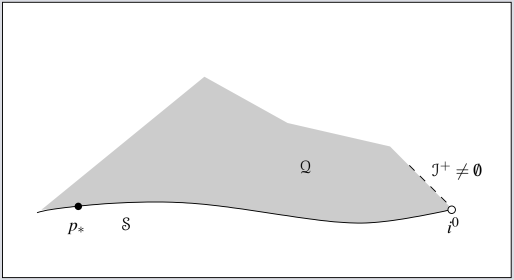
Suppose $\mathcal{S}$ is an asymptotically flat spherically symmetric spacelike hypersurface on which initial data for the Einstein-Higgs field equations is given.

Assume that the radial function r is strictly increasing and tends to ∞ along $\mathcal{S}$ towards the asymptotically flat end, and that the Hawking mass m is uniformly bounded above and κ is bounded above and away from zero on $\mathcal{S}$.



Let $\mathcal{Q}$ denote the 2-dimensional Lorentzian quotient of the future Cauchy development of $\mathcal{S}$.

Let us assume further that $\mathcal{I}^+ \neq \emptyset$ and that $\mathcal{S}$ contains a trapped surface. We denote by p_* the point corresponding to the outermost marginally trapped sphere along $\mathcal{S}$.



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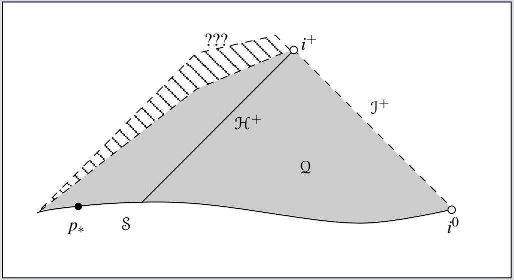
Let us assume further that $\mathcal{I}^+ \neq \emptyset$ and that $\mathcal{S}$ contains a trapped surface. We denote by p_* the point corresponding to the outermost marginally trapped sphere along $\mathcal{S}$.

Then $\mathcal{Q}$ contains a black hole region with event horizon $\mathcal{H}^+$ [Dafermos 2005].



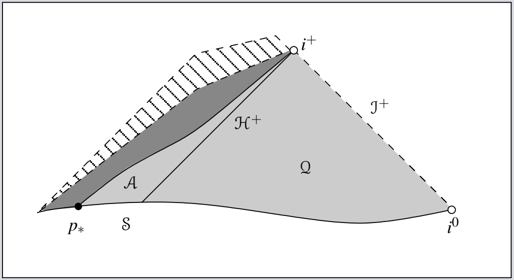
Theorem 4 — statement

Theorem 4

Set $m_1 := m(p_*)$ and $M := \sup_{q\in\mathcal{S}} m(q)$. Assume that $V(x)\geq 0$, $|V''(x)|\leq B$, and $\phi\to\phi_+$ along $\mathcal{S}$. Then there exist constants $\varepsilon>0$ and $\rho\in(0,1)$, depending only on M, B, and initial data, such that if

$$\frac{m_1}{M} > \rho \qquad \text{and} \qquad |\phi-\phi_+| \leq \frac{\varepsilon}{\sqrt{r}} \quad \text{on } \mathcal{S},$$

then $\mathcal{Q}$ contains a connected, achronal, marginally trapped tube $\mathcal{A}$ which intersects $\mathcal{S}$ at p_* and is asymptotic to the event horizon $\mathcal{H}^+$.



Theorem 4 — sketch of proof

Recall that the key estimates in the proof of Theorem 1 involved quantities κ and α, where here

$$\alpha = m - r^3 V(\phi).$$

As in that proof, if we can establish positive lower bounds for κ and α near i^+, then the result will follow from the extension principle and a contradiction argument.

Theorem 4 — sketch of proof

Recall that the key estimates in the proof of Theorem 1 involved quantities κ and α, where here

$$\alpha = m - r^3 V(\phi).$$

As in that proof, if we can establish positive lower bounds for κ and α near i^+, then the result will follow from the extension principle and a contradiction argument.

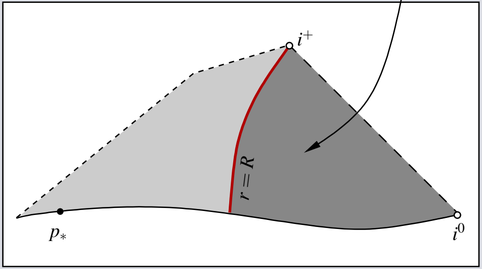
To control α it suffices to control $\phi - \phi_+$, since $V(\phi_+) = V'(\phi_+) = 0$ & $|V''| \leq B$, together with the mean value theorem, imply that

$$V(\phi) \leq B(\phi - \phi_+)^2.$$

Bounding $\phi - \phi_+$ and κ is straightforward for large r, but to control them near $\mathcal{H}^+$, we need the quantity $\left|\frac{r\partial_u \phi}{\partial_u r}\right|$ to be uniformly bounded above.

Theorem 4 — sketch of proof (cont'd)

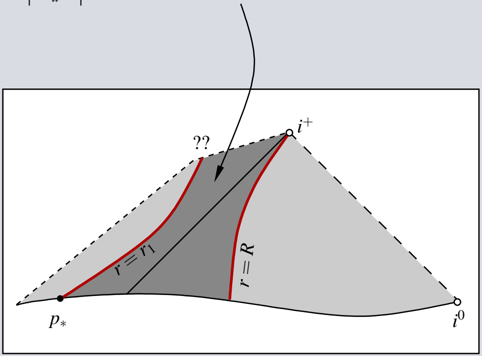
So: we use smallness of the initial data, Raychaudhuri, & an energy estimate to control κ and $|\phi - \phi_+|$ (and hence α) in $\{r \geq R\}\ldots$



Theorem 4 — sketch of proof (cont'd)

So: we use smallness of the initial data, Raychaudhuri, & an energy estimate to control κ and $|\phi - \phi_+|$ (and hence α) in $\{r \geq R\}\ldots$

$\ldots$ and use a bootstrap argument to obtain bounds for κ, α, and $\left|\frac{r\partial_u\phi}{\partial_u r}\right|$ in $\{r_1 \leq r \leq R\}$, and in particular, near i^+.



□