The Star Product for Differential Forms on Symplectic Manifolds [1] [2]

Shannon McCurdy, Anthony Tagliaferro, and Bruno Zumino

University of California, Berkeley

October 30, 2008

[1] arXiv: 0808.4717

[2] This work was supported in part by DE-AC03-76SF00098, and in part by NSF PHY-0457315.

The Star Product

- The star product is a formal non-commutative deformation of the usual point-wise product of functions on a manifold M.

The Star Product

- The star product is a formal non-commutative deformation of the usual point-wise product of functions on a manifold M.
- Deform $C^\infty(M)$ to a noncommutative ring of functions, $C^\infty(M, *)[[\hbar]]$.

The Star Product

- The star product is a formal non-commutative deformation of the usual point-wise product of functions on a manifold M.
- Deform $C^\infty(M)$ to a noncommutative ring of functions, $C^\infty(M,*)[[\hbar]]$.
- $f(x) \in C^\infty(M,*)[[\hbar]]$ are formal power series in $\hbar$ with coefficients in $C^\infty(M)$:

$$f(x) = f_0(x) + \sum_{n=1}^{\infty} \hbar^n f_n(x).$$

The Star Product

- The noncommutative product of $f, g \in C^\infty(M,*)[[\hbar]]$ is denoted by the star product, $*$:

$$f * g = fg + \sum_{n=1}^{\infty} \hbar^n C_n(f,g)$$

The Star Product

- The noncommutative product of $f, g \in C^\infty(M, *)[[\hbar]]$ is denoted by the star product, $*$:

$$f * g = fg + \sum_{n=1}^{\infty} \hbar^n C_n(f, g)$$

- $C_1(f, g) = \{f, g\} = \pi^{ij} \partial_i f \partial_j g$

The Star Product

- The noncommutative product of $f, g \in C^\infty(M, *)[[\hbar]]$ is denoted by the star product, $*$:

$$f * g = fg + \sum_{n=1}^{\infty} \hbar^n C_n(f, g)$$

- $C_1(f, g) = \{f, g\} = \pi^{ij} \partial_i f \partial_j g$
- The star product is associative: $(f * g) * h = f * (g * h)$.

The Star Product

- The noncommutative product of $f, g \in C^\infty(M, *)[[\hbar]]$ is denoted by the star product, $*$:

$$f * g = fg + \sum_{n=1}^{\infty} \hbar^n C_n(f, g)$$

- $C_1(f, g) = \{f, g\} = \pi^{ij} \partial_i f \partial_j g$
- The star product is associative: $(f * g) * h = f * (g * h)$.
- Associativity at order n can be expressed as a condition on the Hochschild coboundary of C_n. For:

$$(\delta C_n)(u, v, w) \equiv u C_n(v, w) - C_n(uv, w) + C_n(u, vw) - C_n(u, v) w,$$

The Star Product

- The noncommutative product of $f, g \in C^\infty(M,*)[[\hbar]]$ is denoted by the star product, $*$:

$$f * g = fg + \sum_{n=1}^{\infty} \hbar^n C_n(f,g)$$

- $C_1(f,g) = \{f,g\} = \pi^{ij}\partial_i f \partial_j g$
- The star product is associative: $(f * g) * h = f * (g * h)$.
- Associativity at order n can be expressed as a condition on the Hochschild coboundary of C_n. For:

$$(\delta C_n)(u,v,w) \equiv uC_n(v,w) - C_n(uv,w) + C_n(u,vw) - C_n(u,v)w,$$

$$(\delta C_n)(u,v,w) = \sum_{r+s=n;r,s>0} (C_r(C_s(u,v),w) - C_r(u,C_s(v,w)))\,\forall n \geq 1.$$

Star Product Equivalence

- Star products are equivalent if there exists a linear operator T of the form:

$$T = 1 + \sum_{n=1}^{\infty} \hbar^n T_n$$

such that

$$T\left(f *_2 g\right) = T\left(f\right) *_1 T\left(g\right).$$

Star Product Equivalence

- Star products are equivalent if there exists a linear operator T of the form:

$$T = 1 + \sum_{n=1}^{\infty} \hbar^n T_n$$

  such that

$$T\left(f *_2 g\right) = T\left(f\right) *_1 T\left(g\right).$$

- Equivalence classes of star products on symplectic manifolds are in one-to-one correspondence with second deRham cohomology $H^2_{dR}(M)[[\hbar]]$.

The Star Product for Functions, Constant π

- The Weyl-Moyal star product can be written in a compact form to all orders (1949):

The Star Product for Functions, Constant π

- The Weyl-Moyal star product can be written in a compact form to all orders (1949):
- $(f * g)(x) = e^{\frac{i\hbar}{2}\pi^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial y^j}} f(x)g(y)|_{y=x}$

The Star Product for Functions, Constant π

- The Weyl-Moyal star product can be written in a compact form to all orders (1949):
- $(f * g)(x) = e^{\frac{i\hbar}{2}\pi^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial y^j}} f(x)g(y)|_{y=x}$
  - $f * g = fg + \frac{i\hbar}{2}\pi^{ij}\partial_i f \partial_j g + \frac{1}{2}(\frac{i\hbar}{2})^2\pi^{ij}\pi^{mn}\partial_i\partial_m f\partial_j\partial_n g + \mathcal{O}(\hbar^3)$.

The Star Product for Functions, Constant π

- The Weyl-Moyal star product can be written in a compact form to all orders (1949):
- $(f * g)(x) = e^{\frac{i\hbar}{2}\pi^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial y^j}} f(x)g(y)|_{y=x}$
  - $f * g = fg + \frac{i\hbar}{2}\pi^{ij}\partial_i f \partial_j g + \frac{1}{2}(\frac{i\hbar}{2})^2\pi^{ij}\pi^{mn}\partial_i\partial_m f \partial_j \partial_n g + \mathcal{O}(\hbar^3)$.
- Associativity:

$$((f * g) * h)(z) = e^{\frac{i\hbar}{2}\pi^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial z^j}} (f * g)(x)h(z)|_{x=z}$$

The Star Product for Functions, Constant π

- The Weyl-Moyal star product can be written in a compact form to all orders (1949):
- $(f * g)(x) = e^{\frac{i\hbar}{2}\pi^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial y^j}} f(x)g(y)|_{y=x}$
  - $f * g = fg + \frac{i\hbar}{2}\pi^{ij}\partial_i f \partial_j g + \frac{1}{2}(\frac{i\hbar}{2})^2\pi^{ij}\pi^{mn}\partial_i\partial_m f \partial_j\partial_n g + \mathcal{O}(\hbar^3)$.
- Associativity:

$$
\begin{aligned}
((f * g) * h)(z) &= e^{\frac{i\hbar}{2}\pi^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial z^j}} (f * g)(x)h(z)|_{x=z} \\
&= e^{\frac{i\hbar}{2}\pi^{ij}\left(\frac{\partial}{\partial x^i}+\frac{\partial}{\partial y^i}\right)\frac{\partial}{\partial z^j}} e^{\frac{i\hbar}{2}\pi^{kl}\frac{\partial}{\partial x^k}\frac{\partial}{\partial y^l}} f(x)g(y)h(z)|_{x=y=z}
\end{aligned}
$$

The Star Product for Functions, Constant π

- The Weyl-Moyal star product can be written in a compact form to all orders (1949):
- $(f * g)(x) = e^{\frac{i\hbar}{2}\pi^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial y^j}} f(x)g(y)|_{y=x}$
  - $f * g = fg + \frac{i\hbar}{2}\pi^{ij}\partial_i f \partial_j g + \frac{1}{2}(\frac{i\hbar}{2})^2\pi^{ij}\pi^{mn}\partial_i\partial_m f\partial_j\partial_n g + \mathcal{O}(\hbar^3)$.
- Associativity:

$$
\begin{aligned}
((f * g) * h)(z) &= e^{\frac{i\hbar}{2}\pi^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial z^j}} (f * g)(x)h(z)|_{x=z} \\
&= e^{\frac{i\hbar}{2}\pi^{ij}\left(\frac{\partial}{\partial x^i}+\frac{\partial}{\partial y^i}\right)\frac{\partial}{\partial z^j}} e^{\frac{i\hbar}{2}\pi^{kl}\frac{\partial}{\partial x^k}\frac{\partial}{\partial y^l}} f(x)g(y)h(z)|_{x=y=z} \\
&= e^{\frac{i\hbar}{2}\pi^{ij}\left(\frac{\partial}{\partial x^i}\frac{\partial}{\partial z^j}+\frac{\partial}{\partial y^i}\frac{\partial}{\partial z^j}+\frac{\partial}{\partial x^i}\frac{\partial}{\partial y^j}\right)} f(x)g(y)h(z)|_{x=y=z}
\end{aligned}
$$

The Star Product for Functions, Constant π

- The Weyl-Moyal star product can be written in a compact form to all orders (1949):
- $(f * g)(x) = e^{\frac{i\hbar}{2}\pi^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial y^j}} f(x)g(y)|_{y=x}$
  - $f * g = fg + \frac{i\hbar}{2}\pi^{ij}\partial_i f \partial_j g + \frac{1}{2}(\frac{i\hbar}{2})^2\pi^{ij}\pi^{mn}\partial_i\partial_m f \partial_j\partial_n g + \mathcal{O}(\hbar^3)$.
- Associativity:

$$
\begin{aligned}
((f * g) * h)(z) &= e^{\frac{i\hbar}{2}\pi^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial z^j}} (f * g)(x)h(z)|_{x=z} \\
&= e^{\frac{i\hbar}{2}\pi^{ij}\left(\frac{\partial}{\partial x^i}+\frac{\partial}{\partial y^i}\right)\frac{\partial}{\partial z^j}} e^{\frac{i\hbar}{2}\pi^{kl}\frac{\partial}{\partial x^k}\frac{\partial}{\partial y^l}} f(x)g(y)h(z)|_{x=y=z} \\
&= e^{\frac{i\hbar}{2}\pi^{ij}\left(\frac{\partial}{\partial x^i}\frac{\partial}{\partial z^j}+\frac{\partial}{\partial y^i}\frac{\partial}{\partial z^j}+\frac{\partial}{\partial x^i}\frac{\partial}{\partial y^j}\right)} f(x)g(y)h(z)|_{x=y=z} \\
&= e^{\frac{i\hbar}{2}\pi^{ij}\frac{\partial}{\partial x^i}\left(\frac{\partial}{\partial y^i}+\frac{\partial}{\partial z^j}\right)} e^{\frac{i\hbar}{2}\pi^{kl}\frac{\partial}{\partial y^k}\frac{\partial}{\partial z^l}} f(x)g(y)h(z)|_{x=y=z}
\end{aligned}
$$

The Star Product for Functions, Constant π

- The Weyl-Moyal star product can be written in a compact form to all orders (1949):
- $(f * g)(x) = e^{\frac{i\hbar}{2}\pi^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial y^j}} f(x)g(y)|_{y=x}$
  - $f * g = fg + \frac{i\hbar}{2}\pi^{ij}\partial_i f \partial_j g + \frac{1}{2}(\frac{i\hbar}{2})^2 \pi^{ij}\pi^{mn}\partial_i\partial_m f \partial_j \partial_n g + \mathcal{O}(\hbar^3)$.
- Associativity:

$$
\begin{aligned}
((f * g) * h)(z) &= e^{\frac{i\hbar}{2}\pi^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial z^j}} (f * g)(x)h(z)|_{x=z} \\
&= e^{\frac{i\hbar}{2}\pi^{ij}\left(\frac{\partial}{\partial x^i}+\frac{\partial}{\partial y^i}\right)\frac{\partial}{\partial z^j}} e^{\frac{i\hbar}{2}\pi^{kl}\frac{\partial}{\partial x^k}\frac{\partial}{\partial y^l}} f(x)g(y)h(z)|_{x=y=z} \\
&= e^{\frac{i\hbar}{2}\pi^{ij}\left(\frac{\partial}{\partial x^i}\frac{\partial}{\partial z^j}+\frac{\partial}{\partial y^i}\frac{\partial}{\partial z^j}+\frac{\partial}{\partial x^i}\frac{\partial}{\partial y^j}\right)} f(x)g(y)h(z)|_{x=y=z} \\
&= e^{\frac{i\hbar}{2}\pi^{ij}\frac{\partial}{\partial x^i}\left(\frac{\partial}{\partial y^i}+\frac{\partial}{\partial z^j}\right)} e^{\frac{i\hbar}{2}\pi^{kl}\frac{\partial}{\partial y^k}\frac{\partial}{\partial z^l}} f(x)g(y)h(z)|_{x=y=z} \\
&= (f * (g * h))(z)
\end{aligned}
$$

The Star Product for Functions, Non-constant π

- π is constrained by the Jacobi Identity:
  $\pi^{im}\partial_m\pi^{jk} + \circlearrowleft (i,j,k) = 0.$

The Star Product for Functions, Non-constant π

- π is constrained by the Jacobi Identity: $\pi^{im}\partial_m\pi^{jk} + \circlearrowleft (i, j, k) = 0.$
- Strong condition:$(f * g) * h - f * (g * h) = 0$:

The Star Product for Functions, Non-constant π

- π is constrained by the Jacobi Identity: $\pi^{im}\partial_m\pi^{jk} + \circlearrowleft (i,j,k) = 0$.
- Strong condition:$(f * g) * h - f * (g * h) = 0$:
  $\mathcal{O}(\hbar^0)$: $fgh - fgh = 0$

The Star Product for Functions, Non-constant π

- π is constrained by the Jacobi Identity:
  $\pi^{im}\partial_m\pi^{jk} + \circlearrowleft (i,j,k) = 0.$
- Strong condition:$(f * g) * h - f * (g * h) = 0$:
  $\mathcal{O}(\hbar^0)$: $fgh - fgh = 0$
  $\mathcal{O}(\hbar)$: $f\{g,h\} + \{fg,h\} - \{f,gh\} - \{f,g\}h = 0$

The Star Product for Functions, Non-constant π

- π is constrained by the Jacobi Identity:
  $\pi^{im}\partial_m\pi^{jk} + \circlearrowleft (i,j,k) = 0.$
- Strong condition: $(f * g) * h - f * (g * h) = 0$:
  $\mathcal{O}(\hbar^0)$: $fgh - fgh = 0$
  $\mathcal{O}(\hbar)$: $f\{g,h\} + \{fg,h\} - \{f,gh\} - \{f,g\}h = 0$
  $\mathcal{O}(\hbar^2)$: $fC_2(g,h) + C_2(f,gh) - C_2(fg,h) - C_2(f,g)h = \{\{f,g\},h\} - \{f,\{g,h\}\}$

The Star Product for Functions, Non-constant π

- π is constrained by the Jacobi Identity:
  $\pi^{im}\partial_m\pi^{jk} + \circlearrowleft (i,j,k) = 0.$
- Strong condition:$(f*g)*h - f*(g*h) = 0$:
  $\mathcal{O}(\hbar^0)$: $fgh - fgh = 0$
  $\mathcal{O}(\hbar)$: $f\{g,h\} + \{fg,h\} - \{f,gh\} - \{f,g\}h = 0$
  $\mathcal{O}(\hbar^2)$: $fC_2(g,h) + C_2(f,gh) - C_2(fg,h) - C_2(f,g)h =$
  $\{\{f,g\},h\} - \{f,\{g,h\}\}$
- Solution: $f*g = fg + \hbar\pi^{ij}\partial_i f\partial_j g + \hbar^2\Big(\frac{1}{2}\pi^{ij}\pi^{mn}\partial_i\partial_m f\partial_j\partial_n g +$

  $\frac{1}{3}\pi^{ij}\partial_j\pi^{mn}\Big(\partial_i\partial_m f\partial_n g - \partial_m f\partial_i\partial_n g\Big)\Big) + \mathcal{O}(\hbar^3).$

The Star Product for Functions, Non-constant π

- π is constrained by the Jacobi Identity:
  $\pi^{im}\partial_m\pi^{jk} + \circlearrowleft (i,j,k) = 0$.
- Strong condition:$(f*g)*h - f*(g*h) = 0$:
  $\mathcal{O}(\hbar^0)$: $fgh - fgh = 0$
  $\mathcal{O}(\hbar)$: $f\{g,h\} + \{fg,h\} - \{f,gh\} - \{f,g\}h = 0$
  $\mathcal{O}(\hbar^2)$: $fC_2(g,h) + C_2(f,gh) - C_2(fg,h) - C_2(f,g)h =$
  $\{\{f,g\},h\} - \{f,\{g,h\}\}$
- Solution: $f*g = fg + \hbar\pi^{ij}\partial_i f\partial_j g + \hbar^2\Big(\frac{1}{2}\pi^{ij}\pi^{mn}\partial_i\partial_m f\partial_j\partial_n g +$
  $\frac{1}{3}\pi^{ij}\partial_j\pi^{mn}\big(\partial_i\partial_m f\partial_n g - \partial_m f\partial_i\partial_n g\big)\Big) + \mathcal{O}(\hbar^3)$.

- To all orders,
  Diagramatically[3]:

$$f * g = fg + \sum_{n=1}^{\infty} \hbar^n \sum_{\Gamma \in G_{n,2}} \omega_\Gamma B_\Gamma(f, g)$$

[3] K: q-alg/9709040
[4] CF: math/9902090

The Star Product for Functions, Non-constant π

- To all orders,
  Diagramatically[3]:

$$f * g = fg + \sum_{n=1}^{\infty} \hbar^n \sum_{\Gamma \in G_{n,2}} \omega_\Gamma B_\Gamma(f, g)$$

Path Integral[4]:

$$(f * g)(x) = \int_{X(\infty)=x} f(X(1))g(X(0))e^{\frac{i}{\hbar}S[X,\eta]}dX\, d\eta$$

$$S[X, \eta] = \int_D \left(\eta_i(u) \wedge dX^i(u) + \frac{1}{2}\alpha^{ij}(X(u))\eta_i(u) \wedge \eta_j(u)\right)$$

[3] K: q-alg/9709040
[4] CF: math/9902090

Deformation Quantization

- In conventional quantum mechanics, $\lim_{\hbar \to 0} \frac{1}{i\hbar}[\hat{f}, \hat{g}] \nleftrightarrow \{f, g\}$

[5] Flato, Bayen, Fronsdal, Lichnerowicz, Sternheimer, 1977

Deformation Quantization

- In conventional quantum mechanics, $\lim_{\hbar\to 0} \frac{1}{i\hbar}[\hat{f},\hat{g}] \not\leftrightarrow \{f,g\}$
- Alternative: deform classical mechanics and use $*$-products.[5] Require:

$$\lim_{\hbar\to 0} \frac{1}{i\hbar}[f,g]_* = \lim_{\hbar\to 0} \frac{1}{i\hbar}(f*g - g*f) = \{f,g\}.$$

[5] Flato, Bayen, Fronsdal, Lichnerowicz, Sternheimer, 1977

Deformation Quantization

- In conventional quantum mechanics, $\lim_{\hbar \to 0} \frac{1}{i\hbar}[\hat{f}, \hat{g}] \nleftrightarrow \{f, g\}$
- Alternative: deform classical mechanics and use $*$-products.[5] Require:

$$\lim_{\hbar \to 0} \frac{1}{i\hbar}[f, g]_* = \lim_{\hbar \to 0} \frac{1}{i\hbar}(f * g - g * f) = \{f, g\}.$$

-

Deformation Quantization	Quantum Mechanics
$\frac{1}{2\pi\hbar}\int \rho(q,p)\, dq\, dp = 1$	$\mathrm{tr}\hat{\rho} = 1$
$(\rho * \rho)(q,p) = \rho(q,p)$	$\hat{\rho}^2 = \hat{\rho}$
$\langle A \rangle = \frac{1}{2\pi\hbar}\int (A * \rho)(q,p)\, dq\, dp$	$\langle A \rangle = \mathrm{tr}(\hat{\rho}\hat{A})$

[5] Flato, Bayen, Fronsdal, Lichnerowicz, Sternheimer, 1977

Deformation Quantization

- Time-independent:

$$(H * \rho_n)(q, p) = E_n \, \rho_n(q, p)$$

Deformation Quantization

- Time-independent:

$$(H * \rho_n)(q, p) = E_n \, \rho_n(q, p)$$

$$E_n = \frac{1}{2\pi\hbar} \int (H * \rho_n)(q, p) \, dq \, dp = \frac{1}{2\pi\hbar} \int H(q, p) \rho_n(q, p) \, dq \, dp.$$

Deformation Quantization

- Time-independent:

$$(H * \rho_n)(q,p) = E_n\, \rho_n(q,p)$$

$$E_n = \frac{1}{2\pi\hbar}\int (H*\rho_n)(q,p)\, dq\, dp = \frac{1}{2\pi\hbar}\int H(q,p)\rho_n(q,p)\, dq\, dp.$$

  - Classical mechanics: $E = \int H(q,p)\, \delta^{(2)}(q-q_0, p-p_0)\, dq\, dp$

Deformation Quantization

- Time-independent:

$$(H * \rho_n)(q,p) = E_n\, \rho_n(q,p)$$

$$E_n = \frac{1}{2\pi\hbar}\int (H{*}\rho_n)(q,p)\,dq\,dp = \frac{1}{2\pi\hbar}\int H(q,p)\rho_n(q,p)\,dq\,dp.$$

  - Classical mechanics: $E = \int H(q,p)\,\delta^{(2)}(q-q_0, p-p_0)\,dq\,dp$

- Time-evolution:

$$i\hbar\frac{d}{dt}Exp(Ht) = H * Exp(Ht)$$

where $Exp(Ht) = \sum_{m=0}^{\infty} \frac{1}{m!}\left(\frac{-it}{\hbar}\right)^m (H*)^m$ and $(H*)^m = \underbrace{H * H * \cdots * H}_{m}$

String Theory [SW: hep-th/9908142]

- Take a string ending on a D-brane, with constant B_{ij} in the $\alpha', g_{ij} \to 0$ limit.

String Theory [SW: hep-th/9908142]

- Take a string ending on a D-brane, with constant B_{ij} in the $\alpha', g_{ij} \to 0$ limit.
- The action governing the boundary degrees of freedom is:

$$S = -\frac{i}{2} \int_{\partial\Sigma} B_{ij} x^i \partial_t x^j .$$

String Theory [SW: hep-th/9908142]

- Take a string ending on a D-brane, with constant B_{ij} in the $\alpha', g_{ij} \to 0$ limit.
- The action governing the boundary degrees of freedom is:

$$S = -\frac{i}{2} \int_{\partial\Sigma} B_{ij} x^i \partial_t x^j .$$

- When $\Sigma = D$, the boundary propogator is:

$$\langle x^i(\tau) | x^j(\tau') \rangle = \frac{i}{2} (B^{-1})^{ij} \epsilon(\tau - \tau') \qquad \Rightarrow \qquad [x^i, x^j] = i (B^{-1})^{ij}$$

String Theory [SW: hep-th/9908142]

- Take a string ending on a D-brane, with constant B_{ij} in the $\alpha', g_{ij} \to 0$ limit.
- The action governing the boundary degrees of freedom is:

$$S = -\frac{i}{2} \int_{\partial \Sigma} B_{ij} x^i \partial_t x^j.$$

- When $\Sigma = D$, the boundary propogator is:

$$\left\langle x^i(\tau) | x^j(\tau') \right\rangle = \frac{i}{2} (B^{-1})^{ij} \epsilon(\tau - \tau') \qquad \Rightarrow \qquad [x^i, x^j] = i(B^{-1})^{ij}$$

- Correlation functions:

$$\lim_{\tau \to \tau'^+} : f(x(\tau)) :: g(x(\tau')) :=: (f * g)(x(\tau')) :$$

String Theory [SW: hep-th/9908142]

- Take a string ending on a D-brane, with constant B_{ij} in the $\alpha', g_{ij} \to 0$ limit.
- The action governing the boundary degrees of freedom is:

$$S = -\frac{i}{2} \int_{\partial\Sigma} B_{ij} x^i \partial_t x^j.$$

- When $\Sigma = D$, the boundary propogator is:

$$\langle x^i(\tau) | x^j(\tau') \rangle = \frac{i}{2} (B^{-1})^{ij} \epsilon(\tau - \tau') \qquad \Rightarrow \qquad [x^i, x^j] = i(B^{-1})^{ij}$$

- Correlation functions:

$$\lim_{\tau \to \tau'^+} : f(x(\tau)) :: g(x(\tau')) :=: (f * g)(x(\tau')) :$$

- The world-volume of a string ending on a D-brane is a non-commutative space.

Motivation

- We are interested in gauge theories on noncommutative spaces.

Motivation

- We are interested in gauge theories on noncommutative spaces.
- Gauge theories involve differential forms, such as the gauge potential $A_\mu dx^\mu$.

Motivation

- We deform the graded exterior algebra of differential forms, $\Omega^\bullet(M)$, to a noncommutative graded exterior algebra $\Omega^\bullet(M,*)[[\hbar]]$.

Motivation

- We deform the graded exterior algebra of differential forms, $\Omega^\bullet(M)$, to a noncommutative graded exterior algebra $\Omega^\bullet(M,*)[[\hbar]]$.
- $\alpha(x) \in \Omega^\bullet(M,*)[[\hbar]]$ are formal power series in $\hbar$ with coefficients in $\Omega^\bullet(M)$:

$$\alpha(x) = \alpha_0(x) + \sum_{n=1}^{\infty} \hbar^n \alpha_n(x).$$

Motivation

- We deform the graded exterior algebra of differential forms, $\Omega^\bullet(M)$, to a noncommutative graded exterior algebra $\Omega^\bullet(M,*)[[\hbar]]$.
- $\alpha(x) \in \Omega^\bullet(M,*)[[\hbar]]$ are formal power series in $\hbar$ with coefficients in $\Omega^\bullet(M)$:

$$\alpha(x) = \alpha_0(x) + \sum_{n=1}^{\infty} \hbar^n \alpha_n(x).$$

- We extend the star product so that it includes differential forms:

$$\alpha * \beta = \alpha\beta + \sum_{n=1}^{\infty} \hbar^n C_n(\alpha, \beta)$$

Motivation

- We deform the graded exterior algebra of differential forms, $\Omega^\bullet(M)$, to a noncommutative graded exterior algebra $\Omega^\bullet(M,*)[[\hbar]]$.
- $\alpha(x) \in \Omega^\bullet(M,*)[[\hbar]]$ are formal power series in $\hbar$ with coefficients in $\Omega^\bullet(M)$:

$$\alpha(x) = \alpha_0(x) + \sum_{n=1}^{\infty} \hbar^n \alpha_n(x).$$

- We extend the star product so that it includes differential forms:

$$\alpha * \beta = \alpha\beta + \sum_{n=1}^{\infty} \hbar^n C_n(\alpha,\beta)$$

- We require associativity: $(\alpha * \beta) * \gamma = \alpha * (\beta * \gamma)$.

The Poisson Bracket

- Must generalize the Poisson bracket to act on differential forms $\{\alpha, \beta\}$

The Poisson Bracket

- Must generalize the Poisson bracket to act on differential forms $\{\alpha, \beta\}$

$$\alpha * \beta = \alpha\beta + \hbar\{\alpha, \beta\} + \hbar^2 C_2(\alpha, \beta) + \mathcal{O}(\hbar^3).$$

The Poisson Bracket

- Must generalize the Poisson bracket to act on differential forms $\{\alpha, \beta\}$

$$\alpha * \beta = \alpha\beta + \hbar\{\alpha, \beta\} + \hbar^2 C_2(\alpha, \beta) + \mathcal{O}(\hbar^3).$$

- For functions:
  1. Skew-symmetry: $\{f, g\} = -\{g, f\}$
  2. Jacobi identity: $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$
  3. Leibniz rule: $\{f, gh\} = \{f, g\}h + g\{f, h\}$.

Graded Differential Poisson Algebra [CH: q-alg/9612031]

- Bracket degree:

$$|\{\alpha, \beta\}| = |\alpha| + |\beta| \tag{1}$$

Graded Differential Poisson Algebra [CH: q-alg/9612031]

- Bracket degree:

$$|\{\alpha, \beta\}| = |\alpha| + |\beta| \tag{1}$$

- Graded symmetry:

$$\{\alpha, \beta\} = (-1)^{|\alpha||\beta|+1}\{\beta, \alpha\} \tag{2}$$

Graded Differential Poisson Algebra [CH: q-alg/9612031]

- Bracket degree:

$$|\{\alpha, \beta\}| = |\alpha| + |\beta| \quad (1)$$

- Graded symmetry:

$$\{\alpha, \beta\} = (-1)^{|\alpha||\beta|+1}\{\beta, \alpha\} \quad (2)$$

- Graded product rule:

$$\{\alpha, \beta\gamma\} = \{\alpha, \beta\}\gamma + (-1)^{|\alpha||\beta|}\beta\{\alpha, \gamma\} \quad (3)$$

Graded Differential Poisson Algebra [CH: q-alg/9612031]

- Bracket degree:

$$|\{\alpha, \beta\}| = |\alpha| + |\beta| \tag{1}$$

- Graded symmetry:

$$\{\alpha, \beta\} = (-1)^{|\alpha||\beta|+1}\{\beta, \alpha\} \tag{2}$$

- Graded product rule:

$$\{\alpha, \beta\gamma\} = \{\alpha, \beta\}\gamma + (-1)^{|\alpha||\beta|}\beta\{\alpha, \gamma\} \tag{3}$$

- Leibniz rule for the exterior derivative, d:

$$d\{\alpha, \beta\} = \{d\alpha, \beta\} + (-1)^{|\alpha|}\{\alpha, d\beta\} \tag{4}$$

Graded Differential Poisson Algebra [CH: q-alg/9612031]

- Bracket degree:
$$|\{\alpha, \beta\}| = |\alpha| + |\beta| \tag{1}$$
- Graded symmetry:
$$\{\alpha, \beta\} = (-1)^{|\alpha||\beta|+1}\{\beta, \alpha\} \tag{2}$$
- Graded product rule:
$$\{\alpha, \beta\gamma\} = \{\alpha, \beta\}\gamma + (-1)^{|\alpha||\beta|}\beta\{\alpha, \gamma\} \tag{3}$$
- Leibniz rule for the exterior derivative, *d*:
$$d\{\alpha, \beta\} = \{d\alpha, \beta\} + (-1)^{|\alpha|}\{\alpha, d\beta\} \tag{4}$$
- Graded Jacobi identity:
$$\{\alpha, \{\beta, \gamma\}\} + (-1)^{|\alpha|(|\beta|+|\gamma|)}\{\beta, \{\gamma, \alpha\}\} + (-1)^{|\gamma|(|\alpha|+|\beta|)}\{\gamma, \{\alpha, \beta\}\} = 0 \tag{5}$$

The Poisson Bracket Between a Function and a Form

- $\boxed{\{f,\alpha\} = \nabla_V \alpha \text{ with } V^j = \pi^{ij}\partial_i f}$

The Poisson Bracket Between a Function and a Form

- $\boxed{\{f, \alpha\} = \nabla_V \alpha \text{ with } V^j = \pi^{ij}\partial_i f}$
- Properties of the covariant derivative ∇_V:
  1. $\nabla_V(g) = V(g)$
  2. $\nabla_V(\alpha + \beta) = \nabla_V \alpha + \nabla_V \beta$
  3. $\nabla_{(V+W)}\alpha = \nabla_V \alpha + \nabla_W \alpha$
  4. $\nabla_V(\alpha \wedge \beta) = (\nabla_V \alpha) \wedge \beta + \alpha \wedge (\nabla_V \beta)$
  5. $\nabla_{(gV)}\alpha = g\nabla_V \alpha.$

The Poisson Bracket Between a Function and a Form

- $\boxed{\{f, \alpha\} = \nabla_V \alpha \text{ with } V^j = \pi^{ij}\partial_i f}$
- Properties of the covariant derivative ∇_V:
  1. $\nabla_V(g) = V(g)$
  2. $\nabla_V(\alpha + \beta) = \nabla_V \alpha + \nabla_V \beta$
  3. $\nabla_{(V+W)}\alpha = \nabla_V \alpha + \nabla_W \alpha$
  4. $\nabla_V(\alpha \wedge \beta) = (\nabla_V \alpha) \wedge \beta + \alpha \wedge (\nabla_V \beta)$
  5. $\nabla_{(gV)}\alpha = g\nabla_V \alpha.$

The Poisson Bracket Between a Function and a Form

- $\boxed{\{f, \alpha\} = \nabla_V \alpha \text{ with } V^j = \pi^{ij}\partial_i f}$
- Properties of the covariant derivative ∇_V:
  1. $\nabla_V(g) = V(g)$
  2. $\nabla_V(\alpha + \beta) = \nabla_V\alpha + \nabla_V\beta$
  3. $\nabla_{(V+W)}\alpha = \nabla_V\alpha + \nabla_W\alpha$
  4. $\nabla_V(\alpha \wedge \beta) = (\nabla_V\alpha) \wedge \beta + \alpha \wedge (\nabla_V\beta)$
  5. $\nabla_{(gV)}\alpha = g\nabla_V\alpha.$

The Poisson Bracket Between a Function and a Form

- $\boxed{\{f, \alpha\} = \nabla_V \alpha \text{ with } V^j = \pi^{ij}\partial_i f}$
- Properties of the covariant derivative ∇_V:
  1. $\nabla_V(g) = V(g)$
  2. $\nabla_V(\alpha + \beta) = \nabla_V \alpha + \nabla_V \beta$
  3. $\nabla_{(V+W)}\alpha = \nabla_V \alpha + \nabla_W \alpha$
  4. $\nabla_V(\alpha \wedge \beta) = (\nabla_V \alpha) \wedge \beta + \alpha \wedge (\nabla_V \beta)$
  5. $\nabla_{(gV)}\alpha = g\nabla_V \alpha.$

The Poisson Bracket Between a Function and a Form

- $\boxed{\{f,\alpha\} = \nabla_V \alpha \text{ with } V^j = \pi^{ij}\partial_i f}$
- Properties of the covariant derivative ∇_V:
  1. $\nabla_V(g) = V(g)$
  2. $\nabla_V(\alpha + \beta) = \nabla_V\alpha + \nabla_V\beta$
  3. $\nabla_{(V+W)}\alpha = \nabla_V\alpha + \nabla_W\alpha$
  4. $\nabla_V(\alpha \wedge \beta) = (\nabla_V\alpha) \wedge \beta + \alpha \wedge (\nabla_V\beta)$
  5. $\nabla_{(gV)}\alpha = g\nabla_V\alpha.$
- $\{f,\alpha\} = \partial_i f\{x^i,\alpha\}$

The Poisson Bracket Between a Function and a Form

- $\boxed{\{f, \alpha\} = \nabla_V \alpha \text{ with } V^j = \pi^{ij}\partial_i f}$
- Properties of the covariant derivative ∇_V:
  1. $\nabla_V(g) = V(g)$
  2. $\nabla_V(\alpha + \beta) = \nabla_V\alpha + \nabla_V\beta$
  3. $\nabla_{(V+W)}\alpha = \nabla_V\alpha + \nabla_W\alpha$
  4. $\nabla_V(\alpha \wedge \beta) = (\nabla_V\alpha) \wedge \beta + \alpha \wedge (\nabla_V\beta)$
  5. $\nabla_{(gV)}\alpha = g\nabla_V\alpha.$
- $\{f, \alpha\} = \partial_i f\{x^i, \alpha\} = \omega_{ij}V^i\{x^j, \alpha\}$

The Poisson Bracket Between a Function and a Form

- $\boxed{\{f, \alpha\} = \nabla_V \alpha \text{ with } V^j = \pi^{ij}\partial_i f}$
- Properties of the covariant derivative ∇_V:
  1. $\nabla_V(g) = V(g)$
  2. $\nabla_V(\alpha + \beta) = \nabla_V\alpha + \nabla_V\beta$
  3. $\nabla_{(V+W)}\alpha = \nabla_V\alpha + \nabla_W\alpha$
  4. $\nabla_V(\alpha \wedge \beta) = (\nabla_V\alpha) \wedge \beta + \alpha \wedge (\nabla_V\beta)$
  5. $\nabla_{(gV)}\alpha = g\nabla_V\alpha$.
- $\{f, \alpha\} = \partial_i f\{x^i, \alpha\} = \omega_{ij}V^i\{x^j, \alpha\}$
- $\nabla_{(gV)}\alpha = \omega_{ij}(gV^i)\{x^j, \alpha\}$

The Poisson Bracket Between a Function and a Form

- $\boxed{\{f, \alpha\} = \nabla_V \alpha \text{ with } V^j = \pi^{ij}\partial_i f}$
- Properties of the covariant derivative ∇_V:
  1. $\nabla_V(g) = V(g)$
  2. $\nabla_V(\alpha + \beta) = \nabla_V\alpha + \nabla_V\beta$
  3. $\nabla_{(V+W)}\alpha = \nabla_V\alpha + \nabla_W\alpha$
  4. $\nabla_V(\alpha \wedge \beta) = (\nabla_V\alpha) \wedge \beta + \alpha \wedge (\nabla_V\beta)$
  5. $\nabla_{(gV)}\alpha = g\nabla_V\alpha.$
- $\{f, \alpha\} = \partial_i f\{x^i, \alpha\} = \omega_{ij}V^i\{x^j, \alpha\}$
- $\nabla_{(gV)}\alpha = \omega_{ij}(gV^i)\{x^j, \alpha\} = g(\omega_{ij}(V^i)\{x^j, \alpha\}) = g\nabla_V\alpha$

The Poisson Bracket Between a Function and a Form

- Some notation:

$$\nabla_i dx^k = -\Gamma^k_{ij} dx^j, \qquad \text{while} \qquad \nabla_i \alpha_k = \partial_i \alpha_k.$$

The Poisson Bracket Between a Function and a Form

- Some notation:

$$\nabla_i dx^k = -\Gamma^k_{ij} dx^j, \qquad \text{while} \qquad \nabla_i \alpha_k = \partial_i \alpha_k.$$

$$\nabla_i(\alpha_k dx^k) = (\partial_i \alpha_k) dx^k + \alpha_k(-\Gamma^k_{ij} dx^j) = (\partial_i \alpha_j - \Gamma^k_{ij} \alpha_k) dx^j.$$

The Poisson Bracket Between a Function and a Form

- Some notation:

$$\nabla_i dx^k = -\Gamma^k_{ij} dx^j, \qquad \text{while} \qquad \nabla_i \alpha_k = \partial_i \alpha_k.$$

$$\nabla_i(\alpha_k dx^k) = (\partial_i \alpha_k) dx^k + \alpha_k(-\Gamma^k_{ij} dx^j) = (\partial_i \alpha_j - \Gamma^k_{ij} \alpha_k) dx^j.$$

- In general, Γ^k_{ij} has torsion:

$$\nabla_i dx^k = -\Gamma^k_{ij} dx^j \qquad \widetilde{\nabla}_i dx^k = -\Gamma^k_{ji} dx^j.$$

The Poisson Bracket Between Arbitrary Forms

- $$\boxed{\{\alpha, \beta\} = \pi^{mn}\nabla_m\alpha\nabla_n\beta + (-1)^{|\alpha|}\widetilde{R}^{mn}(i_m\alpha)(i_n\beta)}$$

The Poisson Bracket Between Arbitrary Forms

- $$\boxed{\{\alpha, \beta\} = \pi^{mn}\nabla_m\alpha\nabla_n\beta + (-1)^{|\alpha|}\widetilde{R}^{mn}(i_m\alpha)(i_n\beta)}$$
- Definitions:

$$\widetilde{R}^{ij} \equiv \frac{1}{2}\pi^{im}\widetilde{R}^{j}_{mab}dx^a dx^b$$

$$[\widetilde{\nabla}_a, \widetilde{\nabla}_b]dx^i = \widetilde{R}^i_{mab}dx^m = \left(\partial_a\Gamma^i_{mb} - \partial_b\Gamma^i_{ma} + \Gamma^i_{\ell a}\Gamma^\ell_{mb} - \Gamma^i_{\ell b}\Gamma^\ell_{ma}\right) dx^m$$

The Poisson Bracket Between Arbitrary Forms

- $$\boxed{\{\alpha, \beta\} = \pi^{mn}\nabla_m\alpha\nabla_n\beta + (-1)^{|\alpha|}\widetilde{R}^{mn}(i_m\alpha)(i_n\beta)}$$
- Definitions:

$$\widetilde{R}^{ij} \equiv \frac{1}{2}\pi^{im}\widetilde{R}^{j}_{mab}dx^a dx^b$$
$$[\widetilde{\nabla}_a, \widetilde{\nabla}_b]dx^i = \widetilde{R}^{i}_{mab}dx^m = \left(\partial_a\Gamma^i_{mb} - \partial_b\Gamma^i_{ma} + \Gamma^i_{\ell a}\Gamma^\ell_{mb} - \Gamma^i_{\ell b}\Gamma^\ell_{ma}\right) dx^m$$

- $i_m = i_{\frac{\partial}{\partial x^m}}$ is the interior product. It is an anti-derivation. It's action on a form $\alpha = \frac{1}{p!}\alpha_{i_1 \ldots i_p} dx^{i_1} \ldots dx^{i_p}$ is:

The Poisson Bracket Between Arbitrary Forms

- $$\boxed{\{\alpha,\beta\} = \pi^{mn}\nabla_m\alpha\nabla_n\beta + (-1)^{|\alpha|}\widetilde{R}^{mn}(i_m\alpha)(i_n\beta)}$$
- Definitions:

$$\widetilde{R}^{ij} \equiv \frac{1}{2}\pi^{im}\widetilde{R}^j_{mab}dx^a dx^b$$
$$[\widetilde{\nabla}_a,\widetilde{\nabla}_b]dx^i = \widetilde{R}^i_{mab}dx^m = \left(\partial_a\Gamma^i_{mb} - \partial_b\Gamma^i_{ma} + \Gamma^i_{\ell a}\Gamma^\ell_{mb} - \Gamma^i_{\ell b}\Gamma^\ell_{ma}\right)dx^m$$

- $i_m = i_{\frac{\partial}{\partial x^m}}$ is the interior product. It is an anti-derivation. It's action on a form $\alpha = \frac{1}{p!}\alpha_{i_1\ldots i_p}dx^{i_1}\ldots dx^{i_p}$ is:

$$i_m\alpha = \frac{1}{(p-1)!}\alpha_{mi_2\ldots i_p}dx^{i_2}\ldots dx^{i_p}.$$

The Poisson Bracket Between Arbitrary Forms

- $$\boxed{\{\alpha, \beta\} = \pi^{mn}\nabla_m\alpha\nabla_n\beta + (-1)^{|\alpha|}\widetilde{R}^{mn}(i_m\alpha)(i_n\beta)}$$
- Definitions:

$$\widetilde{R}^{ij} \equiv \frac{1}{2}\pi^{im}\widetilde{R}^j_{mab}dx^a dx^b$$
$$[\widetilde{\nabla}_a, \widetilde{\nabla}_b]dx^i = \widetilde{R}^i_{mab}dx^m = \left(\partial_a\Gamma^i_{mb} - \partial_b\Gamma^i_{ma} + \Gamma^i_{\ell a}\Gamma^\ell_{mb} - \Gamma^i_{\ell b}\Gamma^\ell_{ma}\right) dx^m$$

- $i_m = i_{\frac{\partial}{\partial x^m}}$ is the interior product. It is an anti-derivation. It's action on a form $\alpha = \frac{1}{p!}\alpha_{i_1\ldots i_p}dx^{i_1}\ldots dx^{i_p}$ is:

$$i_m\alpha = \frac{1}{(p-1)!}\alpha_{mi_2\ldots i_p}dx^{i_2}\ldots dx^{i_p}.$$

On a basis 1-form:

$$i_m dx^k = \delta^k_m.$$

Conditions on Γ^i_{jk} and π^{ij}:

- Leibniz rule: $\widetilde{\nabla}_\ell$ is symplectic:

$$\widetilde{\nabla}_\ell \pi^{ij} = 0 \tag{6}$$

Conditions on Γ^i_{jk} and π^{ij}:

- Leibniz rule: $\widetilde{\nabla}_\ell$ is symplectic:

$$\widetilde{\nabla}_\ell \pi^{ij} = 0 \tag{6}$$

- Jacobi identity: π^{ij} satisfies the Jacobi Identity:

$$(\pi^{ab}\partial_b \pi^{mn} + \pi^{mb}\partial_b \pi^{na} + \pi^{nb}\partial_b \pi^{am}) = 0 \tag{7}$$

Conditions on Γ^i_{jk} and π^{ij}:

- Leibniz rule: $\widetilde{\nabla}_\ell$ is symplectic:

$$\widetilde{\nabla}_\ell \pi^{ij} = 0 \qquad (6)$$

- Jacobi identity: π^{ij} satisfies the Jacobi Identity:

$$(\pi^{ab}\partial_b \pi^{mn} + \pi^{mb}\partial_b \pi^{na} + \pi^{nb}\partial_b \pi^{am}) = 0 \qquad (7)$$

- Jacobi identity: ∇_i has vanishing curvature:

$$[\nabla_m, \nabla_n]\alpha = 0 \qquad (8)$$

Conditions on Γ^i_{jk} and π^{ij}:

- Leibniz rule: $\widetilde{\nabla}_\ell$ is symplectic:

$$\widetilde{\nabla}_\ell \pi^{ij} = 0 \tag{6}$$

- Jacobi identity: π^{ij} satisfies the Jacobi Identity:

$$(\pi^{ab}\partial_b\pi^{mn} + \pi^{mb}\partial_b\pi^{na} + \pi^{nb}\partial_b\pi^{am}) = 0 \tag{7}$$

- Jacobi identity: ∇_i has vanishing curvature:

$$[\nabla_m, \nabla_n]\alpha = 0 \tag{8}$$

- Jacobi identity: $\widetilde{R}^{mn}$ is covariantly constant under ∇_a:

$$(\nabla_a \widetilde{R})^{mn} = 0 \tag{9}$$

Conditions on Γ^i_{jk} and π^{ij}:

- Leibniz rule: $\widetilde{\nabla}_\ell$ is symplectic:

$$\widetilde{\nabla}_\ell \pi^{ij} = 0 \tag{6}$$

- Jacobi identity: π^{ij} satisfies the Jacobi Identity:

$$(\pi^{ab}\partial_b \pi^{mn} + \pi^{mb}\partial_b \pi^{na} + \pi^{nb}\partial_b \pi^{am}) = 0 \tag{7}$$

- Jacobi identity: ∇_i has vanishing curvature:

$$[\nabla_m, \nabla_n]\alpha = 0 \tag{8}$$

- Jacobi identity: $\widetilde{R}^{mn}$ is covariantly constant under ∇_a:

$$(\nabla_a \widetilde{R})^{mn} = 0 \tag{9}$$

- Jacobi identity: $\widetilde{R}^{mn}$ satisfies

$$\widetilde{R}^{ab}(i_b \widetilde{R}^{mn}) + \widetilde{R}^{mb}(i_b \widetilde{R}^{na}) + \widetilde{R}^{nb}(i_b \widetilde{R}^{am}) = 0 \tag{10}$$

Conditions on Γ^i_{jk} and π^{ij}:

- Leibniz rule: $\widetilde{\nabla}_\ell$ is symplectic:

$$\widetilde{\nabla}_\ell \pi^{ij} = 0 \tag{6}$$

- Jacobi identity: π^{ij} satisfies the Jacobi Identity:

$$(\pi^{ab}\partial_b \pi^{mn} + \pi^{mb}\partial_b \pi^{na} + \pi^{nb}\partial_b \pi^{am}) = 0 \tag{7}$$

- Jacobi identity: ∇_i has vanishing curvature:

$$[\nabla_m, \nabla_n]\alpha = 0 \tag{8}$$

- Jacobi identity: $\widetilde{R}^{mn}$ is covariantly constant under ∇_a:

$$(\nabla_a \widetilde{R})^{mn} = 0 \tag{9}$$

- Jacobi identity: $\widetilde{R}^{mn}$ satisfies

$$\widetilde{R}^{ab}(i_b \widetilde{R}^{mn}) + \widetilde{R}^{mb}(i_b \widetilde{R}^{na}) + \widetilde{R}^{nb}(i_b \widetilde{R}^{am}) = 0 \tag{10}$$

Conditions on Γ^i_{jk} and π^{ij}:

- Consider:

$$d\left(\{x^i,\{dx^j,dx^k\}\}+\{dx^j,\{dx^k,x^i\}\}-\{dx^k,\{x^i,dx^j\}\}\right)$$
$$=\{dx^i,\{dx^j,dx^k\}\}+\{dx^j,\{dx^k,dx^i\}\}-\{dx^k,\{dx^i,dx^j\}\}+\dots$$

Conditions on Γ^i_{jk} and π^{ij}:

- This gives:

$$
\begin{aligned}
&\widetilde{R}^{ib}\left(i_b\widetilde{R}^{jk}\right)+\widetilde{R}^{jb}\left(i_b\widetilde{R}^{ki}\right)+\widetilde{R}^{kb}\left(i_b\widetilde{R}^{ij}\right)=\\
&\quad-\left(\pi^{ab}\partial_b\pi^{mn}+\pi^{mb}\partial_b\pi^{na}+\pi^{nb}\partial_b\pi^{am}\right)\nabla_a dx^i\nabla_m dx^j\nabla_n dx^k\\
&\quad-\pi^{ab}\pi^{mn}\big([\nabla_a,\nabla_m]dx^i\nabla_b dx^j\nabla_n dx^k\\
&\qquad+\nabla_n dx^i[\nabla_a,\nabla_m]dx^j\nabla_b dx^k+\nabla_b dx^i\nabla_n dx^k[\nabla_a,\nabla_m]dx^k\big)\\
&\quad+\pi^{ab}\left(\nabla_b\widetilde{R}\right)^{jk}\nabla_a dx^i+\pi^{ab}\left(\nabla_b\widetilde{R}\right)^{ki}\nabla_a dx^j+\pi^{ab}\left(\nabla_b\widetilde{R}\right)^{ij}\nabla_a dx^k\\
&\quad+d\big(\left(\pi^{ib}\partial_b\pi^{mn}+\pi^{mb}\partial_b\pi^{ni}+\pi^{nb}\partial_b\pi^{im}\right)\nabla_m dx^j\nabla_n dx^k\big)\\
&\quad+d\big(\pi^{ab}\pi^{mi}\left([\nabla_a,\nabla_m]dx^i\nabla_b dx^k+\nabla_b dx^j[\nabla_a,\nabla_m]dx^k\right)\big)\\
&\quad-d\big(\pi^{ib}\left(\nabla_b\widetilde{R}\right)^{jk}\big)+\dots
\end{aligned}
$$

The Star Product to $\mathcal{O}(\hbar^2)$:

- Associativity at $\mathcal{O}(\hbar^2)$ gives the expression:

$$f\,C_2(\alpha,\beta) - C_2(f\alpha,\beta) + C_2(f,\alpha\beta) - C_2(f,\alpha)\beta = \{\{f,\alpha\},\beta\} - \{f,\{\alpha,\beta\}\}$$

The Star Product to $\mathcal{O}(\hbar^2)$:

- Associativity at $\mathcal{O}(\hbar^2)$ gives the expression:

  $$f C_2(\alpha, \beta) - C_2(f\alpha, \beta) + C_2(f, \alpha\beta) - C_2(f, \alpha)\beta = \{\{f, \alpha\}, \beta\} - \{f, \{\alpha, \beta\}\}$$

  - Calculable using $\{,\}$.

The Star Product to $\mathcal{O}(\hbar^2)$:

- Associativity at $\mathcal{O}(\hbar^2)$ gives the expression:

$$f C_2(\alpha,\beta) - C_2(f\alpha,\beta) + C_2(f,\alpha\beta) - C_2(f,\alpha)\beta = \{\{f,\alpha\},\beta\} - \{f,\{\alpha,\beta\}\}$$

  - Calculable using $\{,\}$.
  - $\partial \rightarrow \nabla$ in star product of functions:

$$C_2(f,\bullet) = \frac{1}{2}\pi^{ij}\pi^{mn}\partial_i\partial_m f \nabla_j \nabla_n \bullet + \frac{1}{3}\pi^{ia}\partial_a \pi^{mn} \left(\partial_i \partial_m f \nabla_n \bullet - \partial_m f \nabla_i \nabla_n \bullet\right)$$

The Star Product to $\mathcal{O}(\hbar^2)$:

- Associativity at $\mathcal{O}(\hbar^2)$ gives the expression:

$$f\,C_2(\alpha,\beta) - C_2(f\alpha,\beta) + C_2(f,\alpha\beta) - C_2(f,\alpha)\beta = \{\{f,\alpha\},\beta\} - \{f,\{\alpha,\beta\}\}$$

  - Calculable using $\{,\}$.
  - $\partial \to \nabla$ in star product of functions:

$$C_2(f,\bullet) = \frac{1}{2}\pi^{ij}\pi^{mn}\partial_i\partial_m f\nabla_j\nabla_n\bullet + \frac{1}{3}\pi^{ia}\partial_a\pi^{mn}\left(\partial_i\partial_m f\nabla_n\bullet - \partial_m f\nabla_i\nabla_n\bullet\right)$$

- $C_2(\alpha,\beta) = C_2^{\nabla}(\alpha,\beta) + C_2^{\widetilde{R}}(\alpha,\beta)$

The Star Product to $\mathcal{O}(\hbar^2)$:

- Associativity at $\mathcal{O}(\hbar^2)$ gives the expression:

$$f C_2(\alpha,\beta) - C_2(f\alpha,\beta) + C_2(f,\alpha\beta) - C_2(f,\alpha)\beta = \{\{f,\alpha\},\beta\} - \{f,\{\alpha,\beta\}\}$$

  - Calculable using $\{,\}$.
  - $\partial \to \nabla$ in star product of functions:

$$C_2(f,\bullet) = \frac{1}{2}\pi^{ij}\pi^{mn}\partial_i\partial_m f \nabla_j\nabla_n \bullet + \frac{1}{3}\pi^{ia}\partial_a\pi^{mn}\left(\partial_i\partial_m f \nabla_n \bullet - \partial_m f \nabla_i\nabla_n \bullet\right)$$

- $C_2(\alpha,\beta) = C_2^{\nabla}(\alpha,\beta) + C_2^{\widetilde{R}}(\alpha,\beta)$

$$C_2^{\nabla}(\alpha,\beta) = \frac{1}{2}\pi^{ij}\pi^{mn}\nabla_i\nabla_m\alpha\nabla_j\nabla_n\beta + \frac{1}{3}\pi^{ia}\partial_a\pi^{mn}\left(\nabla_i\nabla_m\alpha\nabla_n\beta - \nabla_m\alpha\nabla_i\nabla_n\beta\right)$$

The Star Product to $\mathcal{O}(\hbar^2)$:

- In full glory,

$$\begin{aligned}
\alpha * \beta = \alpha\beta + \hbar\Big(\pi^{mn}\nabla_m\alpha\nabla_n\beta + (-1)^{|\alpha|}\widetilde{R}^{mn}(i_m\alpha)(i_n\beta)\Big) \\
+ \hbar^2\Big(\frac{1}{2}\pi^{ij}\pi^{mn}\nabla_i\nabla_m\alpha\nabla_j\nabla_n\beta \\
+ \frac{1}{3}\pi^{ia}\partial_a\pi^{mn}\left(\nabla_i\nabla_m\alpha\nabla_n\beta - \nabla_m\alpha\nabla_i\nabla_n\beta\right) \\
- \frac{1}{2}\widetilde{R}^{ij}\widetilde{R}^{mn}\left[(i_i i_m\alpha)(i_j i_n\beta)\right] \\
- \frac{1}{3}\widetilde{R}^{i\ell}(i_\ell\widetilde{R}^{mn})\Big[(-1)^{|\alpha|}(i_i i_m\alpha)(i_n\beta) + (i_m\alpha)(i_i i_n\beta)\Big] \\
+ (-1)^{|\alpha|}\pi^{ij}\widetilde{R}^{mn}(i_m\nabla_i\alpha)(i_n\nabla_j\beta)\Big) + \mathcal{O}(\hbar^3).
\end{aligned}$$

- In full glory,

$$
\begin{aligned}
\alpha * \beta = \alpha\beta + \hbar\Big(&\pi^{mn}\nabla_m\alpha\nabla_n\beta + (-1)^{|\alpha|}\widetilde{R}^{mn}(i_m\alpha)(i_n\beta)\Big) \\
+ \hbar^2\Big(&\frac{1}{2}\pi^{ij}\pi^{mn}\nabla_i\nabla_m\alpha\nabla_j\nabla_n\beta \\
&+ \frac{1}{3}\pi^{ia}\partial_a\pi^{mn}\left(\nabla_i\nabla_m\alpha\nabla_n\beta - \nabla_m\alpha\nabla_i\nabla_n\beta\right) \\
&- \frac{1}{2}\widetilde{R}^{ij}\widetilde{R}^{mn}\left[(i_i i_m\alpha)(i_j i_n\beta)\right] \\
&- \frac{1}{3}\widetilde{R}^{i\ell}(i_\ell\widetilde{R}^{mn})\Big[(-1)^{|\alpha|}(i_i i_m\alpha)(i_n\beta) + (i_m\alpha)(i_i i_n\beta)\Big] \\
&+ (-1)^{|\alpha|}\pi^{ij}\widetilde{R}^{mn}(i_m\nabla_i\alpha)(i_n\nabla_j\beta)\Big) + \mathcal{O}(\hbar^3).
\end{aligned}
$$

The Star Product to $\mathcal{O}(\hbar^2)$:

- In full glory,

$$\begin{aligned}\alpha * \beta = \alpha\beta + \hbar\Big(&\pi^{mn}\nabla_m\alpha\nabla_n\beta + (-1)^{|\alpha|}\widetilde{R}^{mn}(i_m\alpha)(i_n\beta)\Big)\\
+ \hbar^2\Big(&\frac{1}{2}\pi^{ij}\pi^{mn}\nabla_i\nabla_m\alpha\nabla_j\nabla_n\beta\\
&+\frac{1}{3}\pi^{ia}\partial_a\pi^{mn}\left(\nabla_i\nabla_m\alpha\nabla_n\beta - \nabla_m\alpha\nabla_i\nabla_n\beta\right)\\
&-\frac{1}{2}\widetilde{R}^{ij}\widetilde{R}^{mn}\left[(i_i i_m\alpha)(i_j i_n\beta)\right]\\
&-\frac{1}{3}\widetilde{R}^{i\ell}(i_\ell\widetilde{R}^{mn})\Big[(-1)^{|\alpha|}(i_i i_m\alpha)(i_n\beta) + (i_m\alpha)(i_i i_n\beta)\Big]\\
&+(-1)^{|\alpha|}\pi^{ij}\widetilde{R}^{mn}(i_m\nabla_i\alpha)(i_n\nabla_j\beta)\Big) + \mathcal{O}(\hbar^3).\end{aligned}$$

Future Directions

- It should be possible (although tedious) to find the star product between differential forms to $\mathcal{O}(\hbar^3)$.

Future Directions

- It should be possible (although tedious) to find the star product between differential forms to $\mathcal{O}(\hbar^3)$.
- It may also be possible to generalize the star product to Poisson manifolds, where π^{ij} is not invertible.

Future Directions

- It should be possible (although tedious) to find the star product between differential forms to $\mathcal{O}(\hbar^3)$.
- It may also be possible to generalize the star product to Poisson manifolds, where π^{ij} is not invertible.
- Apply to physics: study gauge theories on noncommutative spaces, generalize the Seiberg-Witten map.

Future Directions

- Thank you.