2-Groups and their Representations

Derek Wise

Based on joint work with John Baez, Aristide Baratin, and Laurent Freidel

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2-Groups

From Groups to 2-Groups: The rough idea

An ordinary group G can be thought of as a category with one object $\star$, morphisms labeled by elements of G, and composition defined by multiplication in G:

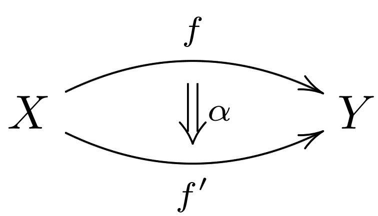
$$\star \xrightarrow{g_1} \star \xrightarrow{g_2} \star \quad = \quad \star \xrightarrow{g_2 g_1} \star$$

In fact, a **group** may be defined to be a category with a unique object and all morphisms invertible.

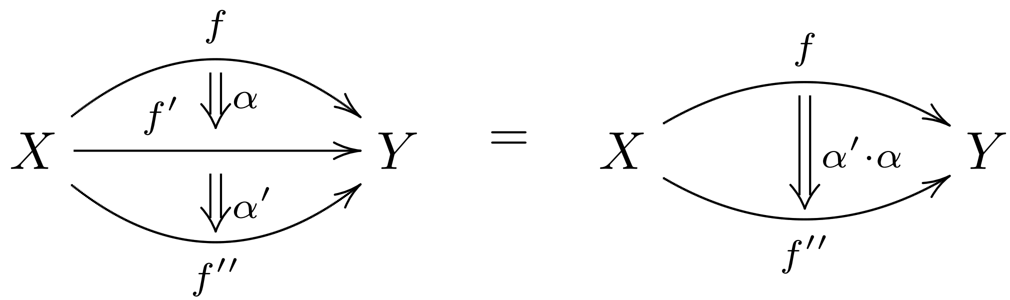
Likewise: a **2-group** is a "2-category with a unique object, all morphisms invertible and all 2-morphisms invertible".

What does this mean? A "2-category" consists of:

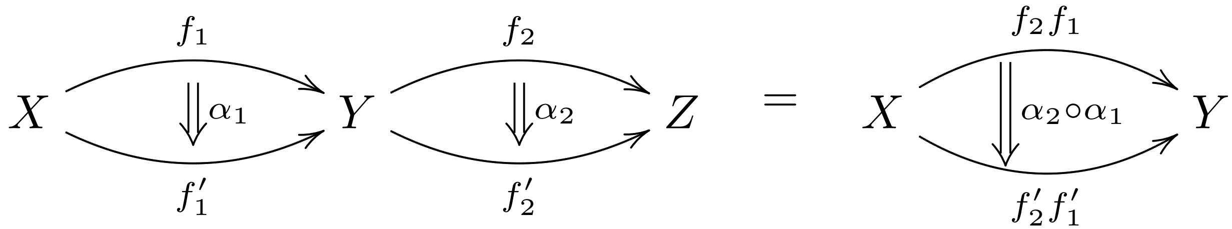
- objects: $X, Y, Z, \ldots$
- morphisms: $X \xrightarrow{f} Y$
- 2-morphisms:



Morphisms can be composed, as in a category, and 2-morphisms can be composed in two distinct ways: vertically:



and horizontally:



And there are some properties to satisfy, but we'll skip those....

So, very roughly, a **2-group** is a gadget with one 'object', invertible 'arrows', and invertible '2-arrows' that can be composed as above.

There are actually various versions of this idea:

- If the objects and morphisms form an honest group, the 2-group is **strict**
- If associativity, unit laws, invertibility hold only *up to 2-isomorphism*, it's **weak**

But, to understand this talk, you only need a rough idea of 2-groups!

Why 2-groups?

Some answers from physics and topology:

- Homotopy theory; "fundamental 2-groups"
- Nonabelian generalization of "2-form electromagnetism"; parallel transport of strings.
- State sum models

Some answers from category theory:

- A *set* has a *group* of automorphisms, but...
- a *category* has a *2-group* of autoequivalences.

  (Likewise if we replace 'set' with 'space', 'manifold', etc., and replace 'category with 'topological category', 'smooth category', etc.)

  Special case: If G is a group, $\mathrm{Aut}(G)$ deserves to be a (strict) 2-group, though people often ignore the 2-morphisms and pretend it is a group.

Group Representations and 2-group Representations

A linear **representation** of a group is a functor $\rho\colon G \to \mathrm{Vect}$ from G to the category of vector spaces. Why? Such a functor gives:

$$\begin{array}{rl} \rho(\star) = V & \text{a vector space} \\ \rho(g)\colon V \to V & \text{a linear operator for each } g \in G \end{array}$$

Saying that ρ is a functor just means that it preserves identities and composition:

$$\rho(1) = 1_V$$
$$\rho(gh) = \rho(g)\rho(h)$$

Similarly, an **intertwiner** between representations ρ_1, ρ_2 is a natural transformation: a linear operator $\phi = \phi(\star)$ such that

$$\begin{array}{ccc} V_1 & \xrightarrow{\rho_1(g)} & V_1 \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \phi} \\ V_2 & \xrightarrow[\rho_2(g)]{} & V_2 \end{array}$$

commutes.

It's easy to generalize the idea of representation theory, replacing the category Vect with any other category C we wish...

- **Representations** in C are functors $\rho\colon G \to C$.
- '**Intertwiners**' between reps are natural transformations.

Let's generalize this story to 2-groups! From now on, I'll consider only *strict* 2-groups, where associativity, unit and inverse laws for morphisms hold 'on the nose'.

A sketch of the representation theory of a 2-group $\mathcal{G}$ in a (strict) 2-category $\mathcal{C}$ is then:

- **Representations** are (strict) '2-functors' $\rho\colon \mathcal{G} \to \mathcal{C}$.
- **Intertwiners** are 'pseudonatural transformations'.

But now there are also '2-intertwiners' between intertwiners:

- **2-intertwiners** are 'modifications'.

Let's explain what these terms mean, and *then* discuss good candidates for the generic target 2-category $\mathcal{C}$.

Representations

A **representation** is a (strict) 2-functor $\rho\colon \mathcal{G} \to \mathcal{C}$. This gives:

- an object $\rho(\star) = V$ of $\mathcal{C}$
- for each morphism $g \in G$, a morphism in $\mathcal{C}$ from V to itself:

$$V \xrightarrow{\rho(g)} V$$

- for each 2-morphism $u\colon g \Rightarrow g'$, a 2-morphism in $\mathcal{C}$

$$V \underset{\rho(g')}{\overset{\rho(g)}{\rightrightarrows}} V \qquad \Downarrow \rho(u)$$

To be a 2-functor, this must preserve the relevant structure:

- for all morphisms g, g': $\quad \rho(g'g) = \rho(g')\,\rho(g)$
- for all vertically composable 2-morphisms u and u':

$$\rho(u' \cdot u) = \rho(u') \cdot \rho(u)$$

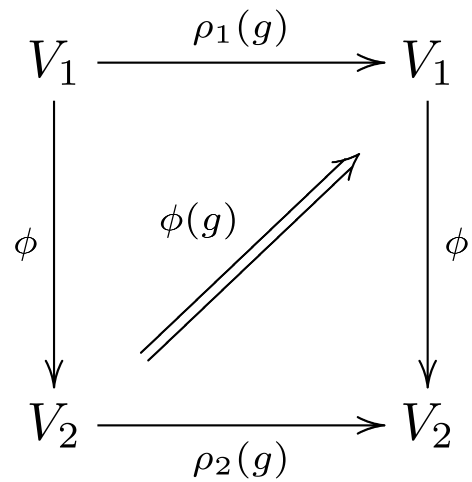
- for all 2-morphisms u, u':

$$\rho(u' \circ u) = \rho(u') \circ \rho(u)$$

(Since $\mathcal{G}$ is a 2-group, preserving identities follows from preserving composition.)

Intertwiners

An **intertwiner** $\phi\colon \rho_1 \to \rho_2$ is a 'pseudonatural transformation'—its naturality squares commute only *up to specified 2-isomorphism*:

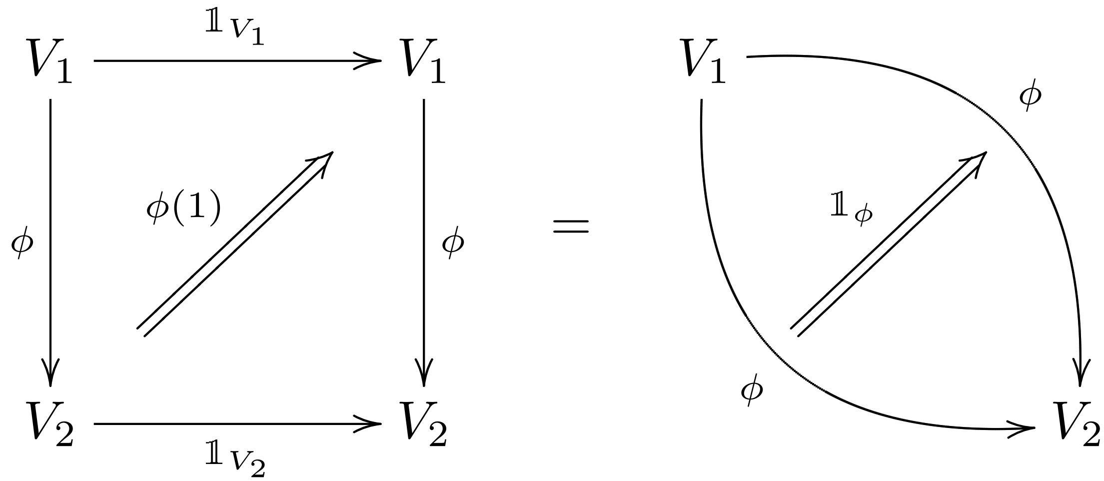


So: an intertwiner $\phi\colon \rho_1 \to \rho_2$ consists of:

- a morphism $\phi\colon V_1 \to V_2$
- for each $g \in G$, a 2-isomorphism $\phi(g)\colon \rho_2(g)\,\phi \xrightarrow{\sim} \phi\,\rho_1(g)$

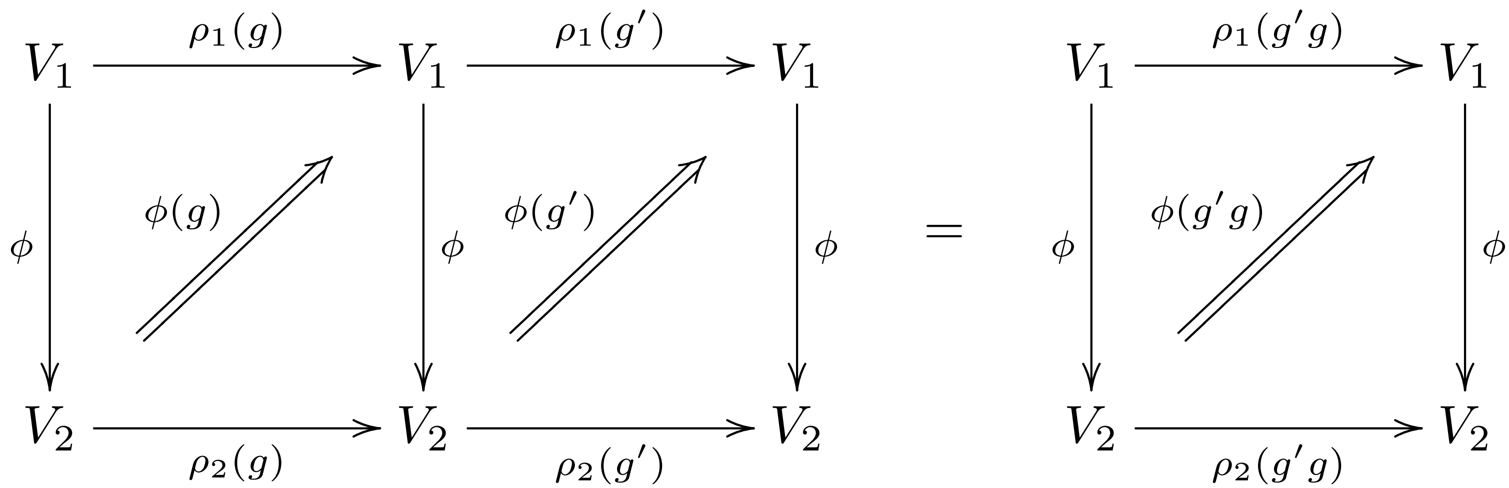
To be 'pseudonatural', these data must satisfy some 'coherence laws':

- Compatibility with identities:



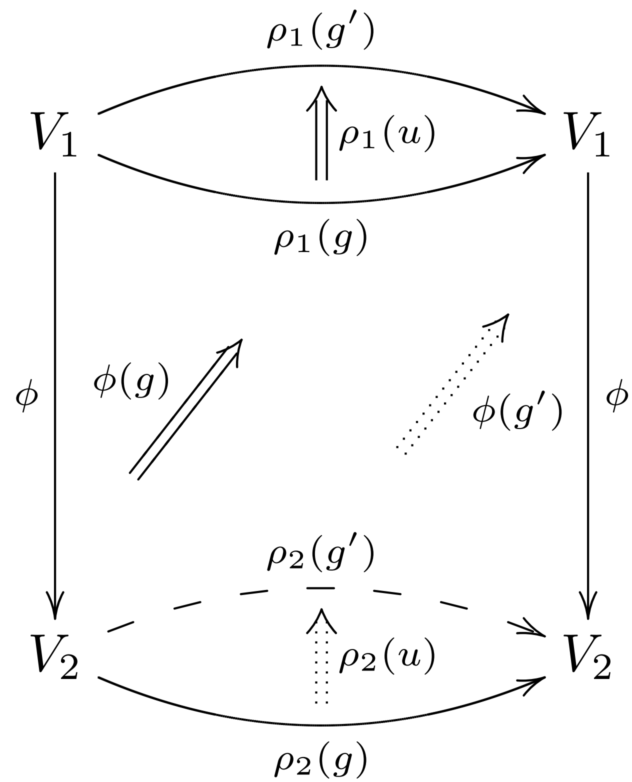
i.e. $\phi(1) = \mathbb{1}_\phi$.

- Compatibility with composition of morphisms in G:



i.e. $$\left[\phi(g') \circ \mathbb{1}_{\rho_1(g)}\right] \cdot \left[\mathbb{1}_{\rho_2(g')} \circ \phi(g)\right] = \phi(g'g)$$

- Finally, ϕ should satisfy a higher-dimensional analogue of 'naturality': For each 2-morphism $u\colon g \Rightarrow g'$ in the 2-group we demand that the "pillow" diagram:

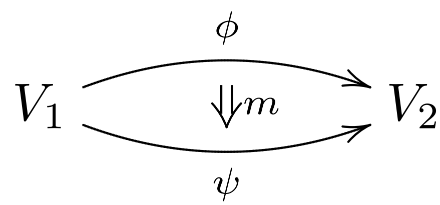


commutes. I.e.:

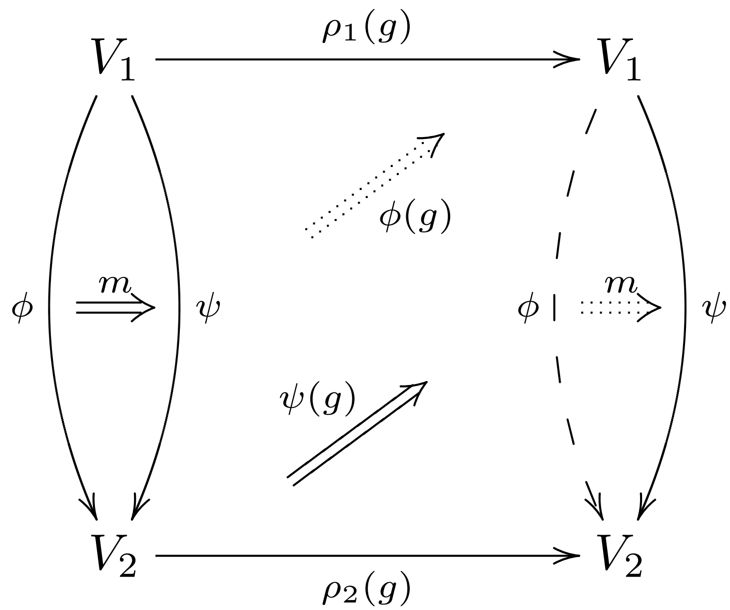
$$\left[\mathbb{1}_\phi \circ \rho_1(u)\right] \cdot \phi(g) = \phi(g') \cdot \left[\rho_2(u) \circ \mathbb{1}_\phi\right]$$

2-Intertwiners

A **2-intertwiner** $m\colon \phi \Rightarrow \psi$ is a 'modification' between pseudonatural transformations. This consists of a 2-morphism in $\mathbb{C}$:



in $\mathcal{C}$ such that this diagram commutes:



I.e. $$\psi(g) \cdot \left[\mathbb{1}_{\rho_2(g)} \circ m\right] = \left[m \circ \mathbb{1}_{\rho_1(g)}\right] \cdot \phi(g)$$

We've defined:

- representations of a 2-group,
- intertwiners,
- 2-intertwiners,

in an arbitrary 2-category $\mathcal{C}$.

Problem: *What's a good target 2-category $\mathcal{C}$ to use?*

2-Linear Algebra

Kapranov–Voevodsky 2-vector spaces

Kapranov and Voevodsky invented certain "2-vector spaces"—categories analogous to vector spaces—by replacing the "ground field" $\mathbb{C}$ by the category Vect of finite-dimensional complex vector spaces, and exploiting this analogy:

linear algebra	2-linear algebra
$\mathbb{C}$	Vect
$+$	$\oplus$
$\times$	$\otimes$
0	$\{0\}$
1	$\mathbb{C}$

So: just as every finite-dim. vector space is isomorphic to some $\mathbb{C}^N$, every finite-dim. **2-vector space** is equivalent to some Vect^N.
An *element* of $\mathbb{C}^N$ (a **vector**) is an N-tuple of numbers;
An *object* of Vect^N (a **2-vector**) is an N-tuple of vector spaces.
There are also morphisms between 2-vectors: N-tuples of linear maps!

Next:

A linear map $T\colon \mathbb{C}^M \to \mathbb{C}^N$ is equal to one given by a $N \times M$ matrix of numbers, $T_{n,m}$, and composition of maps is accomplished by matrix multiplication:

$$(UT)_{k,m} = \sum_{n=1}^{N} U_{k,n} T_{n,m}$$

So, likewise:

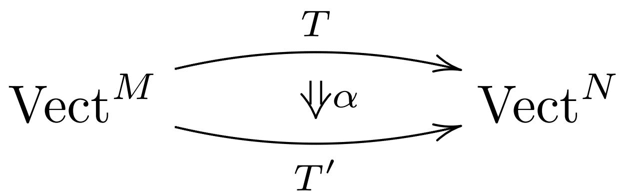
A linear map $T\colon \mathrm{Vect}^M \to \mathrm{Vect}^N$ is isomorphic to one given by an $N \times M$ matrix of vector spaces, $T_{n,m}$, and composition of maps is accomplished by matrix multiplication:

$$(UT)_{k,m} = \bigoplus_{n=1}^{N} U_{k,n} \otimes T_{n,m}$$

for $T\colon \mathrm{Vect}^M \to \mathrm{Vect}^N$ and $U\colon \mathrm{Vect}^N \to \mathrm{Vect}^K$.

Most importantly, we have a new layer of structure not present in ordinary linear algebra:

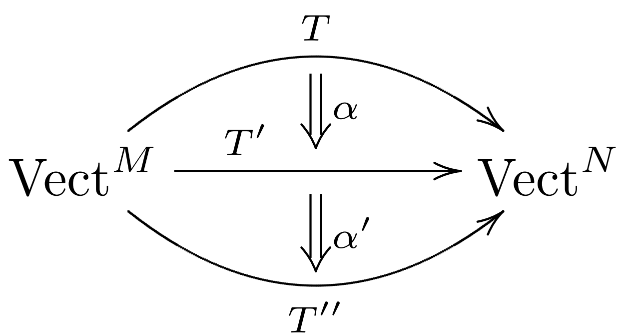
Given 'matrices' $T, T' \colon \mathrm{Vect}^M \to \mathrm{Vect}^N$, a 2-map α between these:



is an $N \times M$ matrix of linear maps of vector spaces, with components

$$\alpha_{n,m} \colon T_{n,m} \to T'_{n,m}.$$

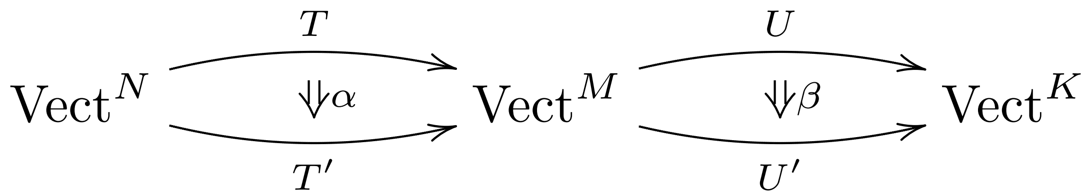
Such 2-maps can be composed *vertically*:



simply by composing componentwise the linear maps:

$$(\alpha' \cdot \alpha)_{n,m} = {\alpha'}_{n,m} \alpha_{n,m}.$$

They can also be composed *horizontally*:



by using 'matrix multiplication' with respect to tensor product and direct sum of maps:

$$(\beta \circ \alpha)_{k,m} = \bigoplus_{n=1}^{N} \beta_{k,n} \otimes \alpha_{n,m}.$$

In fact, we can form a 2-category **2Vect** with:

- 2-vector spaces as objects
- linear maps as morphisms
- linear 2-maps as 2-morphisms

Representations of 2-Groups in 2Vect

We can now study the '2-linear representation theory' of a 2-group by taking $\mathbb{C} = \mathbf{2Vect}$.

A representation $\rho\colon \mathcal{G} \to \mathbf{2Vect}$ gives:

- a 2-vector space, say $\rho(\star) = \mathrm{Vect}^N$
- for each morphism g in $\mathcal{G}$, a linear map:

$$\rho(g)\colon \mathrm{Vect}^N \to \mathrm{Vect}^N$$

  i.e. a matrix of vector spaces $\rho(g)_{n',n}$

- for each 2-morphism $u\colon g \Rightarrow g'$ in $\mathcal{G}$, a linear 2-map:

$$\rho(u)\colon \rho(g) \Rightarrow \rho(g')$$

  i.e. a matrix of linear maps $\rho(u)_{n',n}\colon \rho(g)_{n',n} \to \rho(g')_{n',n}$

For this to be a representation, we need compatibility with composition. In particular, for any morphisms g, g' in our 2-group, we need:

$$\rho(g'g) = \rho(g')\rho(g)$$

At the level of matrices, this says:

$$\rho(g'g)_{n'',n} = \bigoplus_{n'} \rho(g')_{n'',n'} \otimes \rho(g)_{n',n}$$

But here lies an interesting problem:

Since g is invertible, $\rho(g)$ must be invertible. For a complex vector space V, the operation "tensoring with V" is invertible iff $V \cong \mathbb{C}$.

In fact, this implies the only matrices of complex vector spaces that are (even weakly) invertible with respect to matrix multiplication are ones that look like "$\mathbb{C}$ times a permutation matrix".

So: any automorphism of Vect^N is essentially described by some permutation of the basis 2-vectors

$$e_i = (0, \ldots, \underbrace{\mathbb{C}}_{i\text{th place}}, \ldots, 0) \qquad i = 1 \ldots N.$$

The compatibility equations:

$$\rho(1) = 1$$
$$\rho(g'g) = \rho(g')\rho(g)$$

imply we get an action of the group G of morphisms in $\mathcal{G}$ on the set of basis vectors!
This is fine if we are interested mainly in 2-groups with *finite* G. But if G is, say, a *Lie group* or other *topological group*, it will typically have very few nontrivial actions on finite sets!

Moral: *We need a better version of* **2Vect** *if we want a (collectively) faithful representation theory for 'topological 2-groups'.*

Next time:

- Generalize Vect^N so the index set $\{0, 1, 2, \ldots, N-1\}$ becomes an infinite set of some sort (in fact, a measurable space or topological space) on which a topological group has more interesting actions.
- Form a 2-category from such 'infinite dimensional 2-vector spaces', analogous to the construction of **2Vect**, but with the 'direct sums' in matrix multiplication becoming 'direct integrals'
- Study representations of 2-groups in this 2-category, as well as intertwiners, and 2-intertwiners.