Davis Math Lab Project Report, Spring 2026
Exploration of Chiral Central Charge
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1 Introduction

Project Goal: We are exploring how to microscopically define chiral central charge. The exploration follows a proposal described by Kitaev.
The work below builds background and foundations to approach this goal. The main physical setup is studying and getting familiar with the Integer Quantum Hall Effect (IQHE), all of the below calculations and readings is related to this topic.

2 Progress

- Read up on background information about Quantum Hall Effect, particularly Integer Quantum Hall Effect (IQHE) using David Tong's lecture notes [1, 2], alongside foundational papers on thermal transport [3].
- Followed the setup of lecture notes and recreate calculations in the lecture notes.
- Calculated the spectrum of the Landau Hamiltonian with implemented edge at y=0, obtaining energy spectrum with k dependence (Edge Modes.)
- Calculated and found an expression for the spectrum of the fancy edge.

Target Hamiltonian (Fancy Edge):

$$\begin{aligned} H_{FE} &= \frac{1}{2}\{Y, [(\vec{p}-\vec{A})^2 - E_F]\} \\ &= \frac{1}{2}\left(Y[(\vec{p}-\vec{A})^2 - E_F] + [(\vec{p}-\vec{A})^2 - E_F]Y\right) \end{aligned} \tag{1}$$

2.1 Power Series Expansion

Working with Landau Gauge $\vec{A} = (By, 0, 0)$, and looking at fibered form $p_x = k$ gives:

$$H_{FE}(k) = \frac{1}{2}\{Y, [(k - By)^2 - \partial_y^2 - E_F]\}$$

Insert plane wave ansatz $\Psi(x, y) = e^{ikx}\psi_k(y)$:

$$\begin{aligned} H_{FE}(k)\Psi(x,y) &= E(k)\Psi(x,y) \\ &= \frac{1}{2}\left(2k^2y + 2B^2y^3 - 4kBy^2 - 2yE_F - \partial_y^2 y - y\partial_y^2\right)\psi_k(y) \end{aligned}$$

Group terms and expanding the chain rule gives

$$\Rightarrow \left([k^2 - E_F] - 2kBy + B^2y^2 - \frac{\partial_y}{y} - \partial_y^2 - \frac{E(k)}{y}\right)\psi_k(y) = 0 \tag{2}$$

This is the differential equation I will attempt to solve to obtain the spectrum (eigenvalues) of this Hamiltonian.

Extracting asymptote behavior of the differential equation, at large y, the y^2 term dominates

$$\begin{aligned} \Rightarrow B^2y^2\psi_k(y) &\approx \partial_y^2\psi_k(y) \\ \psi_k(y) &\approx Ae^{-By^2/2} + Be^{By^2/2} \end{aligned}$$

We drop the exponentially growing term ($Be^{By^2/2}$) to satisfy normalization requirements as $y \to \infty$.

$$\begin{aligned} \partial_y^2\psi_k(y) &= B(By^2 - 1)e^{-By^2/2} \approx B^2y^2e^{-By^2/2} \\ \psi_k(y) &\to f(y)e^{-By^2/2} \text{ at large } y \end{aligned}$$

Taking $\psi_k(y)$ to be of this form

$$\begin{aligned} \psi_k(y) &= f(y)e^{-By^2/2} \\ \partial_y\psi_k(y) &= \frac{df}{dy}e^{-By^2/2} - f(y)Bye^{-By^2/2} = \left(\frac{df}{dy} - fBy\right)e^{-By^2/2} \\ \partial_y^2\psi_k(y) &= \left(\frac{d^2f}{dy^2} - \frac{df}{dy}By - fB\right)e^{-By^2/2} - By\left(\frac{df}{dy} - fBy\right)e^{-By^2/2} \\ &= \left(\frac{d^2f}{dy^2} - 2\frac{df}{dy}By + fB(By^2 - 1)\right)e^{-By^2/2} \end{aligned}$$

Plug back into equation (2)

$$\left([k^2 - E_F] - 2kBy + B^2y^2 - \frac{\partial_y}{y} - \partial_y^2 - \frac{E(k)}{y}\right)\psi_k(y) = 0$$

$$\begin{aligned} \Rightarrow [k^2 - E_F]f(y) - 2kByf(y) + B^2y^2f(y) - \frac{1}{y}\left(\frac{df}{dy} - fBy\right) \\ - \left(\frac{d^2f}{dy^2} - 2\frac{df}{dy}By + fB(By^2 - 1)\right) - \frac{E(k)}{y}f(y) = 0 \end{aligned} \tag{3}$$

Let $f(y) = \sum_{n=0}^{\infty} c_n y^n$

$$\begin{aligned} \frac{df}{dy} &= \sum_{n=1}^{\infty} nc_ny^{n-1} \\ \frac{d^2f}{dy^2} &= \sum_{n=0}^{\infty}(n+1)(n+2)c_{n+2}y^n \end{aligned}$$

Which gives

$$
\begin{aligned}
&[k^2 - E_F + 2B]\sum_{n=0}^{\infty} c_n y^{n+1} - 2kB\sum_{n=0}^{\infty} c_n y^{n+2} - \sum_{n=1}^{\infty} n c_n y^{n-1} \\
&- \sum_{n=0}^{\infty}(n+1)(n+2)c_{n+2}y^{n+1} + 2B\sum_{n=1}^{\infty} n c_n y^{n+1} - E(k)\sum_{n=0}^{\infty} c_n y^n = 0
\end{aligned}
$$

Re-indexing and grouping terms then gives

$$
\begin{aligned}
&[k^2 - E_F + 2B]\sum_{n=1}^{\infty} c_{n-1} y^{n} - 2kB\sum_{n=2}^{\infty} c_{n-2} y^{n} - \sum_{n=0}^{\infty}(n+1)^2 c_{n+1} y^{n} \\
&+ 2B\sum_{n=2}^{\infty}(n-1)c_{n-1}y^n - E(k)\sum_{n=0}^{\infty} c_n y^n = 0
\end{aligned}
$$

Which yields recursion relation:

$$
[k^2 - E_F + 2B]c_{n-1} - 2kBc_{n-2} - (n+1)^2 c_{n+1} + 2B(n-1)c_{n-1} - E(k)c_n = 0 \tag{4}
$$

Unfortunately a simple proof can show that this recursion relation does not terminate, I have also double checked with Hermite polynomial expansion, which also does not terminate, validating the result.

2.2 Hermite Polynomial Expansion

The Biconfluent Heun Equation (BHE) has the following form [6]:

$$
zy'' + (\gamma + \delta z + \epsilon z^2)y' + (\alpha z - q)y = 0
$$

$$
\text{where } \gamma, \delta, \epsilon, \alpha, q \in \mathbb{C}, \text{ and } y \text{ is a function of } z
$$

Which can be matched with expression (3) by matching y with z, and f(y) with y.

$$
\begin{aligned}
&yf''(y) \to zy'' \\
&(1 - 2By^2)f'(y) \to (\gamma + \delta z + \epsilon z^2)y' \\
&[2kBy^2 - (k^2 - E_f + 2B)y + E(k)]f(y) \nrightarrow (\alpha z - q)y
\end{aligned}
$$

The issue lies in the $2kBy^2 f(y)$ term, but for now let's assume it is small enough to account for it using perturbation theory.

Taking this step leaves us with:

$$
yf''(y) + (1 - 2By^2)f'(y) + [-(k^2 - E_f + 2B)y + E(k)]f(y) = 0
$$

Solution of BHE $u'' + (\frac{\gamma}{z} + \delta + \epsilon z)u' + \frac{\alpha z - q}{z}u = 0$ is [5]:

$$
\begin{aligned}
&u = \sum_n C_n u_n \\
&u_n = H_{\alpha_0 + n}(s_0(z + z_0)) \\
&\gamma = 1, \delta = 0, \epsilon = -2B, \alpha = -(k^2 - E_f + 2B), q = -E(k) \\
&2s_0^2 = -\epsilon \to 2s_0^2 = 2B \to s_0 = \sqrt{B} \\
&z_0 = \frac{\delta}{\epsilon} = 0 \\
&\alpha_0 = \gamma - \frac{\alpha}{\epsilon} = 1 - \frac{-(k^2 - E_f + 2B)}{-2B} = \frac{E_f - k^2}{2B}
\end{aligned}
$$

With recursion relation[5]:

$$\begin{aligned}
&R_n C_n + Q_{n-1} C_{n-1} + P_{n-2} C_{n-2} = 0 \\
&R_n = \sqrt{\frac{2}{-\epsilon}}(\alpha_0 + n)(\alpha + (\alpha_0 + n - \gamma)\epsilon) \\
&Q_n = -\frac{\alpha\delta + (q + (\alpha_0 + n)\delta)\epsilon}{\epsilon} \\
&P_n = \frac{\alpha + (\alpha_0 + n)\epsilon}{\sqrt{-2\epsilon}}
\end{aligned}$$

Termination condition [5] requires $\gamma = -N$ (where N $\in \mathbb{Z}_{\geq 0}$), but since $\gamma = 1$ this condition is not satisfied, leaving us with the infinite sum still. This suggests to tackle the problem numerically.

2.3 Numerical Approach

Starting with the usual normalized Harmonic Oscillator Eigenstate [4]:

$$\Psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} H_n(\xi) e^{-\xi^2/2}$$

Landau Eigenstate (up to normalization factor):

$$\Psi_{n,k}(x, y) \sim e^{iky} H_n(x + k l_B^2) e^{-(x+k l_B^2)^2/2 l_B^2}$$

We can quickly normalize this by comparing with the Harmonic Oscillator Eigenstate with the physical constants of the system.

$$l_B = \sqrt{\frac{\hbar}{eB}} \quad \omega_B = \frac{eB}{m}$$

$$\left(\frac{m\omega_B}{\pi\hbar}\right)^{\frac{1}{4}} = \left(\frac{meB}{\pi\hbar m}\right)^{\frac{1}{4}} = \left(\frac{eB}{\pi\hbar}\right)^{\frac{1}{4}} = \left(\frac{1}{\pi}\right)^{\frac{1}{4}} \sqrt{\frac{1}{l_B}}$$

Normalized Landau Eigenstate:

$$\Psi_{n,k}(x, y) = \frac{1}{\sqrt{2^{n+1} n! l_B \pi^{3/2}}} e^{iky} H_n(x + k l_B^2) e^{-(x+k l_B^2)^2/2 l_B^2} \tag{5}$$

Goal: Aim to use the completeness relation ($\sum_n |n\rangle\langle n| = I$) of the Landau eigenstates to solve for spectrum of H_{FE}. Note: Landau eigenstates span a complete orthonormal basis in Hilbert Space.

Landau Hamiltonian with the units convention I work with is

$$H_L = (\vec{p} - \vec{A})^2$$

H_L is equipped with its eigenstates $|n\rangle$ (5), and discrete eigenvalues E_n:

$$H_L |n\rangle = E_n |n\rangle$$

If we substitute H_L into H_{FE}, and promote y to the Y operator in the Landau Basis, we obtain:

$$H_{FE} = \frac{1}{2}\{Y, H_L - E_F\}$$

Using the completeness relation of the landau eigenstates

$$\begin{aligned}
H_{FE}|\Psi(k)\rangle &= E|\Psi(k)\rangle \\
(I)H(I)|\Psi(k)\rangle &= E|\Psi(k)\rangle \\
\sum_{m,n} |m\rangle\langle m|H|n\rangle\langle n|\Psi(k)\rangle &= E|\Psi(k)\rangle \\
\sum_{m,n} H_{m,n}|m\rangle\langle n|\Psi(k)\rangle &= E|\Psi(k)\rangle
\end{aligned}$$

Where

$$\begin{aligned}
H_{m,n} = \langle m|H_{FE}|n\rangle &= \frac{1}{2}\langle m|\{Y, H_L - E_F\}|n\rangle \\
&= \frac{1}{2}\langle m|Y(H_L - E_F) + (H_L - E_F)Y|n\rangle \\
&= \frac{1}{2}\left(\langle m|YH_L|n\rangle - 2E_F\langle m|Y|n\rangle + \langle m|H_LY|n\rangle\right) \\
&= \frac{1}{2}\left(\langle m|Y|n\rangle E_{L,n} - 2E_F\langle m|Y|n\rangle + E_{L,m}\langle m|Y|n\rangle\right) \\
&= \frac{1}{2}(E_{L,n} + E_{L,m} - 2E_F)\langle m|Y|n\rangle
\end{aligned}$$

We can find Y in basis of landau eigenvalues in terms of creation & annihilation operators (details of this is in David Tong's lecture notes [1, 2]).

$$\begin{aligned}
\pi_x = p_x + By \quad \pi_y &= p_y + Ay \quad \vec{A} = (-By, 0, 0) \\
a &= \frac{1}{\sqrt{2\hbar eB}}(p_x + By - ip_y) \\
a^\dagger &= \frac{1}{\sqrt{2\hbar eB}}(p_x + By + ip_y)
\end{aligned}$$

Inserting $p_x = -i\hbar\partial_x$ yields :

$$\implies Y = \frac{l_B}{\sqrt{2}}(a + a^\dagger) + il_B^2\partial_x$$

$$\begin{aligned}
\langle m|Y|n\rangle &= \langle m|\frac{l_B}{\sqrt{2}}(a + a^\dagger) + il_B^2\partial_x|n\rangle \\
&= \frac{l_B}{\sqrt{2}}(\langle m|a|n\rangle + \langle m|a^\dagger|n\rangle) + il_B^2\langle m|\partial_x|n\rangle \\
&= \frac{l_B}{\sqrt{2}}(\sqrt{n}\langle m|n-1\rangle + \sqrt{n+1}\langle m|n+1\rangle) - l_B^2 k\delta_{m,n} \\
&= \frac{l_B}{\sqrt{2}}(\sqrt{n}\delta_{m,n-1} + \sqrt{n+1}\delta_{m,n+1}) - l_B^2 k\delta_{m,n}
\end{aligned}$$

$$\implies \langle m|H_{FE}|n\rangle = \frac{1}{2}(E_{L,n} + E_{L,m} - 2E_F)\left[\frac{l_B}{\sqrt{2}}(\sqrt{n}\delta_{m,n-1} + \sqrt{n+1}\delta_{m,n+1}) - l_B^2 k\delta_{m,n}\right]$$

Which gives rise to Tridiagonal matrix:

$$\begin{aligned}
\implies \Bigg(\sum_{m,n}\frac{1}{2}(E_{L,n} + E_{L,m} - 2E_F)\Bigg[&\frac{l_B}{\sqrt{2}}(\sqrt{n}\delta_{m,n-1} + \sqrt{n+1}\delta_{m,n+1}) \\
&- l_B^2 k\delta_{m,n}\Bigg]|m\rangle\langle n|\Bigg)|\Psi(k)\rangle = E|\Psi(k)\rangle
\end{aligned}$$

$$\begin{aligned}
\implies \Bigg(&\sum_n \frac{1}{2}(E_{L,n} + E_{L,n-1} - 2E_F)\frac{l_B}{\sqrt{2}}\sqrt{n}|n-1\rangle\langle n| \\
&+ \sum_n \frac{1}{2}(E_{L,n} + E_{L,n+1} - 2E_F)\frac{l_B}{\sqrt{2}}\sqrt{n+1}|n+1\rangle\langle n| \\
&- \sum_n \frac{1}{2}(2E_{L,n} - 2E_F)l_B^2 k|n\rangle\langle n|\Bigg)|\Psi(k)\rangle \\
= {}& E|\Psi(k)\rangle
\end{aligned}$$

$$\left(\sum_{m,n} H_{m,n}|m\rangle\langle n|\right)|\Psi(k)\rangle = E|\Psi(k)\rangle \tag{6}$$

We know components of the matrix $(H_{m,n})$, so we can set it up and find its eigenvalue & eigenvector numerically. Doing so and computing the energy spectrum for a range of k values between -5 and +5 yields the following plot:

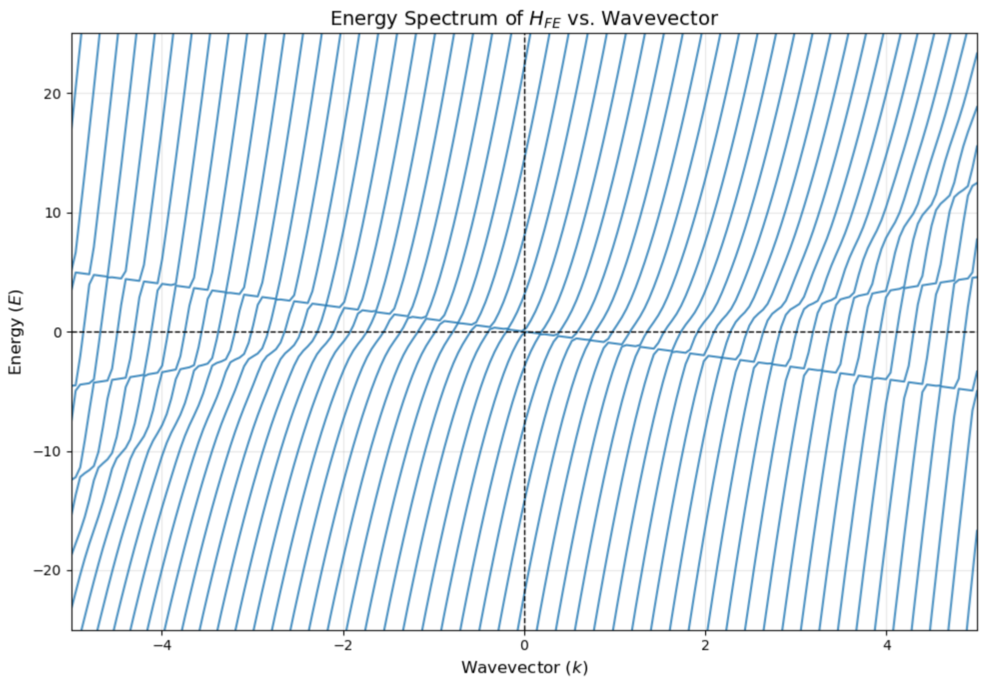

Figure 1: Numerical energy spectrum of the Fancy Edge Hamiltonian (H_{FE}) plotted against the wavevector k, computed using a basis size of $N = 150$.

3 Future Plans

- Verify and analyze the numerical approach output

References

[1] D. Tong, "Lectures on the Quantum Hall Effect," Chapter 1: Landau Levels. [Online]. Available: https://www.damtp.cam.ac.uk/user/tong/qhe/one.pdf

[2] D. Tong, "Lectures on the Quantum Hall Effect," Chapter 2: The Integer Quantum Hall Effect. [Online]. Available: https://www.damtp.cam.ac.uk/user/tong/qhe/two.pdf

[3] C. L. Kane and M. P. A. Fisher, "Quantized thermal transport in the fractional quantum Hall effect," *Phys. Rev. B*, vol. 55, no. 23, pp. 15832–15837, 1997.

[4] D. J. Griffiths and D. F. Schroeter, *Introduction to Quantum Mechanics*, 3rd ed. Cambridge: Cambridge University Press, 2018.

[5] T. A. Ishkhanyan and A. M. Ishkhanyan, "Solutions of the bi-confluent Heun equation in terms of the Hermite functions," *Annals of Physics*, vol. 383, pp. 79–91, 2017. [Online]. Available: `https://arxiv.org/abs/1608.02245`

[6] Wikipedia contributors, "Heun function," *Wikipedia, The Free Encyclopedia*. [Online]. Available: `https://en.wikipedia.org/wiki/Heun_function`