BUILDING UP TO KHOVANOV SKEIN LASAGNA MODULES

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Abstract. This is an expository article presenting the Khovanov skein lasagna module, with an undergraduate audience in mind. We start off with some preliminaries in low-dimensional topology, homological algebra, and category theory, before diving into a discussion of Khovanov homology and skein lasagna modules. We end with an overview of some example computations for skein lasagna modules. Note that this article is largely a result of working through Manolescu–Neithalath's paper ([MN22]).

Contents

1. Introduction

To even highly skilled low-dimensional topologists, smooth 4-manifolds are particularly elusive. From seemingly bizarre discoveries like Taubes' uncountably many exotic $\mathbb{R}^4$'s ([Tau87]), to the notoriously difficult 4-dimensional smooth Poincaré conjecture, investigating smooth structures in dimension 4 has been, by no means, an easy venture. For decades, the smooth 4-dimensional topologist's toolbox has primarily been composed of tools with a more analytical flavor; in particular, the Seiberg–Witten equations ([SW94a], [SW94b], [SW94c]) and Donaldson invariants ([Don83]). Introduced in [MWW22], Morrison–Walker–Wedrich's *skein lasagna modules* offer a new perspective. Built from the Khovanov–Rozansky

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homology KhR_N of links $L \subset S^3$, the skein lasagna module has its foundations in quantum algebra and representation theory, and benefits from the depth of work in these fields.

In particular, the skein lasagna module takes as its input a pair (W, L), where W is a smooth 4-manifold and $L \subset \partial W$ is a link in its boundary. This invariant then outputs a collection of $\mathbb{Q}$-vector spaces graded by a homological grading i, a quantum grading j, and a relative homological level $\alpha \in H_2(W; L)$. Then the skein lasagna module of (W, L) is

$$\mathcal{S}_0^2(W; L) = \bigoplus_{(i,j,\alpha)\in\mathbb{Z}^2\times H_2(W;L)} \mathcal{S}_{0,i,j}^2(W; L, \alpha).$$

Already, the skein lasagna module has given way to some exciting results. Manolescu–Neithalath showed that the skein lasagna module is sensitive to orientation reversal ([MN22]); Ren–Willis found an exotic pair of 4-manifolds with non-isomorphic skein lasagna modules ([RW25]); and Sullivan showed that this invariant can also detect exotically knotted pairs of surfaces ([Sul25]). With these results in mind, the skein lasagna module can be viewed as a promising new tool that could help low-dimensional topologists grapple with smooth structures in dimension four.

In this article, we give an expository presentation of the Khovanov skein lasagna module, i.e. the skein lasagna module built off of the original Khovanov homology package. Note that in this article when we use ”skein lasagna module” we are referring specifically to the Khovanov skein lasagna module (as similar invariants have been built from other knot homology theories). This exposition is done with the typical undergraduate in mind; a background in group and ring theory, as well as some topology may be helpful throughout. First, we will present some necessary topological, algebraic, and categorial preliminaries. Then we will discuss Khovanov homology, from both the computational and theoretical perspectives, before heading into our main topic: the Khovanov skein lasagna module and how to compute it. Finally, we will end with some examples showing how one might compute the skein lasagna module.

1.1. **Acknowledgements.** I would like to thank my friends and parents for supporting me throughout my academic career and for taking the time to listen to my research and ideas as an unwilling audience. I also thank the staff and faculty at the Undergraduate Research Center, and the MURPPS and LSAMP/CAMP programs for their support with getting started in undergraduate research.

Most of all, I would like to thank my faculty advisor, Melissa Zhang, for helping me develop my interests in low-dimensional topology and for her unwavering support throughout this project.

2. Preliminaries

2.1. **Topological preliminaries.** We adapt the material in this section from [Ada04], [GS99], and [Sco05].

A *knot* $K \subset S^3$ is a smooth embedding $S^1 \hookrightarrow S^3$. Similarly, an *link* $L \subset S^3$ with n components is an embedding of n disjoint copies of S^1, namely $\amalg_{i=1}^n S_i^1 \hookrightarrow S^3$. We will often say that L is an n-component link. A *link diagram* of L, which we will denote by D is a projection of L onto the plane. We say that two link diagrams D, D' are *isotopic* if they are related by a finite sequence of Reidemeister moves and ambient isotopy.

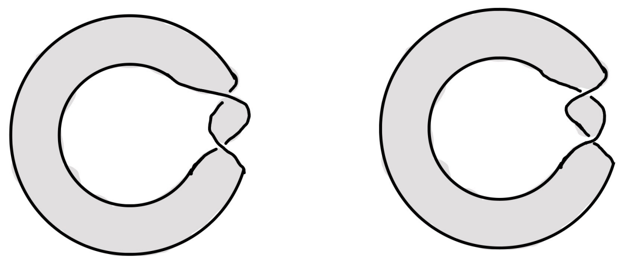


Figure 1. Example of integral framing for the unknot. The diagram on the left has framing $p = -1$ and the diagram on the right has framing $p = +1$.

We may endow a knot K with an integral *framing* $p \in \mathbb{Z}$. To this, we embed a ribbon $S^1 \times [0,1] \hookrightarrow S^3$, with $K = S^1 \times \{0\}$ (i.e., we "thicken up" K), such that this ribbon has p full twists (Figure 1).

Another way to think of framing, perhaps more intuitively, is in the following way. Let $N(K) \cong S^1 \times B^2$ denote a tubular neighborhood of K with K the "core". Then the framing p corresponds to a longitudinal curve which runs parallel to K and wraps meridionally around the core K a total of p times. If K has framing p, we will denote this by K^p. If L is a n-component link, with components L_i for $i = 1, \ldots, n$, then we can assign a framing to L by assigning to each component L_i a framing p_i. As is the case with knots, we will denote framed links by L^p, but it should be understood that in this context, p is an n-tuple $(p_1, \ldots, p_n)$.

In (smooth) 4-manifold topology, knots are often useful when examining handle decompositions of 4-manifolds.

Definition 1. Let W be an n-manifold. An *n-dimensional k-handle*, for $0 \leq k \leq n$, is a copy of $B^k \times B^{n-k}$, attached to W via the embedding $\eta : \partial B^k \times B^{n-k} \to \partial W$. It is also helpful to be familiar with some handle anatomy. Consider a handle $h = B^k \times B^{n-k}$.

- The *core* of h is $B^k \times \{0\}$.
- The *cocore* of h is $\{0\} \times B^{n-k}$.
- The *belt sphere* of h is $\{0\} \times \partial B^{n-k} \cong \{0\} \times S^{n-k-1}$.
- The *attaching sphere* of h is $\partial B^k \times \{0\} \cong S^{k-1} \times \{0\}$.
- The *attaching region* of h onto a n-manifold W is $\partial B^k \times B^{n-k} \cong S^{k-1} \times B^{n-k}$.

Remark 2. From now on, unless otherwise specified, we will only be discussing handle decompositions of 4-manifolds. Thus, rather than saying "4-dimensional k-handle", we will simply use the term "k-handle".

We will largely be working with a class of smooth 4-manifolds which we will call "2-handlebodies". For simplicity, these are 4-manifolds built from a single 0-handle $B^0 \times B^4$, finitely many 2-handles $B^2 \times B^2$, and (potentially) a 4-handle $B^4 \times B^0$. To understand 2-handlebodies, one should understand how the 2-handle is glued onto the 0-handle, in particular, via the map

$$\eta : S^1 \times B^2 \hookrightarrow S^3.$$

From this, note that the attaching region is an embedded solid torus $S^1 \times B^2$ and the attaching sphere is a subspace $S^1 \subset S^1 \times B^2$. Thus, we have a series of embeddings

$$S^1 \subset S^1 \times B^2 \subset S^3.$$

From this, the canonical perspective is to view the gluing information for a 2-handle attachment as a framed knot, and we communicate this information through *Kirby diagrams* (see Figure 16 for an example).

Definition 3. An (n,n)-*tangle*, T, is a 1-manifold properly embedded in $[0,1]\times[0,1]\times(-\frac{1}{2},\frac{1}{2})$ that has $2n$-boundary points embedded in $[0,1]\times\{0,1\}\times\{0\}$. A *tangle diagram* (of T) is a projection of T onto $[0,1]\times[0,1]$.

Remark 4. Note that tangles may be defined more generally (e.g., (n,m) tangles). Additionally, we will consider tangles that do not have closed components.

In particular, we will denote by Id_n the identity tangle on n strands and by $FT_n^{\otimes p}$ the tangle formed by doing p full twists on n strands.

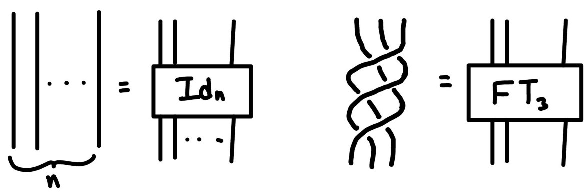


FIGURE 2. The identity tangle Id_{2n} (on the left) and a full twist on three strands FT_3 (on the right).

We will also extensively discuss cobordisms in this article.

Definition 5. Given oriented n-manifolds W and Y, a *cobordism* is an $(n+1)$-manifold Z such that $\partial Z = \overline{W} \sqcup Y$, where $\overline{W}$ denotes W with the reverse orientation.

For our purposes, cobordisms will typically be surfaces between links.

Remark 6. Note that our convention is to "read" cobordisms from bottom to top. Refer to 2.1. This is an important convention to state; the reader should note that, for later use in Section 3.2, Bar-Natan's convention (in [BN05]) is to read cobordisms top to bottom.

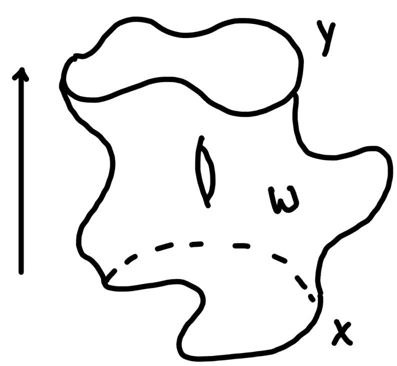


FIGURE 3. Our convention for reading cobordisms (from bottom to top).

2.2. Algebraic and Categorical Preliminaries.

2.2.1. *Some Homological Algebra.* We adapt the following material from ([Rot09]) and ([Wei94]).

Definition 7. A *chain complex* consists of the sequence of data $(C_i, \partial_i)_{i\in\mathbb{Z}}$, where

- the *i-th chain group* C_i is an abelian groups (i.e. a $\mathbb{Z}$-module),
- the *i-th differential* $\partial_i : C_i \to C_{i+1}$ is a group homomorphisms satisfying $\partial_i \circ \partial_{i-1} = 0$.

Remark 8. Importantly, note that we have a slight difference in convention throughout our discussion of chain complexes. Traditionally, differentials *decrease* the index in the sequence, but for us the differential *increases* the index. This is simply to keep our definitions in line with the conventions for Khovanov homology. The reader may also observe that our Khovanov homology groups are denoted with superscripts rather than the subscripts used in this sections; this is merely a matter of sticking with conventional notation for Khovanov homology.

Although not directly related to homological algebra, we will be working with *bi-graded modules*.

Definition 9. A $(\mathbb{Z}\oplus\mathbb{Z})$-graded abelian group (or bi-graded $\mathbb{Z}$-module), V, is an abelian group (or $\mathbb{Z}$-module) with distinguished generators $v_\pm$, endowed with a bi-grading $(0, \pm 1)$. We may write V as

$$V = \mathbb{Z}v_+ \oplus \mathbb{Z}v_-.$$

When building the Khovanov chain complex, our bi-graded $\mathbb{Z}$-modules will be graded by $\mathrm{gr} = (\mathrm{gr_h}, \mathrm{gr_q})$, where $\mathrm{gr_h}$ is the *homological grading* and $\mathrm{gr_q}$ is the *quantum grading*. Furthermore, the $\mathbb{Z}$-module $V^{\otimes k}$, $k \geq 0$, has the 2^k-cardinality set of distinguished generators

$$\{v_\pm \otimes \cdots \otimes v_\pm\},$$

where each generator is a k-bit pure tensor.

Definition 10. Given a bi-graded module $C_{i,j}$, the *height shift* operator $[p]$ and *degree shift* operator $\{q\}$ are defined by

$$C_{i,j}[p]\{q\} = C_{i-p,j-q}$$

In particular, these shifts shift the bi-gradings of the distinguished generators.

To motivate the following definition, we claim that given a chain complex (C_i, ∂_i), we have $\operatorname{img}\partial_{i-1} \subset \ker\partial_i$. This immediately follows form the condition on the differential that $\partial_i \circ \partial_{i-1} = 0$.

Definition 11. The i-th homology group of a chain complex (C_i, ∂_i) is the quotient group

$$H_i(\mathcal{C}) = \ker\partial_i / \operatorname{img}\partial_{i-1}.$$

Definition 12. Let (C_i, ∂_i) and (C_i', ∂_i') be chain complexes. A *chain map* $f : (C_i, \partial_i) \to (C_i', \partial_i')$ is a collection of maps $\{f_i : C_i \to C_i'\}$ such that

$$\partial_i' f_i = f_{i+1}\partial_i$$

for all $i \in \mathbb{Z}$. It may be helpful to keep the following diagram in mind:

$$\begin{tikzcd}
\cdots \arrow[r] & C_{i-1} \arrow[r, "\partial_{i-1}"] \arrow[d, "f_{i-1}"] & C_i \arrow[r, "\partial_i"] \arrow[d, "f_i"] & C_{i+1} \arrow[r] \arrow[d, "f_{i+1}"] & \cdots \\
\cdots \arrow[r] & C_{i-1}' \arrow[r, "\partial_{i-1}'"] & C_i' \arrow[r, "\partial_i'"] & C_{i+1}' \arrow[r] & \cdots
\end{tikzcd}$$

Definition 13. Let (C_i, ∂_i), (C'_i, ∂'_i) be chain complexes and $f, g : (C_i, \partial_i) \to (C'_i, \partial'_i)$ chain maps. A *chain homotopy from f to g*, $h : (C_i, \partial_i) \to (C'_i, \partial'_i)$, is a collection of maps $\{h_i : C_i \to C_{i-1}\}$ such that

$$f_i - g_i = h_{i+1}\partial_i + \partial'_{i-1}h_i$$

for all $i \in \mathbb{Z}$. If there is a chain homotopy from f to g, then we say that f and g are *homotopy equivalent*.

As with chain maps, it is helpful to keep the following diagram in mind when thinking about a chain homotopy.

$$\begin{tikzcd}
\cdots \arrow[r] & C_{i-1} \arrow[r, "\partial_{i-1}"] \arrow[d, shift right, "f_{i-1}"'] \arrow[d, shift left, "g_{i-1}"] & C_i \arrow[r, "\partial_i"] \arrow[d, shift right, "f_i"'] \arrow[d, shift left, "g_i"] \arrow[dl, "h_i"'] & C_{i+1} \arrow[r] \arrow[d, shift right, "f_{i+1}"'] \arrow[d, shift left, "g_{i+1}"] \arrow[dl, "h_{i+1}"'] & \cdots \\
\cdots \arrow[r] & C'_{i-1} \arrow[r, "\partial'_{i-1}"'] & C'_i \arrow[r, "\partial'_i"'] & C'_{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

An important fact to keep in mind here is that chain homotopy equivalent chain complexes have the same homology.

2.2.2. *Some Category Theory.* We adapt portions of this section from [Ago23] and [Rie16]. As the typical undergraduate curriculum may not include category theory, we will go over some of the basics before discussing the main ideas which the reader should take away from this section: colimits.

Definition 14. A *category* $\mathcal{C}$ consists of the following:

- a collection of objects, denoted $\operatorname{Ob}\mathcal{C}$,
- for every pair of objects $A, B \in \operatorname{Ob}\mathcal{C}$, a (possibly empty) set of *morphisms* $\operatorname{Hom}(A, B)$, that is maps $f : A \to B$, such that
  - morphisms obey a *composition law*, i.e. for objects $A, B, C \in \operatorname{Ob}\mathcal{C}$ and morphisms $f \in \operatorname{Hom}(A, B), g \in \operatorname{Hom}(B, C)$, then $g \circ f \in \operatorname{Hom}(A, C)$,
  - morphisms under composition are *associative*
  - every object $A \in \mathcal{C}$ has an *identity morphism* id_A,
  - given a morphism $f \in \operatorname{Hom}(A, B)$, we have

$$\mathrm{id}_B \circ f = f \circ \mathrm{id}_A = f.$$

Example 15. The two main categories we will be working with are the following:

- The category $\mathbf{LinkDiag}(\mathbb{R}^2)$, in which
  - objects are link diagrams in $\mathbb{R}^2$
  - morphisms are "movies" between links consisting of Reidemeister moves and elementary cobordisms (see Fig. 4).
- The category $gg\operatorname{Mod}(\mathbb{Z})$, where
  - objects are bi-graded $\mathbb{Z}$-modules $M_{i,j}$ with bi-degree $(i, j) \in \mathbb{Z}^2$,
  - morphisms are module homomorphisms $m_{k,l} : M_{i,j} \to M_{k,j}$ that have degree (i.e., change the bi-grading) by $(k - i, l - j)$.

Definition 16. Let $\mathcal{C}$ and $\mathcal{C}'$ be categories. A *functor* $F : \mathcal{C} \to \mathcal{C}'$ is an assignment given by

- mapping each $A \in \operatorname{Ob}\mathcal{C}$ to $F(A) \in \operatorname{Ob}\mathcal{C}'$,

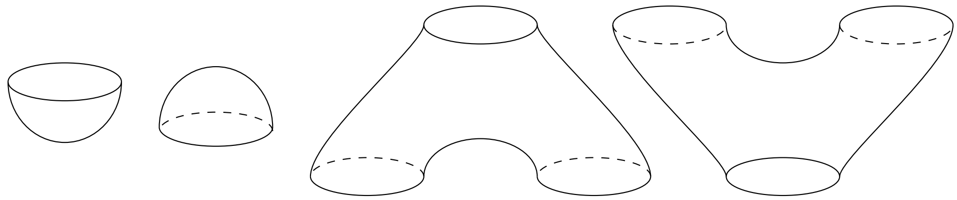


Figure 4. Elementary cobordisms used to build morphisms in the category **LinkDiag**($\mathbb{R}^2$). From left to right, these are called "birth", "death", "merge", and "split".

- mapping each morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ to a morphism $F(f) \in \mathrm{Hom}_{\mathcal{C}'}(F(A), F(B))$, such that
  - the identity morphisms maps to the identity morphism, i.e. for all $A \in \mathrm{Ob}(\mathcal{C})$, we have $\mathrm{id}_A \mapsto \mathrm{id}_{F(A)}$,
  - for every $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$, we have

$$F(g \circ f) = F(g) \circ F(f).$$

Definition 17. A *directed system* is a functor $F : \mathcal{C} \to \mathbb{N}^n$, for $n \geq 1$, Objects of $\mathcal{C}$ are denoted by C_x, where $x \in \mathbb{N}^n$, and morphisms are collections of maps $f^k_{x,y} : C_x \to C_y$, for $1 \leq k \leq n$, that increase the index of the k-th coordinate by one. Note that the objects of $\mathbb{N}^n$ are n-tuples of natural numbers and morphisms are given by the standard ordering on $\mathbb{N}^n$. Furthermore, note that our convention is that $0 \in \mathbb{N}$. Likewise, we will denote the zero vector by $0 \in \mathbb{N}^n$.

Remark 18. Technically, directed systems should be defined more generally. However, we intentionally define directed systems indexed by $\mathbb{N}^n$ both for concreteness and because this is largely the context in which we will see and use directed systems when discussing cabled knot homology theories and skein lasagna modules. Here is a rough "sketch" of a directed system $\mathbb{N} \to \mathcal{C}$: $C_0 \xrightarrow{f_0} C_1 \xrightarrow{f_1} C_2 \xrightarrow{f_2} \cdots$.

Definition 19. Let F be a directed system. The *colimit* of F is an object $C \in \mathcal{C}$, together with a collection of morphisms $\{g_i : C_i \to C\}$ which satisfy the universal property. Namely, if $C' \in \mathcal{C}$ is another object and $\{g'_i : \mathcal{C}_i \to C'\}$ another collection of morphisms, then there is a unique isomorphism $h : C \to C'$ such that the following diagram commutes for all $i \in \mathbb{N}^n$.

$$\begin{tikzcd}
C_i \arrow[rr, "f_i"] \arrow[dr, "g_i"] \arrow[ddr, bend right, "g'_i"'] & & C_{i+1} \arrow[dl, "g_{i+1}"'] \arrow[ddl, bend left, "g'_{i+1}"] \\
& C \arrow[d, dashed, "h"] & \\
& C' &
\end{tikzcd}$$

3. Khovanov homology

3.1. **Example: Khovanov Homology of the Hopf Link.** The following section is loosely adapted from [Zha25] and [BN02].

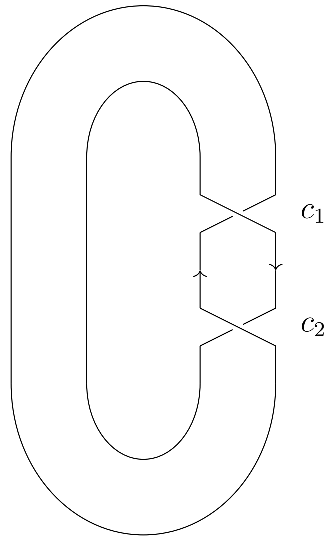


FIGURE 5. Our chosen diagram of the Hopf link, D_H. From the chosen orientations, this diagram has $n_+ = 0$, $n_- = 2$, and $n = 2$.

Let $L \subset \mathbb{R}^3$ be an oriented link and $D \subset \mathbb{R}^2$ a diagram of L (with the same orientation). Roughly put, Khovanov homology assigns a collection of bi-graded abelian groups (classically, bi-graded $\mathbb{Z}$-modules) to D, denote $\mathrm{Kh}(D)$. Later on, we will discuss how Khovanov homology is, in fact, a link invariant.

Throughout this section, we will compute the Khovanov homology of the Hopf link H as a guiding example in constructing this theory. In particular, we will use the following diagram of H, which we will denote by D_H (refer to Figure 5).

3.1.1. *Constructing the Cube of Resolutions.* Given an oriented link diagram D, we define n to be the total number of crossings, n_+ the number of positively oriented crossings and n_- the number of negatively oriented crossings . Additionally, we assign some ordering to the crossings, denoting crossing i by c_i.

As noted in Section 2.2, we compute the homology of a chain complex, not a topological space. Thus, we first describe how to construct the *Khovanov chain complex* $\mathrm{CKh}(D)$ from a link diagram.

The Khovanov chain complex will be constructed from a collection of planar circles, meaning that we need to describe an organized way to convert link diagrams to planar circles. To do this, consider the set $\{0,1\}^n$ consisting of bit-strings of length n. For a bit-string u, let $\mathrm{wt}(u)$ denote the sum of bits in u. For example, if $u = 01101 \in \{0,1\}^5$, then $\mathrm{wt}(u) = 3$. We will organize the set of all bit-strings in $\{0,1\}^n$ into a 2^n-dimensional cube with $\mathrm{wt}(u)$ increasing from left to right, and and a lexicographical ordering from bottom to top. Additionally, draw a directed edge from $u \to v$ if u and v differ at exactly one bit. For example, we would draw an edge $100 \to 101$, but not one between bits 010 and 101. Now, consider such an edge $u \to v$ such that u and v differ at the i-th bit. We label the edge $u \to v$ with a bit-string of length n from $\{0,1,*\}^n$ by

$$u_1 \cdots u_{i-1} * u_{i+1} \cdots u_n = v_1 \cdots v_{i-1} * v_{i+1} \cdots v_n. \tag{1}$$

Now, we claim there are two ways to resolve an (unoriented) crossing, into what we will call the *0-resolution* and the *1-resolution*:

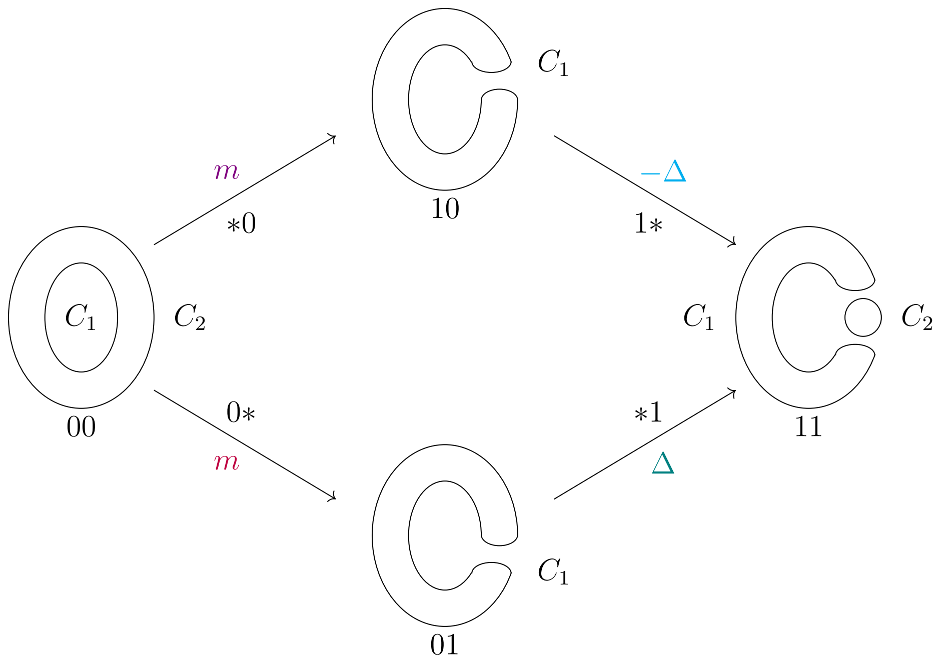
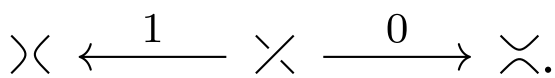


FIGURE 6. The full cube of resolutions for D_H, including (signed) edge maps.



For a bit-string $u = u_1 \cdots u_n$, we will assign, to crossing c_i the resolution corresponding to bit u_i. After resolving all the crossings in D according to u, we will assign the *resolved diagram* D_u to vertex u in the bit-string cube. After repeating this process for every $u \in \{0,1\}^n$, we have constructed the *cube of resolutions* for D. (Figure 6 for our Hopf link example). Finally, for each vertex in the cube of resolutions, let k_u denote the number of planar circles in the resolution D_u; enumerate these circles by $C_1 \ldots C_{k_u}$. Note that our final computation of Khovanov homology will be independent of this choice of labelling.

3.1.2. *Moving into the Algebraic Setting.* As we move into the algebraic setting, we will be working with the $\mathbb{Z}$-module

$$\mathcal{A} = \mathbb{Z}[X]/(X^2) \cong \mathbb{Z} \cdot 1 \oplus \mathbb{Z} \cdot X,$$

which has *distinguished generators* 1 and X Then, to each vertex assign the $\mathbb{Z}$-module $\mathcal{A}^{\otimes k_u}$.

Aside. As previously mentioned, Khovanov homology outputs *bi-graded* $\mathbb{Z}$-modules. In particular, to each $\mathbb{Z}$-module we assign the bi-grading $(\mathrm{gr_h}, \mathrm{gr_q})$, where $\mathrm{gr_h}$ is the *homological grading* and $\mathrm{gr_q}$ is the *quantum grading*. Working off of our cube of resolutions, the following rules will be helpful when computing bi-gradings.

(1) For each generator of $\mathcal{A}$ at vertex u, assign the "base grading" $(\mathrm{gr_h}, \mathrm{gr_q}) = (\mathrm{wt}(u), \mathrm{wt}(u))$.
(2) The generators of $\mathcal{A}$ have the following quantum gradings:
$$\mathrm{gr_q}(1) = +1, \quad \mathrm{gr_q}(X) = -1.$$
(3) If $a \in \mathcal{A}^{\otimes(k-1)}$ and v is a distinguished generator of $\mathcal{A}$, then $a \otimes v \in \mathcal{A}^{\otimes k_u}$ has bi-grading
$$\mathrm{gr}(a \otimes v) = \mathrm{gr}(a) + \mathrm{gr}(v).$$

In practice, when computing bi-gradings (at this stage), we will start off at Rule 1, then adjust quantum grading using Rules 2 and 3. Note that gradings will become important when we want to start discussing maps induced by Khovanov homology between different link diagrams (which may be of the same link or of different links). Additionally, we may use the notation $v_\pm$ to denote a distinguished generator. When working specifically with the module $\mathcal{A}$, our convention will be that $v_+ = 1$ and $v_- = X$.

Of course, the other important ingredient when constructing a chain complex is defining our differentials. Observe that, by referring to Figure 6, we can either have planar circles *merge* or *split*. We can, canonically, encode this information in the merge and split maps $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}\{1\}$ and $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}\{1\}$, defined by

$$m(1 \otimes 1) = 1, \quad m(1 \otimes X), m(X \otimes 1) = X, \quad m(X \otimes X) = 0$$

and

$$\Delta(1) = 1 \otimes X + X \otimes 1, \quad \Delta(X) = X \otimes X.$$

For now, we will refer to m and Δ as our *edge maps*. Additionally, note that m and Δ correspond, respectively, to the merge and split cobordisms in Figure 4.

Remark 20. Observe that we defined the merge (resp. split) map to map into $\mathcal{A}\{1\}$ (resp. $\mathcal{A} \otimes \mathcal{A}\{1\}$). This is to ensure that that maps preserve quantum grading.

Note that in our Hopf link example, we have $k_u = 1$ or $k_u = 2$. In general, we typically have more than two planar circles in D_u (i.e. $k_u > 2$). For the merge map, suppose circles C_i and C_j in D_u, where $i, j = 1, \ldots, k_u$ and $i \neq j$, merge. Then in the tensor product

$$v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_j \otimes \cdots \otimes v_{k_u},$$

the merge map will only operate on the components v_i and v_j; for the remaining components, the edge map is given by the identity. The idea is similar for the split map.

Finally, when setting up our differentials, we must have that $\partial^2 = 0$. Observe that, leaving the edge maps as defined, we instead have that the edge maps commute; we want the edge maps to anti-commute. In order to do this, to the edge $D_u \to D_v$, we add the sign

$$s_{uv} = (-1)^{\sum_{j=1}^{i-1} u_j}$$

(see Notation 1) to the edge maps. Namely, s_{uv} counts the number of ones before the $*$ in our labeling of $u \to v$.

In order to be fully explicit, refer to Figure 6 for a fully detailed cube of resolutions for the Hopf link, and Figure 7 for the explicit edge maps.

3.1.3. *The Khovanov Bracket.* We are now ready to form the Khovanov chain complex. We do this by taking a direct sum across all vertices at the same homological grading to form both the chain groups and differentials. Following Bar-Natan's notation ([BN02]), this "flattened" complex is often referred to as the *Khovanov bracket* of D, denoted $[\![D]\!]$.

Before (or after) taking homology, we want to incorporate global grading shifts. These shifts are denoted by $[\cdot], \{\cdot\}$ for the homological and quantum gradings, respectively. In particular, our canonical grading shifts are given by $[-n_-]\{n_+ - 2n_-\}$. Namely, if we imagine our chain groups as sitting in the $\mathbb{Z}^2$-lattice, then these shifts correspond to moving the entire complex by the vector $-n_-e_1 + (n_+ - 2n_-)e_2$, where e_1, e_2 are the standard basis vectors.

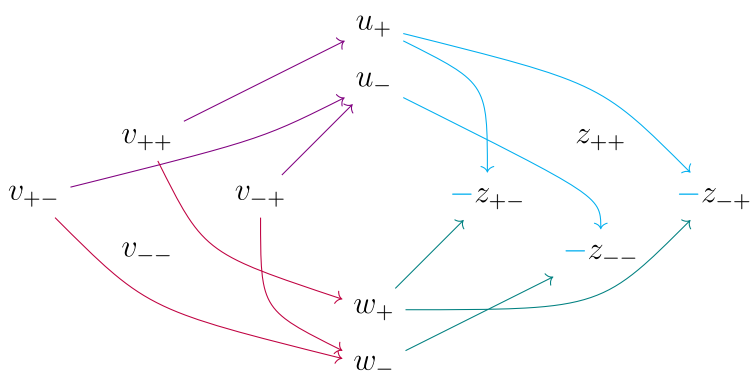


FIGURE 7. The edge maps in our Hopf link computation. Note that we use the notation $v_\pm$ as shorthand for our distinguished generators, with $1 = v_+$ and $X = v_-$, and $v_{\pm\pm} = v_\pm \otimes v_\pm$.

Remark 21. For link diagrams D, D', consider a map $\mathrm{CKh}(D) \to \mathrm{CKh}(D')$. In this context, it's often best practice to incorporate global grading shifts before taking homology to ensure that the map preserves homological grading.

We now finally have the Khovanov chain complex,

$$\mathrm{CKh}(D) = [\![D]\!][-n_-]\{n_+ - 2n_-\}.$$

Note that in $\mathrm{CKh}(D_H)$ (2), we have underlined the chain group at homological level zero. We will continue to use this notation throughout.

$$[\![D_H]\!] : \mathcal{A}^{\otimes 2}[-2]\{-4\} \xrightarrow{m \oplus m} (\mathcal{A} \oplus \mathcal{A})[-2]\{-4\} \xrightarrow{-\Delta \oplus \Delta} \underline{\mathcal{A}^{\otimes 2}[-2]\{-4\}} \tag{2}$$

The Khovanov homology is then given by taking the homology of the complex $\mathrm{CKh}(D)$. Namely,

$$\mathrm{Kh}^{i,j}(D) = H^*(\mathrm{CKh}(D)).$$

The group $\mathrm{Kh}^{i,j}(D)$ denotes the homology group with bi-grading $\mathrm{gr_h} = i$ and $\mathrm{gr_q} = j$ (after shifting). Note that while the Khovanov homology of D is defined by $\mathrm{Kh}(D) = \bigoplus_{i,j\in\mathbb{Z}} \mathrm{Kh}^{i,j}(D)$, we will often represent our homology groups in a table.

$\mathrm{gr_q} \setminus \mathrm{gr_h}$	-2	-1	0
-6	$\mathbb{Z}$		
-4	$\mathbb{Z}$		
-2			$\mathbb{Z}$
0			$\mathbb{Z}$

with homology classes given by

$\mathrm{gr_q} \setminus \mathrm{gr_h}$	-2	-1	0
-6	$[v_{--}]$		
-4	$[v_{+-} - v_{-+}]$		
-2			$[z_{+-}] = [z_{-+}]$
0			$[z_{--}]$

Remark 22. Note that, up until now, we have discussed Khovanov homology as arising from a link diagram. In the following section, we will describe a fairly simple way of showing that Khovanov homology is actually a *link invariant*. Namely, given diagrams D, D' of a link L, then $\mathrm{Kh}(D) \cong \mathrm{Kh}(D')$. Thus, we will often just write $\mathrm{Kh}(L)$. However, a brief warning is in order: the Khovanov chain complexes may be different as it depends entirely on the chosen link diagram, i.e. $\mathrm{CKh}(D) \neq \mathrm{CKh}(D')$.

Before proceeding, we must note that when constructing the skein lasagna module, we will be working over $\mathbb{Q}$. Thus, we must define Khovanov homology with rational coefficients, denoted $\mathrm{Kh}(L;\mathbb{Q})$. We can do this by taking the tensor product

$$\mathrm{Kh}^{*,*}(L;\mathbb{Q}) \cong \mathrm{Kh}^{*,*}(L) \otimes_{\mathbb{Z}} \mathbb{Q}, \tag{3}$$

where $\mathrm{Kh}^{*,*}(L)$ denotes a Khovanov homology group with integer coefficients (described throughout the previous sections).

3.2. **Bar-Natan's Category.** The goal of this section, adapted from [BN05], is to introduce Bar-Natan's *(un-)dotted cobordism category*. In practice, it can often be complicated to translate topological information into algebraic data. Instead, we can circumvent this by working in Bar-Natan's category, where we can instead construct the appropriate cobordisms and then "plug in" cobordisms corresponding to our distinguished generators.

We will also start to briefly discuss why Khovanov homology is a link invariant. Khovanov originally proved this in [Kho00], and then, in [Kho02b], he construed a family of rings from $(2m, 2n)$-tangles that can be used to show Reidemeister invariance on the level of local tangle pictures. Chen–Khovanov ([CK14]) then extended this to sub-quotients of the rings constructed in [Kho02b]. In this section, we will follow Bar-Natan's approach from [BN05], in which he constructed Khovanov chain complexes for local tangle pictures, denoted $[\![T]\!]$; in a later section, we will discuss the foundations of [Kho02b].

We begin by asserting the following result from Bar-Natan.

Theorem 23 ([BN05], Theorem 1 (Invariance Theorem))**.** Let T be a tangle. The isomorphism class of the complex $[\![T]\!]$ is a tangle invariant. Namely, it does not depend on the ordering of the cube of resolutions or the choice of order on the crossings, and it is invariant under the three Reidemeister moves.

Focusing on Reidemeister invariance, Bar-Natan proved his result by constructing a variant of the Khovanov chain complex for local pictures of tangles. Then, treating each Reidemeister move as a local tangle, he showed that the relevant Khovanov chain complexes are chain homotopy equivalent.

Perhaps more important, one should take away the construction of Bar-Natan's (un-)dotted cobordism category, as this will allow us to easily compute the Khovanov differentials in later examples (see Sections 4.2, 4.3).

Definition 24. Bar-Natan's *dotted cobordism category*, which we will denote by $\mathcal{TL}_0^\bullet$, is defined as follows:

- objects in $\mathcal{TL}_0^\bullet$ are finite collections of planar circles (smoothly embedded in $\mathbb{R}^2$) $\{C_i\}$,
- morphisms in $\mathcal{TL}_0^\bullet$ are finite sums of cobordisms $\Sigma : \{C_i\} \to \{C'_j\}$ smoothly embedded in $\mathbb{R}^2 \times I$ such that:
  - $\partial\Sigma \subset \mathbb{R}^2 \times \{0,1\}$,

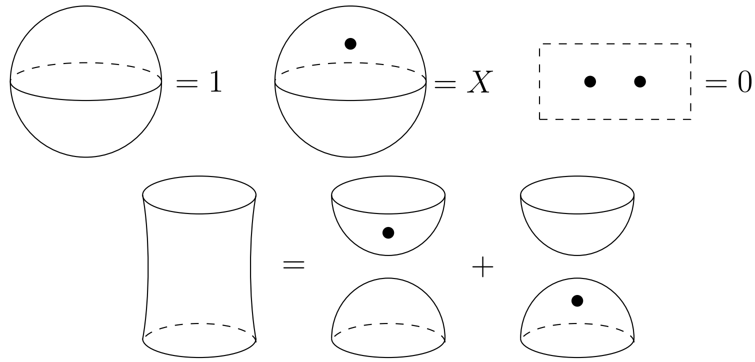


FIGURE 8. Local relations for $\mathcal{TL}_0^\bullet$. Note that the first two relations tell us that any closed component of Σ evaluates to 1 or X. The final relation on the top row tells us that if any region of Σ contains at least two dots, then Σ evaluates to 0. The relation on the bottom row is often referred to as "neck-cutting".

- Σ is decorated with finitely many dots $\bullet$,
- Σ is considered up to isotopy rel boundary,
- Σ is subject to the local relations in Figure 8.

Remark 25. One interpretation of the "$\bullet$" in this category is that decorating a cobordism with a dot corresponds to, algebraically, multiplying by X.

Alternatively, Bar-Natan also constructed an un-dotted cobordism category, $\mathcal{TL}_0$.

Definition 26. Bar-Natan's *un-dotted cobordism category* $\mathcal{TL}_0$ is defined as follows:

- objects are the same as in $\mathcal{TL}_0^\bullet$,
- morphisms are defined similarly as in $\mathcal{TL}_0^\bullet$, with the only differences being:
  - cobordisms are not decorated with dots,
  - cobordisms are subject to the different local relations, shown in Figure 9.

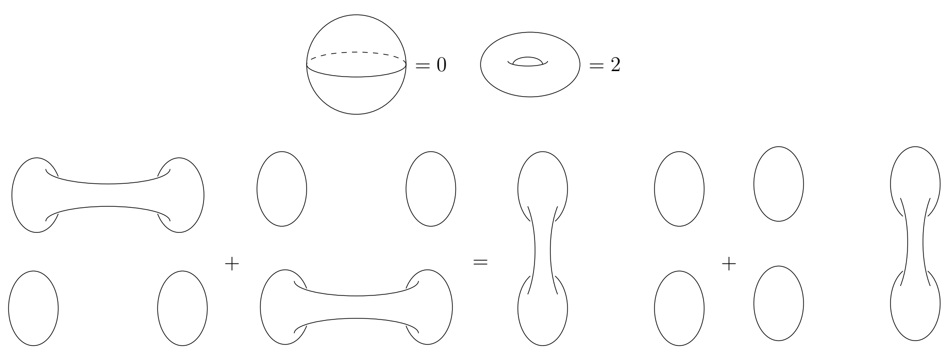


FIGURE 9. Local relations for $\mathcal{TL}_0$. The top two relations are called S ("sphere") and T ("torus"), and the bottom relation is called $4Tu$ ("four tube").

Remark 27. Note that the objects in $\mathcal{TL}_0^\bullet, \mathcal{TL}_0$ can be thought of as tangles with zero endpoints; hence the "0" in our notation. A fun fact, of sorts, is that this construction can be generalized to (n, n)-tangles, in which case we would denote the appropriate categories by $\mathcal{TL}_n^\bullet, \mathcal{TL}_n$. This distinction is important to make because in the statement of Theorem 23, we are working with arbitrary (n, n)-tangles. However, when we later on start using Bar-Natan's results by thinking of tangles as local "pieces" of link diagrams, then we can switch over to $\mathcal{TL}_0^\bullet, \mathcal{TL}_0$.

We now turn to discussing how one would obtain algebraic information from working in the category $\mathcal{TL}_0^\bullet$. Define the maps $\iota : \mathbb{Z} \to \mathcal{A}$ and $\varepsilon : \mathcal{A} \to \mathbb{Z}$ by

$$\iota(1) = 1; \qquad \varepsilon(1) = 0, \quad \varepsilon(X) = 1.$$

Definition 28. Define the assignment $\mathcal{F}_{BN} : \mathcal{TL}_0^\bullet \to gg\,\mathrm{Mod}_{\mathbb{Z}}$ by sending

- The collection of zero planar circles $\varnothing \to \mathbb{Z}$ (the base ring),
- Collections of k planar circles to $\mathcal{A}^{\otimes k}\{-1\}$,
- Cobordisms to linear maps determined by the following pieces (Figure 10):

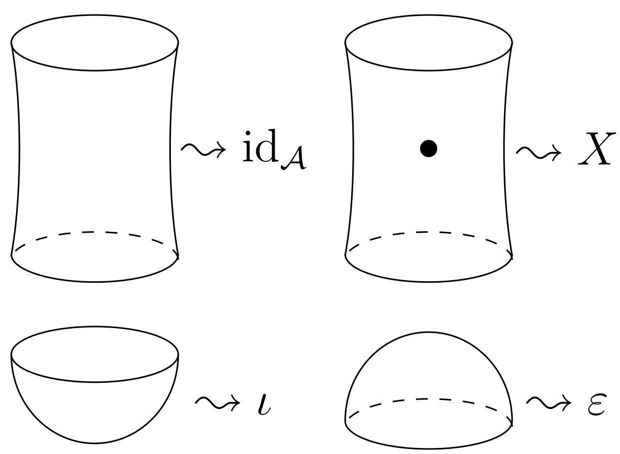


Figure 10. On top, we map an undotted cylinder to the identity and a dotted cylinder to multiplication by X. On the bottom, we have the birth and death cobordisms, corresponding to ι and ε, respectively. Note that, as in $\mathcal{TL}_0^\bullet$, we can think of the dot "$\bullet$" as multiplication by X. Furthermore, while cobordisms may look more complicated in practice, we can use the local relations for $\mathcal{TL}_0^\bullet$ to simplify down to these pieces.

Remark 29. An important note for the last point of Definition 28. A (single) complicated cobordism can be built by composing the maps define above. Specifically, if the topological pieces "stack" on each other then the associated algebraic maps are composed. If we have multiple *disjoint* cobordisms, say $Z = Z_1 \sqcup \cdots \sqcup Z_n$, then we assign the map [1]

$$\mathcal{F}_{BN}(Z) = \mathcal{F}_{BN}(Z_1) \otimes \cdots \otimes \mathcal{F}_{BN}(Z_n).$$

[1] While we will not discuss this in this article, it's worth pointing out that is because $\mathcal{F}_{BN}$ is not only a functor, but a $(1+1)$-dimensional topological quantum field theory (TQFT).

3.3. **Functoriality of Khovanov Homology.** As seen in the previous sections, since Khovanov homology takes a link as an input and outputs a collection of $\mathbb{Z}$-modules, one might expect Khovanov homology to be functorial. Namely, one would expect that if we have a link cobordism from L to L', then there would be an induced morphism $\mathrm{Kh}(L) \to \mathrm{Kh}(L')$. It turns out that Khovanov homology is *projectively functorial* ([Jac04], [BN05]), meaning that the induced morphism is defined up to a sign. Fortunately, this sign issue was fixed in the work of [Cap07], [CMW09], [Bla10], [BHPW23], [Vog20]. Furthermore, [San21] showed that the Lee's canonical homology classes offer another way to fix the sign issue.

Remark 30. Many of the concepts which we will explore in Section 4 (namely, skein lasagna modules and cabled Khovanov–Rozansky homology) were originally defined for Khovanov–Rozansky homology KhR_N. However, we will be restricting our attention to Khovanov homology so any relevant definitions and results will be defined in terms of Khovanov homology. This is fine since the sign issue with Khovanov homology has been fixed. Additionally, both the original and Morrison–Walker–Wedrich's framed version of Khovanov–Rozansky homology are functorial ([KR08], [MWW22]). Moreover, framed Khovanov–Rozansky homology is (non-canonically) isomorphic to Khovanov homology:

$$\mathrm{KhR}_2^{i,j}(L) \cong \mathrm{Kh}^{i,-j-2w}(\overline{L}), \tag{4}$$

where L is a link, $\overline{L}$ its mirror, and w is the writhe of a chosen diagram of L. Thus, while the constructions in Section 4 were originally defined over Khovanov–Rozansky homology KhR_2, we may work over Khovanov homology Kh.

3.4. **Khovanov's Arc Algebra.** In part, this section sets up Khovanov's construction used to show that his link homology theory is a link invariant ([Kho02b]). In the context of this article, this section provides necessary background for some computational tools that will be used later in Section 4.3.

For $r \geq 0$, let $\mathfrak{C}^r$ denote the set of planar matchings on $2r$ collinear points (i.e. crossingless matchings). Given $a, b \in \mathfrak{C}^r$, the composition $\overline{b}a$ is given by "stacking" the mirror image of b on top of a.

In [Kho02b], Khovanov defined

$$a(H^r)b := \mathcal{A}^{\otimes k}\{r\},$$

where $\mathcal{A} = \mathbb{Z}[X]/(X^2)$ (our favorite ring when computing Khovanov homology) and k is the number of planar circles in the composition $\overline{b}a$.

We impose the following multiplication structure

$$a(H^r)b \otimes c(H^r)d \to 0 \quad \text{if } b \neq c, \quad a(H^r)b \otimes c(H^r)d \to a(H^r)d \quad \text{if } b = c, \tag{5}$$

explicitly computed by a sequence of saddle maps taking $\overline{b} \sqcup c \rightsquigarrow Id_{2r}$.

A good way to think about this multiplication structure is that Equation (5) is giving a "target" for where to land. We emphasize this because it's best to think of this as a "codomain assignment" as the actual mappings will vary case-by-case. Refer to Figure 11 for an example.

Proposition 31 ([Kho02b], Proposition 1)**.** The collection

$$H^r = \bigoplus_{a,b \in \mathfrak{C}^r} a(H^r)b$$

is a ring.

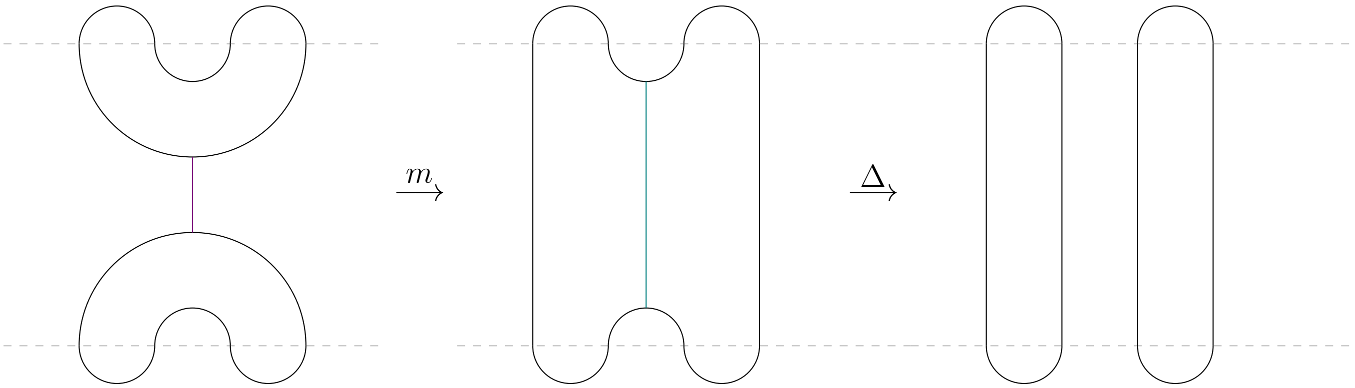


FIGURE 11. The multiplication structure $a(H^2)b \otimes b(H^2)a \to a(H^2)a$. Explicitly, this map is given by $\Delta m(1 \otimes 1) = 1 \otimes X + X \otimes 1$, $\Delta m(1 \otimes X) = \Delta m(X \otimes 1) = X \otimes X$, and $\Delta m(X \otimes X) = 0$.

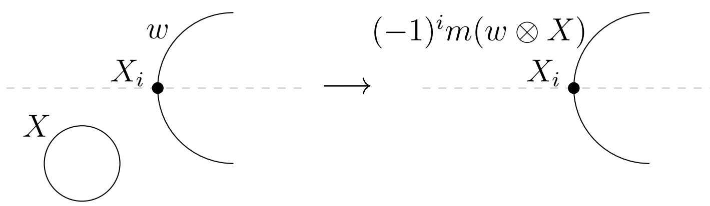


FIGURE 12. Arc algebra basepoint action.

Note that in the same paper, Khovanov generalized this construction to tangles. In particular, given an $(2m, 2r)$-tangle T (where $m, r \geq 0$) we obtain a (H^m, H^r) bi-module. Furthermore, when $T = Id_r$, then the resulting (H^r, H^r) bi-module is simply H^r.

Before proceeding, we must explain some notation. Let D denote the diagram of $a(H^r)b$. On D, label each of the points we are matching by $1, \ldots, 2n$. Near point i, introduce an unknot U, labeled by some $v \in \mathcal{A}$. The action of v on D is induced by the multiplication map (saddle) taking $D \sqcup U \to D$. This is called the "basepoint action", described in [Kho03], [HN13], and [BLS17].

In particular, Manolescu–Neithalath claim that this action allows us to view H^r as an algebra over the module

$$\mathbb{Z}[X_1, \ldots, X_{2r}]/(X_1^2, \ldots, X_{2r}^2),$$

where the variable X_i is induced by the basepoint action at point i. Namely, if a planar circle of D containing point i is labeled with $w \in \mathcal{A}$, then X_i is given by

$$(-1)^i m(w \otimes X).$$

Refer to Figure 12 for a helpful cartoon.

Finally, we may turn to a result that will be crucial in computing the partial skein lasagna module of disk bundles.

Theorem 32 ([Kho02a], Theorems 1 and 2)**.** The center of the ring H^r is isomorphic to

$$Z(H^r) \cong \mathbb{Z}[X_1, \ldots, X_{2r}]/(X_i^2, \Sigma_{|I|=k} X_I),$$

for $i, k = 1, \ldots, 2n$, where the terms X_I are the product of the X_i's across all cardinality k subsets $I \subseteq \{1, \ldots, 2r\}$. Moreover, $\mathrm{gr_q}(X_i) = 2$ for all i.

Theorem 33 ([Sto09], Proposition 1)**.** Let Z_j^r denote the component of $Z(H^r)$ at quantum grading j. Then

$$Z_{2k}^r \cong \mathbb{Z}^{\binom{2r}{k}-\binom{2r}{k-1}}$$

for $k = 0, \ldots, r$, and is zero for all other quantum gradings. Furthermore, $Z(H^r)$ has total rank $\binom{2r}{r}$.

Along with the previous results, we will describe a nice method for finding a basis of Z_{2k}^r, originating from Khovanov's work.

Definition 34 ([Kho02a])**.** Let $r \geq 0$. A subset $I \subseteq \{1, \ldots, 2r\}$ is called *admissible* if

$$|I \cap \{1, \ldots, m\}| \leq \frac{m}{2}$$

for each $m = 1, \ldots, 2r$. Furthermore, there are at most $\binom{2r}{r}$ admissible subsets.

Additionally, using Manolescu–Neithalath's notation, let

$$A_k^r = \{I \subseteq \{1, \ldots, 2r\} \mid I \text{ is admissible, } |I| = k\},$$

for each $k = 1, \ldots, r$.

Example 35. Observe that, trivially, for all $r \geq 0$, the only element of A_0^r is the empty set $\varnothing$. Now consider $r = 1$. Then $I \subseteq \{1, 2\}$, so we have to check the subsets $\varnothing$, $\{1\}$, $\{2\}$, $\{1, 2\}$. Additionally, we know that there are $\binom{2}{1} = 2$ admissible subsets, one of which must be the empty set. By considering the case $m = 1$, we immediately see that $\{1\}, \{1, 2\}$ are not admissible subsets because

$$|\{1\} \cap \{1\}| = |\{1, 2\} \cap \{1\}| = 1 > 1/2.$$

So the only other admissible subset must be $\{2\}$, meaning that

$$A_0^1 = \{\varnothing\}, \quad A_1^1 = \{\{2\}\}.$$

For $r = 2$, similar computations give the collection of admissible subsets

$$A_0^2 = \{\varnothing\},\ A_1^2 = \{\{2\}, \{3\}, \{4\}\},\ A_2^2 = \{\{2, 4\}, \{3, 4\}\}.$$

Thinking about admissible subsets, in this context, is important as we have the following result from Khovanov (as interpreted by Manolescu–Neithalath).

Proposition 36 ([Kho02a], Lemma 8)**.** A basis for the component Z_{2k}^r, $k \geq 0$, is given by the elements X_I, where $I \in A_k^r$.

For later use, we will also want to consider the dual center, $Z(H^r)^\vee$. Recall that, formally, $Z(H^r)^\vee$ is the hom-space $\mathrm{Hom}(Z(H^r); \mathbb{Z})$. This space was explored and formally described in [MN22]. Following our notation for the center, let $(Z_j^n)^\vee$ denote the component of the dual center at quantum grading j. As the X_I are basis elements for the center, we will denote dual basis elements by $X_I^\vee$, subject to the relation

$$X_I^\vee(X_J) = \begin{cases} 1 & I = J \\ 0 & I \neq J \end{cases}.$$

Additionally, note that as Proposition 36 gives us a description of the basis elements for Z_{2k}^r, then we only need to focus on dualizing these basis element to get a sense of $(Z_{2k}^r)^\vee$.

Furthermore, Manolescu–Neithalath define a multiplication on the dual center: $\oplus_k (Z_k^n)^\vee \times \oplus_k (Z_k^n)^\vee \to \oplus_k (Z_k^n)^\vee$ by

$$X_I^\vee \cdot X_J^\vee = \begin{cases} X_{I\cup J}^\vee & I \cap J = \varnothing \\ 0 & \text{otherwise} \end{cases}, \tag{6}$$

and a contraction operation $(Z_\ell^n)^\vee \times Z_k^n \to Z_{k-l}^n$ given by

$$X_I^\vee(X_J) = \begin{cases} X_{J-I} & I \subseteq J \\ 0 & \text{otherwise} \end{cases}. \tag{7}$$

Furthermore, one may show that multiplication and contraction are associative, in the sense that:

$$(X_I^\vee \cdot X_J^\vee)(X_K) = X_I^\vee(X_J^\vee(X_K)).$$

3.5. **Khovanov Homology from Hochschild (Co)Homology.** In this section, we will go over Rozansky's Equation from [Roz10] showing that Khovanov homology can be computed from Hochschild (co)homology. While we will not go into details on Hochschild (co)homology, the following result will be useful later on.

Theorem 37. Given a ring R, the zero-th Hochschild cohomology ring is $HH^0(R) \cong Z(R)$.

For interested readers, see Chapter 9 of [Wei94] for more.

Let $L \subset S^2 \times S^1$ be a link obtained by taking the circular closure of a $(2n, 2n)$-tangle T. Since we can obtain $S^2 \times S^1$ by performing 0-surgery on the unknot, then we may draw L as a link "wearing" a 0-framed unknot as a "belt". Following [Roz10], the Khovanov homology of L is defined by

$$\mathrm{Kh}^{i,*}(L; S^2 \times S^1) := HH_i(\mathcal{F}(T)),$$

where $HH_*(F(T))$ is the Hochschild homology of $\mathcal{F}(T)$. Furthermore, in the same paper, Rozansky showed that

$$\mathrm{Kh}^{i,*}(L; S^2 \times S^1) \cong \mathrm{Kh}^{i,*}(L^p; S^3), \quad i \geq n_+ - 2p + 2, \tag{8}$$

where $L^p \subset S^3$ can be drawn as L with an extra p-full twists (and no belt). In particular, we may think of L^p as the composition of T and $FT_{2n}^{\otimes p}$. Observe that by the previous equation, we have that

$$\mathrm{Kh}^{i,*}(L^p; S^3) \cong HH_i(\mathcal{F}(T)), \quad i \geq n_+ - 2p + 2.$$

4. Skein lasagna modules

Let W be a 4-manifold and $L \subset \partial W$ a link. Morrison–Walker–Wedrich's *skein lasagna module* takes the pair (W, L) as an input and outputs a collection of tri-graded $\mathbb{Q}$-vector spaces, i.e.

$$\mathcal{S}_{0,i,j}^2(W; L) = \bigoplus_{\alpha \in H_2(X,L)} \mathcal{S}_{0,i,j}^2(W; L, \alpha).$$

Note that the tri-grading $(i, j, \alpha) \in \mathbb{Z}^2 \times H_2(W, L)$ consists of the usual homological and quantum gradings (i, j resp.) and the homological level α. Additionally, note that when $L = \varnothing$, the empty link, then $H_2(W; L) \cong H_2(W)$. We now formally describe the construction of the Khovanov skein lasagna module, following [MWW22].

Definition 38. A *lasagna filling* $\mathcal{F}$ of the input pair (W, L) (as described above) consists of the data $(\Sigma, \{B_i, L_i, v_i\}_i)$:

- Σ is a framed, oriented surface properly embedded in $W - \bigcup_i \operatorname{int}(B_i)$ with boundary $\partial\Sigma = L \sqcup \amalg_i L_i$,
- the B_i are a collection of 4-balls embedded in W (*input balls*) with $L_i \subset \partial B_i \cong S^3_i$ (*input links*),
- the label v_i is a generator of a class in $\mathrm{Kh}(L_i; \mathbb{Q})$.

Note that, alternatively, we can think of a lasagna filling of (W, L) as a *skein* (the first two points in the definition above) together with a collection of Khovanov generators associated to the input links.

Definition 39. The *Khovanov skein lasagna module* of (W, L) is the $\mathbb{Q}$-vector space

$$\mathcal{S}^2_0(W; L) = \mathbb{Q}\langle \text{ all lasagna fillings } \mathcal{F} \text{ of } (W, L)\rangle / \sim,$$

where $\sim$ is the equivalence relation given by the following.

- Linear combinations of lasagna fillings are multilinear in the labels v_i.
- Enclosement: given two lasagna fillings $\mathcal{F} = (\Sigma, \{B_i, L_i, v_i\})$ and $\mathcal{F}' = (\Sigma', \{B'_i, L'_i, v'_i\})$ such that the data $(B_i, L_i, v_i) = (B'_i, L'_i, v'_i)$ for all $i \neq 1$. For $i = 1$, we have $B'_1 \subset B_1$ and $\Sigma = \Sigma' - B_1$ up to isotopy rel $\partial\Sigma$. Now, let $\Sigma'' = \Sigma' - \Sigma$. If $\mathrm{KhR}_2(\Sigma'')(v'_i) = v_i$, then we consider $\mathcal{F} \sim \mathcal{F}'$. Here, we may also consider Σ up to isotopy rel boundary.

Figure 13 may be helpful in understanding the enclosement relation.

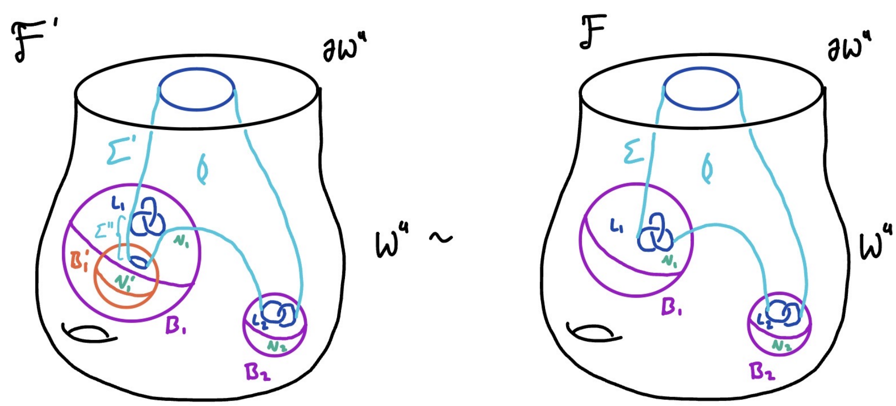


FIGURE 13. Enclosement relation for $\mathcal{S}^2_0(W; L)$.

Remark 40. From now on, we will work with Khovanov homology over rational coefficients, $\mathrm{Kh}(L; \mathbb{Q})$. Most results will be stated in terms of integral Khovanov homology, keeping in line with their original statement, but it should be understood that the result for rational Khovanov homology will be similar. Refer to Equation (3).

4.1. **Cabled Khovanov Homology.** Unfortunately, the skein lasagna module is often not computable from the definition. If we restrict to the class of 4-manifolds which are 2-handlebodies (with the 2-handle attached along a framed link K) and have no link in their boundary, then we may use Manolescu–Neithalath's ([MN22]) *cabled Khovanov homology*, $\underline{\mathrm{KhR}}_2(K; \mathbb{Q})$, along with the following result to compute $\mathcal{S}^2_0(W, \varnothing)$.

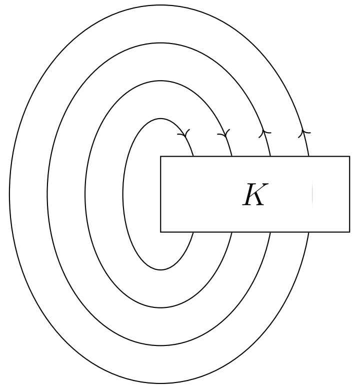


FIGURE 14. This is the $(2,2)$ cable of an arbitrary knot K.

The main result we aim to discuss in this section is the following, from Manolescu–Neithalath.

Theorem 41 ([MN22], Thm. 1.1)**.** Let W be the 4-manifold obtained from attaching 2-handles to B^4 along a framed n-component link K. For each $\alpha \in H_2(W;\mathbb{Z}) \cong \mathbb{Z}^n$ we have an isomorphism

$$\Phi : \underline{\mathrm{KhR}}_{2,\alpha}(K) \to \mathcal{S}_0^2(W;\varnothing,\alpha).$$

In order to define the cabled Khovanov homology, consider a framed oriented link $K \subset S^3$ with components $K_1, \dots K_n$. This link K defines the attaching region for our 2-handle. For the tuples $k^+ = (k_1^+, \dots, k_n^+)$ and $k^- = (k_1^-, \dots, k_n^-)$, let $K(k^-, k^+)$ denote the framed oriented $k^- + k^+$-component link obtained by taking $k_i^- + k_i^+$ parallel copies of each K_i (see Figure 14). Note that each K_i inherits a framing from the framing on K. Additionally, our convention will be to negatively orient the k_i^- strands (clockwise) and positively orient the k_i^+ strands (counter-clockwise).

We will also briefly describe the equivalence relations that will be used in Definition 42. Consider the braid group $B_{k^-+k^+}$. We can interpret a braid $b \in B_{k^-+k^+}$ as a cobordism embedded in $B^2 \times [0,1]$. Taking this product with S^1, we now get a cobordism Σ_b in a "thickened torus" $S^1 \times B^2 \times [0,1]$. By identifying $S^1 \times B^2 \times \{0,1\}$ with a tubular neighborhood of the link $K(k^-,k^+)$, then we may realize Σ_b as a cobordism in $S^3 \times [0,1]$ from $K(k^-,k^+)$ to itself, i.e.

$$\Sigma_b \subset S^1 \times B^2 \times [0,1] \subset S^3 \times [0,1].$$

Then for all $b \in B_{k^-+k^+}$, this cobordism, Σ_b (see Figure **??**), induces an automorphism on the Khovanov homology of $K(k^-,k^+)$:

$$\beta(b) := \mathrm{Kh}(\Sigma_b) : \mathrm{Kh}(K(k^-,k^+)) \to \mathrm{Kh}(K(k^-,k^+)). \tag{9}$$

Additionally, we have the (un-)dotted annulus cobordisms $\psi^{[0]}$ and $\psi^{[1]}$. Let $e_i \in \mathbb{Z}^n$, where $i = 1, \dots, n$, denote a standard basis vector. This cobordism, $Z \subset S^3 \times [0,1]$, maps a link $K(k^-,k^+) \sqcup U$ to $K(k^- + e_i, k^+ + e_i)$, introducing two new, oppositely oriented cables to the component K_i. As before, we obtain a linear map

$$\mathrm{Kh}(Z_i) : \mathrm{Kh}(K(k^-,k^+) \sqcup U) \to \mathrm{Kh}(K(k^- + e_i, k^+ + e_i)).$$

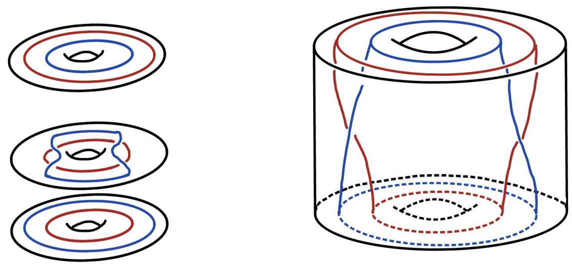


FIGURE 15. Cobordism induced by the braid group action (read bottom to top). On the right is a visualization of this cobordism as a "movie" of toruses; on the left is a visualization as a self-intersecting surface, drawn a dimension down (as it would only be properly embedded in dimension four).

Recall that, since Khovanov homology is a TQFT,

$$\mathrm{Kh}(K(k^-,k^+) \sqcup U) \cong \mathrm{Kh}(K(k^-,k^+)) \otimes \mathrm{Kh}(U).$$

Thus, by pre-composing with the inclusion map, we can encode the information in $\mathrm{Kh}(Z_i)$ to get the maps

$$\varphi, \psi : \mathrm{Kh}(K(k^-,k^+)) \to \mathrm{Kh}(K(k^- + e_i, k^+ + e_i)),$$

defined by

$$\varphi(v) = \mathrm{Kh}(Z_i)(v \otimes 1) \quad \text{and} \quad \psi(v) = \mathrm{Kh}(Z_i)(v \otimes X). \tag{10}$$

Recall that Theorem 41 defines an isomorphism for each homological level $\alpha \in H_2(W;\mathbb{Z}) \cong \mathbb{Z}^n$. For $\alpha = (\alpha_1, \dots, \alpha_n)$, let $\alpha^+, \alpha^- \in \mathbb{Z}^n$ denote the positive and negative components of α, respectively. Additionally, let $|\alpha| = \sum_{i=1}^n |\alpha_i|$. For example, if $\alpha = (-2,1,0) \in \mathbb{Z}^3$, then $\alpha^+ = (0,1,0)$, $\alpha^- = (-2,0,0)$, and $|\alpha| = 3$.

Definition 42 ([MN22], Definition 3.4)**.** The *cabled Khovanov homology* of K at homological level α is

$$\underline{\mathrm{KhR}}_{2,\alpha}(K) = \left(\bigoplus_{r \in \mathbb{N}^n} \mathrm{KhR}_2(K(r-\alpha^-,\ r+\alpha^+))\{-2r-|\alpha|\} \right) / \sim, \tag{11}$$

where $\sim$ is the transitive and linear closure of the following relations. Denote by v a generator of $\mathrm{KhR}_2(K(r-\alpha^-, r+\alpha^+))$.

- For all $b \in B_{r-\alpha^-,\ r+\alpha^+}$, the action $\beta(b)v \sim v$.
- Under the action of un-dotted annulus cobordism, $\varphi(v) \sim 0$.
- Under the action of the dotted annulus cobordism, $\psi(v) \sim v$.

Remark 43. For a given link L, we will refer to $\mathrm{KhR}_2(L)/(\beta(b)v \sim v)$ as the *symmetrized Khovanov homology* of L. To make notation more compact, we will denote the symmetrized Khovanov homology of L by $\mathrm{Sym}(\mathrm{Kh}(L))$.

Note that by using Equation (4) and Proposition 44, we can simplify the construction of cabled Khovanov–Rozansky homology.

Proposition 44 ([GLW18], Prop. 9)**.** The braid group action, as endomorphisms of $\mathrm{Kh}(K(k^-,k^+))$, is given by the equation

$$\beta(\sigma_i) = \mathrm{id} + \mathrm{Kh}(\hat{Z}(i)) \circ \mathrm{Kh}(\hat{Z}^r(i)) = \beta(\sigma_i^{-1}) \tag{12}$$

Remark 45. The exact statement of Proposition 44 in this article follows Manolescu–Neithalath's interpretation, as Proposition 3.6. in [MN22].

The importance of Proposition 44 is that since the actions

$$\beta(\sigma_i) = \beta(\sigma_i^{-1})$$

are equal on $\mathrm{Kh}(K(k^-, k^+))$, then this action factors thorough the entire symmetric group $S_{k^-+k^+}$. In contrast for general KhR_N with $N \neq 2$, we can only guarantee that the braid group action $\beta(\sigma_i)$ factors through $S_{k^-} \times S_{k^+}$. Together with Equation (4), the previous proposition allows us to simplify the construction of cabled Khovanov homology.

Proposition 46 ([MN22], Prop. 3.8)**.** The cabled Khovanov homology over $\mathbb{Q}$ of a framed knot K at homological level α is

$$\underline{\mathrm{KhR}}_{2,\alpha}(K;\mathbb{Q}) = \left(\bigoplus_{r\in\mathbb{N}} \mathrm{Kh}(K(r-\alpha_-, r+\alpha_+);\mathbb{Q})\{-2r-|\alpha|\}\right)/\sim,$$

where $\sim$ is the transitive and linear closure of the relations

$$\beta(b)v \sim v, \quad \psi(v) \sim v.$$

for all $b \in S_{2r+|\alpha|}$ and $v \in \mathrm{Kh}(K(r-\alpha_-, r+\alpha_+);\mathbb{Q})$.

Remark 47. In [MN22], the previous proposition was defined over any field $\Bbbk$ with $2 \in \Bbbk^\times$. Of course, some examples of this are $\mathbb{Q}$ and $\mathbb{Z}/3$. This restriction is imposed in order to ensure that Equation (12) is an isomorphism. Furthermore, we fix $\Bbbk = \mathbb{Q}$ to be consistent with the original construction of the skein lasagna module as a $\mathbb{Q}$-vector space.

Note the differences between this proposition and Definition 42. Firstly, we no longer have the un-dotted annulus relation $\varphi(v) \sim v$. Roughly, this relation is "absorbed" into the relation $\beta(b)v \sim v$. For a more in depth explanation, refer to [MN22].

The following result from Manolescu–Neithalath is also often useful when computing cabled Khovanov homology.

Proposition 48 (Proposition 2.1 in [MN22])**.** Let W, W' be 4-manifolds. The inclusion map $i : W \to W'$ induces a natural map on the skein lasagna module:

$$i_* : \mathcal{S}_0^2(W;\varnothing) \to \mathcal{S}_0^2(W';\varnothing).$$

In particular, if W' is the result of attaching a 3-handle (resp. 4-handle) to W, then i_* is a surjection (resp. isomorphism).

4.2. **Example: Skein Lasagna Module of $S^2 \times B^2$.** In this section, we will compute the skein lasagna module of $S^2 \times B^2$. While originally computed in [MN22] for general KhR_N, here will will do this computation strictly using Khovanov homology. This example is of particular interest as it is the only skein lasagna module which, to date, has been computed solely using the definition of cabled Khovanov homology.

Theorem 49 ([MN22], Thm. 1.2)**.** The skein lasagna module of $S^2 \times B^2$ is

$$\mathcal{S}_0^2(S^2 \times B^2;\varnothing) \cong \mathbb{Q}[x, x^{-1}, y].$$

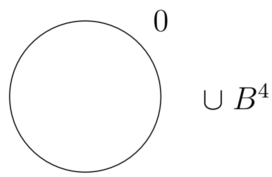


FIGURE 16. Kirby diagram of $S^2 \times B^2$

Recall that $S^2 \times B^2$ is the 2-handlebody with Kirby diagram given by the unknot with zero-framing, U^0, capped off with a B^4 (Figure 16). Note that by Proposition 48, we need only examine $S^2 \times B^2$ as a 2-handlebody.

From the definition of cabled Khovanov homology (42), for $\alpha \in H_2(S^2 \times B^2) \cong \mathbb{Z}$, we know that

$$\underline{\mathrm{KhR}}_{2,\alpha}(U^0) = \left(\bigoplus_{r \in \mathbb{N}} \mathrm{KhR}_2(U^0(r-\alpha^-, r+\alpha^+))\{-2r - |\alpha|\} \right) / \sim .$$

We start by proving some claims on the symmetrized Khovanov–Rozansky homology of the link $U^0(k^-, k^+)$.

Claim 50. The Khovanov–Rozansky homology of U^0 is supported only in homological degree $\mathrm{gr_h} = 0$. Furthermore, $\mathrm{KhR}_2(U^0(k^-, k^+)) \cong \mathcal{A}^{\otimes(k^- + k^+)}$, up to global shifts in quantum grading.

Proof. We will work over Khovanov homology, before using Equation 4 to transition to Khovanov–Rozansky homology.

Since the link $U^0(k^-, k^+)$ has no crossings and is, trivially, a planar resolution with $k^- + k^+$ circles to begin with, then the Khovanov chain complex is

$$0 \to \mathcal{A}^{\otimes(k^- + k^+)} \to 0,$$

where the differentials are zero maps. Additionally, since we have no crossings, then there are no global grading shifts in our computation and no writhe. Thus, the Khovanov homology of $U^0(k^-, k^+)$ is

$$\mathrm{Kh}(U^0(k^-, k^+)) = \mathcal{A}^{\otimes(k^- + k^+)}.$$

By 4, we know that $\mathrm{KhR}_2^{0,j}(U^0(k^-, k^+)) \cong \mathrm{Kh}^{0,-j}(U^0(k^-, k^+))$. Observe that this shift in quantum grading "flips" each chain group in the Khovanov complex because the gradings of our generators are shifted to

$$\mathrm{gr_q}(1) = -1, \quad \mathrm{gr_q}(X) = 1.$$

$\square$

For later use, observe that we get the resulting rational Khovanov homology

$$\mathrm{Kh}(U^0(k^-, k^+); \mathbb{Q}) \cong (\mathbb{Q}[X]/(X^2))^{\otimes(k^- + k^+)} \tag{13}$$

The next step is to symmetrize the Khovanov-Rozansky homology which we just computed. Note that our choice of (unoriented) link diagram of $U^0(k^-, k^+)$ is a collection of $k^- + k^+$ nested circles. As such, our convention will be that the innermost circle is labeled "1" and the outermost circle is labeled "$k^- + k^+$".

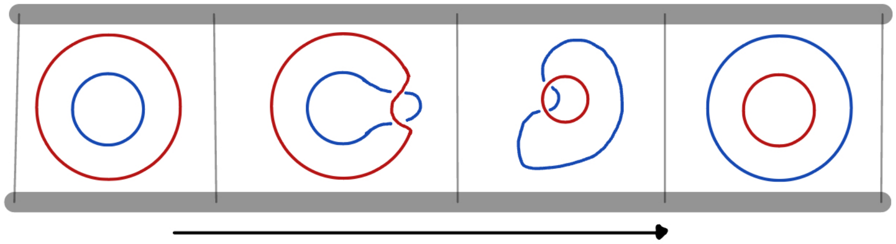


FIGURE 17. The cobordism Σ_{σ_i}. In the first frame, the blue strand is i and the red strand is $i+1$. We see that this cobordism is given by RII, an isotopy, and then a RII-inverse. Label the frames (diagrams) by D_1, D_2, D_3, D_4, from left to right.

Claim 51. Let Σ_{σ_i} denote the cobordism on $U^0(k^-, k^+)$, corresponding to the generator $\sigma_i \in B_{k^-+k^+}$ (which swaps the i-th and $(i+1)$-st stands). As an endomorphism on Khovanov homology, $\beta(\sigma_i) : \mathrm{Kh}(U^0(k^-, k^+)) \to \mathrm{Kh}(U^0(k^-, k^+))$ is defined by

$$1 \otimes 1 \to 1 \otimes 1, \; 1 \otimes X \to -X \otimes 1, \; X \otimes 1 \to -1 \otimes X, \; X \otimes X \to X \otimes X.$$

Moreover, the actions $\beta(\sigma_i)$ for $i = 1, \ldots, k^- + k^+ - 1$ generate the entire action of the symmetric group $S_{k^-+k^+}$ on $\mathrm{Kh}(U^0(k^-, k^+))$.

Proof. Note that throughout the first part of the proof, we will isolate the i-th and $(i+1)$-st strands of $U^0(k^-, k^+)$, before generalizing to the entire $k^- + k^+$ cable of U^0. Denote these two strands by U_i. Additionally, note that we may work with unoriented strands. [2]

As a cobordism from U_i to itself, Σ_{σ_i} may be visualized via the movie in Figure 17. Denote the corresponding Khovanov homologies by $\mathrm{Kh}(D_1), \ldots, \mathrm{Kh}(D_4)$. By using the chain maps (and chain homotopy) from Bar-Natan's proof of invariance under Reidemeister II, we see that Σ_{σ_i} induces the following maps.

- The map $\mathrm{Kh}(D_1) \to \mathrm{Kh}(D_2)$ is given by $((\mathrm{id} \otimes \iota) \circ m) \oplus \mathrm{id}$, which acts on generators by

$$\begin{aligned} 1 \otimes 1 &\mapsto (1 \otimes 1, \; 1 \otimes 1) \\ 1 \otimes X &\mapsto (X \otimes 1, \; 1 \otimes X) \\ X \otimes 1 &\mapsto (X \otimes 1, \; X \otimes 1) \\ X \otimes X &\mapsto (0, X \otimes X). \end{aligned}$$

- The map $\mathrm{Kh}(D_2) \to \mathrm{Kh}(D_3)$ is the identity on all components.
- The map $\mathrm{Kh}(D_3) \to \mathrm{Kh}(D_4)$ is given by $-(\Delta \circ (\varepsilon \otimes \mathrm{id})) + \mathrm{id}$. In particular, the first component of this map acts on generators by

$$\begin{aligned} 1 \otimes 1 &\mapsto 0 \\ 1 \otimes X &\mapsto 0 \\ X \otimes 1 &\mapsto -1 \otimes X - X \otimes 1 \\ X \otimes X &\mapsto -X \otimes X \end{aligned}$$

[2]Refer to Remark 30 on fixing the functoriality of unoriented Khovanov homology.

The composition of all three maps gives us the action $\beta(\sigma_i)$ on Khovanov homology:

$$\begin{aligned} 1\otimes 1 &\mapsto 1\otimes 1\\ 1\otimes X &\mapsto -X\otimes 1\\ X\otimes 1 &\mapsto -1\otimes X\\ X\otimes X &\mapsto X\otimes X \end{aligned}$$

Importantly, observe that this action preserves quantum grading.

From here, let v_j, for $j = 1, \dots, k^- + k^+$ denote the distinguished generator assigned to planar circle (i.e. strand) i. Then our computation tells us that

$$v_1 \otimes \cdots \otimes 1_i \otimes X_{i+1} \otimes \cdots \otimes v_{k^-+k^+} \sim -(v_1 \otimes \cdots \otimes X_i \otimes 1_{i+1} \otimes \cdots \otimes v_{k^-+k^+}), \tag{14}$$

with all other bits fixed. Furthermore, the equivalences on $1_i \otimes 1_{i+1}$ and $X_i \otimes X_{i+1}$ is a self-identification.

From Proposition 44, we know that for Khovanov homology the action of the braid group B_{k^-,k^+} factors through the symmetric group $S_{k^-+k^+}$. Furthermore, we know that the symmetric group is generated by adjacent transpositions $(i \quad i+1)$ for $i = 1, \dots, k^- + k^+ - 1$. Using these facts, we provide an algorithm to compute $\mathrm{Sym}(\mathrm{Kh}(U^0(k^-, k^+)))$.

(1) Consider the set of generators of $\mathrm{Kh}(U^0(k^-, k^+))$ at a fixed quantum grading $\mathrm{gr_q}$.
(2) Starting by considering the action of the transposition $(k^-+k^+-1 \quad k^-+k^+)$, identify generators according to Equation 14 in pairs.
(3) Repeat (2) for each transposition $(i \quad i+1)$ for all $i = 1, \dots, k^- + k^+ - 2$.

This should result in all generators at quantum grading $\mathrm{gr_q}$ being identified together, meaning that we may make a (non-canonical) choice of generator to represent all generators with grading $\mathrm{gr_q}$. Thus, we may describe the symmetrized Khovanov homology by

$$\mathrm{Sym}(\mathrm{Kh}(U^0(k^-, k^+)); \mathbb{Q}) \cong \bigoplus_{j=0}^{k^-+k^+} \mathbb{Q}_{2j},$$

where $\mathbb{Q}_j$ is $\mathbb{Q}$ spanned by a single generator at quantum grading $\mathrm{gr_q} = j$.

□

For the next claim, we will restrict to homological levels $0, 1 \in H_2(S^2 \times B^2)$.

Claim 52. The map $\psi : \mathrm{Sym}(\mathrm{Kh}(U^0(k^-, k^+)); \mathbb{Q}) \to \mathrm{Sym}(\mathrm{Kh}(U^0(k^- + 1, k^+ + 1)); \mathbb{Q})$ is given by

$$\psi(v) = v \otimes X \otimes X$$

Furthermore, $\underline{\mathrm{KhR}}^{0,j}_{2,\alpha}(U^0; \mathbb{Q}) \cong \mathbb{Q}[x]$, where x has bi-grading $(0, 2j)$.

Proof. In the Bar-Natan category, the dotted cobordism taking $U^0(k^-, k^+) \to U^0(k^-+1, k^++1)$ is depicted in Figure 18. Algebraically, this cobordism is given by

$$\mathrm{id}^{\otimes k^-+k^+} \otimes (\Delta \circ \iota),$$

where $\mathrm{id}^{\otimes k^-+k^+}$ denotes $k^- + k^+$ copies of the identity map.

By using the neck-cutting relation on the $\Delta \circ \iota$ component, we get the desired mapping $v \to v \otimes X \otimes X$.

We now want to use this to compute the cabled Khovanov homology $\underline{\mathrm{KhR}}^{0,j}_{2,\alpha}(U^0; \mathbb{Q})$. In particular, at homological level $\alpha = 0$, links are of the form $U^0(r, r)$, where $r \in \mathbb{N}$.

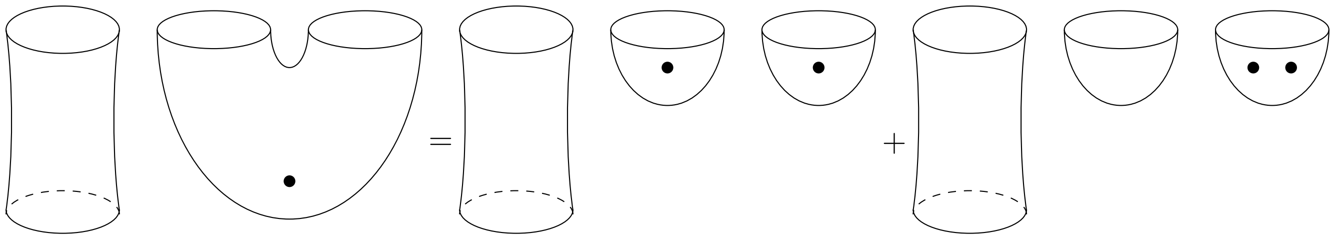


Figure 18. Cobordism corresponding to the colimit map. Note that the second term on the right side vanishes because the cup has two dots.

Additionally, from the previous claim, we know that $\mathrm{Sym}(\mathrm{Kh}(U^0(r,r));\mathbb{Q}) \cong \bigoplus_{j=0}^{r} \mathbb{Q}_{2j}$. So the cabled Khovanov homology of U^0 at homological level $\alpha = 0$ is the colimit of the diagram

$$\mathbb{Q} \xrightarrow{\psi} \mathbb{Q}_0 \oplus \mathbb{Q}_2 \oplus \mathbb{Q}_4 \xrightarrow{\psi} \mathbb{Q}_0 \oplus \mathbb{Q}_2 \oplus \mathbb{Q}_4 \oplus \mathbb{Q}_6 \oplus \mathbb{Q}_8 \xrightarrow{\psi} \cdots$$

where the first term $\mathbb{Z}$ is the symmetrized Khovanov homology of the empty link.

Since these maps preserve quantum grading, then inductively we can see that

$$\mathrm{Sym}(\mathrm{Kh}(U^0(r,r));\mathbb{Q}) \hookrightarrow \mathrm{Sym}(\mathrm{Kh}(U^0(r+1,r+1));\mathbb{Q})$$

for all $r \in \mathbb{N}$. Taking $r \to \infty$, we get the desired result

$$\underline{\mathrm{KhR}}_{2,0}^{0,j}(U^0;\mathbb{Q}) \cong \bigoplus_{r\in\mathbb{N}} \mathbb{Q}_{2r} \cong \mathbb{Q}[x].$$

The case at homological level $\alpha = 1$ is similar, although the directed system looks slightly different. ☐

Observe that we only need to check homological levels $\alpha = 0, 1$ because the skein lasagna module is defined over all homological levels $\alpha \in H_2(S^2 \times B^2)$. By summing over all homological levels $\alpha \in H_2(S^2 \times B^2) \cong \mathbb{Z}$, follows from the previous claim that

$$\mathcal{S}_0^2(S^2 \times B^2; \varnothing) = \bigoplus_{\alpha \in H_2(S^2\times B^2)} \mathcal{S}_0^2(S^2 \times B^2, \varnothing; \alpha) \cong \mathbb{Q}[y, y^{-1}, x],$$

where x has tri-grading $(0, 2, 1)$ and y has tri-grading $(0, 0, 1)$.

4.3. **Example: Skein Lasagna Module of** $B(p)$**.** Let $B(p)$ denote the B^2-bundle over S^2 with Euler number p. The main goal of this section will be to discuss and provide a (relatively) short proof of the following result.

Theorem 53 ([MN22], Proposition 6.1)**.** For $p > 0$, we have that

$$\underline{\mathrm{KhR}}_{2,0}^{0,j}(U^p) = 0$$

for all $j \in \mathbb{Z}$. On the other hand, for $p < 0$, we have

$$\underline{\mathrm{KhR}}_{2,0}^{0,j}(U^p) = \begin{cases} \mathbb{Q} & \text{if } j = 0 \\ 0 & \text{otherwise} \end{cases}.$$

Throughout, let $L \subset S^2 \times S^1$ be a link obtained by taking the circular closure of the tangle Id_{2r} in $S^2 \times S^1$. By Rozansky's Equation (8), we know that

$$\mathrm{Kh}^{i,j}(S^3; L^p) \cong HH_i(H^r), \quad i \geq n_+ - 2p + 2,$$

where L^p is the circular closure of the tangle $FT_{2r}^{\otimes p}$ with an extra $2rp$ twists so that $w(L^p) = 0$, and we recall that H^r is Khovanov's arc algebra on $2r$ points. Observe that then $r_+ = 0$, so to work at homological level $i = 0$, we must have that $p > 0$.

Claim 54. The Khovanov–Rozansky homology of L^p at bigrading $(0, j) \in \mathbb{Z}^2$ is

$$\mathrm{KhR}_2^{0,j}(L^p) \cong \begin{cases} Z^r_{2r+j} & p > 0 \\ (Z^r_{2r-j})^\vee & p < 0 \end{cases},$$

where Z^r_{2r+j} (resp. $(Z^r_{2r-j})^\vee$) is the quantum grading $2r + j$ (resp. $2r - j$) component of the center (resp. dual center) of Khovanov's arc algebra H^r.

Proof. First, consider the case for $p < 0$,

$$\mathrm{KhR}_1^{i,j}(L^p) \cong \mathrm{Kh}^{i,-j}(-L^p) \cong \mathrm{Kh}^{i,-j}(L^{-p}).$$

Since $-p > 0$, then by Rozansky's Equation we have

$$\mathrm{Kh}^{0,-j}(L^{-p}) \cong HH_0(H^r).$$

By Equation 6.13 from [MMSW23], we know that

$$HH_0(H^r) \cong HH^0(H^r)^\vee\{-2r\},$$

so we get that

$$\mathrm{Kh}^{0,-j}(L^{-p}) \cong (Z^r_{2r-j})^\vee$$

by Theorem 37.

Now, when $p > 0$. By 4, we know that

$$\mathrm{KhR}_2^{i,j}(L^p) \cong \mathrm{Kh}^{i,-j}(-L^p) \cong \mathrm{Kh}^{i,-j}(L^{-p}).$$

From Rozansky's Equation (8) and 37, then at homological grading $i = 0$

$$\mathrm{Kh}^{0,-j}(L^{-p}) \cong HH^0(H^r)\{-2n\} \cong Z^r_{2r+j}.$$

□

The following lemma will also be useful for our final computation.

Lemma 55 ([MN22], Proposition 6.14, 6.15)**.** For $p > 0$, the colimiting maps in $\underline{\mathrm{KhR}}_{2,0}^{0,j}(U^p)$ are given by

$$\varphi : Z(H^r)_{2r+j} \to Z(H^{r+1})_{2r+2+j}, \quad X_I \mapsto \pm X_I(X_{2r+2} - X_{2r+1}) \tag{15}$$

$$\psi : Z(H^r)_{2r+j} \to Z(H^{r+1})_{2r+4+j}, \quad X_I \mapsto \pm X_I(X_{2r+1}X_{2r+2}). \tag{16}$$

For $p < 0$, the maps are given by

$$\varphi' : Z(H^r)^\vee_{2r-j} \to Z(H^{r+1})^\vee_{2r+2-j}, \quad \varphi'(f)(X_I) = \pm \begin{cases} f(X_J) & I = J \cup \{2r+2\} \\ -f(X_J) & I = J \cup (2r+1\} \\ 0 & \text{otherwise} \end{cases} \tag{17}$$

$$\psi' : Z(H^r)^\vee_{2n-j} \to Z(H^{r+1})^\vee_{2n-j}, \quad \psi'(f)(X_I) = \pm \begin{cases} f(X_I) & I \subseteq \{1, \dots, 2n\} \\ 0 & \text{otherwise} \end{cases}, \tag{18}$$

where $J \subseteq \{1, \dots, 2r\}$.

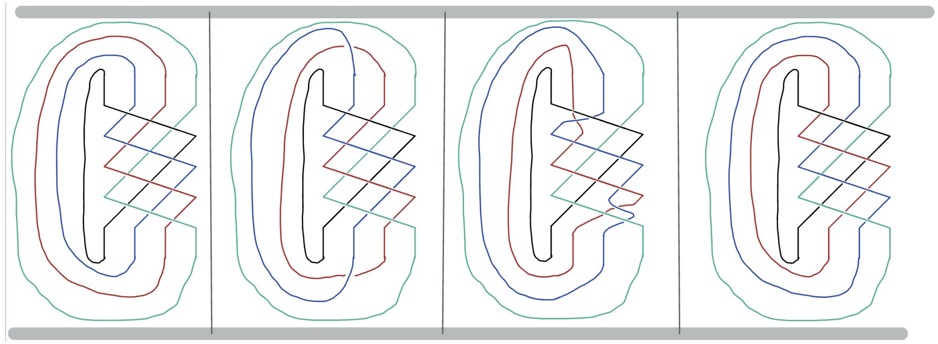


FIGURE 19. Example of the braid group action for $B(p)$. This figure, in particular, is the action of σ_2 on the link L_2^1.

Proof of Theorem 53. By definition, the cabled Khovanov homology of U^p is

$$\underline{\mathrm{KhR}}_{2,0}^{0,j}(U^p) = \left(\bigoplus_{r\in\mathbb{N}} \mathrm{KhR}_2^{0,j}(U^p(r,r))\{-2r\}\right)/\sim$$
$$= \left(\bigoplus_{r\in\mathbb{N}} \mathrm{KhR}_2^{0,j}(L_r^p)\{-2r\}\right)/\sim .$$

We begin with the case where $p > 0$. By Claim 54, we know that $\mathrm{KhR}_2^{0,j}(L_r^p) \cong Z_{2r+j}^r$, so

$$\underline{\mathrm{KhR}}_{2,0}^{0,j}(U^p) \cong \left(\bigoplus_{r\in\mathbb{N}} Z_{4r+j}^r\right)/\sim$$

Recall that we quotient out by the relations $\varphi(v) \sim 0$ and $\psi(v) \sim v$ for $v \in Z_{4r+j}^r$. From Equation (15), we know that $\varphi(X_I) = \pm X_I(X_{2r+2} - X_{2r+1})$. Thus,

$$\pm X_I(X_{2r+2} - X_{2r+1}) \sim 0,$$

so $X_I \sim 0$ or $X_{2r+1} \sim X_{2r+2}$. Furthermore, by Equation (16), we know that $\psi(X_I) = \pm X_I(X_{2r+1}X_{2r+2}) \sim X_I$. It follows from $\varphi(X_I) \sim 0$ that

$$X_I \sim \pm X_I(X_{2r+1}X_{2r+2}) \sim \pm X_I(X_{2r+1}^2) \sim 0,$$

where the final equivalence comes from the fact that $X_{2r+1}^2 \sim 0$ in the computation of the center (see Theorem 32). Since $X_I \sim 0$ for all subsets $I \subseteq \{1, \dots, 2r\}$, then these two relations effectively "kill off" all generators of Z_{4r+j}^r. Thus,

$$\underline{\mathrm{KhR}}_{2,0}^{0,j}(U^p) \cong 0.$$

We now turn to the case where $p < 0$. By Claim 54, we know that

$$\underline{\mathrm{KhR}}_{2,0}^{0,j}(U^p) \cong \left(\bigoplus_{r\in\mathbb{N}} (Z_{-j}^r)^\vee\right)/\sim .$$

From their definition, we know that the maps φ' and ψ' are in bi-degrees $(0,0)$ and $(0,2)$, respectively. After incorporating grading shifts, they are now in bi-dgrees $(0,-2)$ and $(0,0)$, respectively; however we know that $\mathrm{Kh}(L^p)$ is only supported non-negative quantum degrees, so the undotted annulus map φ' necessarily sends everything to 0.

Furthermore, we claim that the maps

$$\psi' : (Z_0^r)^\vee \to (Z_0^{r+1})^\vee$$

are isomorphisms. From Theorem 36 and by dualizing, we know that $(Z_0^r)^\vee$ is generated by $1 \in Z_0^r \cong \mathbb{Z}$, where 1 can be thought of as corresponding to the admissible subset $\varnothing$. By Lemma 55, we know that ψ' maps $f(1)$ to itself (up to multiplication by ± 1). Since $f(1)$ is the only generator of Z_0^{r+1}, then ψ' is an isomorphism. Thus, the cabled Khovanov homology $\underline{\mathrm{KhR}}_{2,0}^{0,0}(U^p)$ is the colimit of the directed system

$$\mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{\cong} \cdots,$$

which is $\mathbb{Z}$. By tensoring with $\mathbb{Q}$, we get

$$\underline{\mathrm{KhR}}_{2,0}^{0,0}(U^p; \mathbb{Q}) \cong \mathbb{Q}.$$

We now consider $(Z_{-j}^r)^\vee$ where $j = -2k$ for $k > 0$. Consider an arbitrary element $f \in (Z_{-j}^r)^\vee$, which can be expressed as a finite sum in terms of the dual basis,

$$f = \sum a_i X_I^\vee,$$

where $a_i \in \mathbb{Z}$ and $1 \leq i \leq \binom{2r}{m} - \binom{2r}{m-1}$, with $m \leq r$, and $I \in A_k^r$. Furthermore, let $J \subseteq \{1, \ldots, 2r+2\}$. By Lemma 55, we can break up the computation of the map $\varphi'(f)$ into three cases.

(1) Suppose $J = I \cup \{2n+2\}$. Then $\varphi'(f)X_J = \pm f(X_I)$, which can be simplified using the contraction operation to

$$\pm \sum a_i X_I^\vee(X_J) = \pm a_i X_{2n+2},$$

for some a_i. By identifying $\pm f(X_I) \sim 0$, we get that $X_{2n+2} \sim 0$ provided that $a_i \neq 0$.

(2) Now suppose $J = I \cup \{2n+1\}$. This case is similar to the last one, although we now have that $\varphi'(f)X_J$ simplifies to $\mp a_i X_{2n+1}$. Thus, $X_{2n+1} \sim 0$.

(3) Otherwise, we have that $\varphi'(f)X_J = 0$.

We now turn to the braid group action on $\mathrm{KhR}_2^{0,j}(L^p)$. Observe that the action of a braid group generator $\sigma_i \in B_{2n}$ can be drawn, diagrammatically, with a Reidemeister II move, a sequence of Reidemeister III moves, and a Reidemeister II-inverse move (refer to Figure add). Namely, we start out with the isotopic tangle structure $\sigma_i^{-1} FT_{2r}^{\otimes p} \circ \sigma_i$. By properties of Bar-Natan's functor $\mathcal{F}_{BN}$, then

$$\mathrm{KhR}_2(\sigma_i^{-1} FT_{2r}^{\otimes p} \circ \sigma_i) = \beta(\sigma_i^{-1}) \otimes \mathrm{KhR}_2(FT_{2r}^{\otimes p}) \otimes \beta(\sigma_i).$$

Recall that $\mathrm{KhR}_2(FT_{2r}^{\otimes p}) \cong (Z_{-j}^r)^\vee$, so $\beta(\sigma_i^{\pm 1})$ gives maps $g, g^{-1} \in (Z_{-j}^r)^\vee$ and $\mathrm{KhR}_2(FT_{2n}^{\otimes p})$ gives a map $f \in (Z_{-j}^r)^\vee$. Then for $v \in Z_{-j}^r$, we have

$$\begin{aligned} \mathrm{KhR}_2(\sigma_i^{-1} FT_{2r}^{\otimes p} \circ \sigma_i)(v) &= (g^{-1} \otimes f \otimes g)v \\ &= g^{-1}(v) \otimes f(v) \otimes g(v) \\ &= f(v). \end{aligned}$$

By our discussion of the map φ', we know that, in the subsequent term of the colimit, we will have $\varphi'(f)v \sim 0$. So we have that $v \sim f(v)$ (under the braid group action) and $\varphi'(f)v \sim 0$ (under the colimiting map), so $f(v) \sim 0$. Thus, the entire dual center $(Z_{-j}^r)^\vee$ collapses to zero, i.e.,

$$\underline{\mathrm{KhR}}_{2,0}^{0,j}(U^p) \cong 0.$$

We obtain the final result for the rational cabled Khovanov homology by tensoring with $\mathbb{Q}$. $\square$

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