Davis Math Lab Project Report, Spring 2026
D-Modules and Habiro Rings

Lucas Bertocchini, Andrew Li
Zachary Ibarra (Graduate Mentor)
Motohico Mulase (Faculty Mentor)

June 15, 2026

Contents

1 Introduction **1**

2 Progress **1**

3 Future Plans **4**

1 Introduction

The main object of our exploration thus far has been the construction of the Habiro ring via the inverse limit of a collection of quotient rings with maps induced by inclusion of ideals, and its relation to the quantized enveloping algebra of the simple Lie algebra of dimension 3. Furthermore, we considered related topics such as quantum invariants of tangles constructed via the Hopf-algebraic structure of the aforementioned algebra. We have also begun to explore additional topics such as sheaves, D-modules, and homology and cohomology.

2 Progress

First, we began by investigating the construction and properties of the Habiro ring. The Habiro ring is defined as the limit $\widehat{\mathbb{Z}[q]}$ of the inverse system of quotient rings $\mathbb{Z}[q]/\langle(q;q)_n\rangle$, with natural homomorphisms from $\mathbb{Z}[q]/\langle(q;q)_m\rangle$ to $\mathbb{Z}[q]/\langle(q;q)_n\rangle$, $m \geq n$, where $(q;q)_n$ is the q-Pochhammer symbol. In particular, the diagram below must commute for all $m \geq n$, and $\widehat{\mathbb{Z}[q]}$ must be universal in the inverse limit sense (the integer ring $\mathbb{Z}$ induces a cone for this system, but is not the Habiro ring!).

$$\begin{tikzcd} & \widehat{\mathbb{Z}[q]} \arrow[dl] \arrow[dr] & \\ \mathbb{Z}[q]/\langle (q;q)_n \rangle & & \mathbb{Z}[q]/\langle (q;q)_m \rangle \arrow[ll] \end{tikzcd} \qquad (m \geq n)$$

This completion is the ring of infinite sequences that is a subset of

$$\prod_{n=1}^{\infty} \mathbb{Z}[q]/\langle (q;q)_n \rangle$$

where the projections into $\mathbb{Z}[q]/\langle (q;q)_n \rangle$ are defined by extracting the nth term, and the terms are determined by the compatability condition that the diagram commutes; or equivalently, the completion is the ring of infinite sums of $(q;q)_n$ over n with polynomial coefficients (it is proved that each element of the Habiro ring can be written as such a series, and that the two rings are isomorphic).

$$\sum_{n=0}^{\infty} a_n(q)(q;q)_n$$

Next, we investigated the quantized enveloping algebra, which (in our case) is a deformation of $U(\mathfrak{sl}_2)$, which is the universal enveloping algebra of the simple Lie algebra with dimension 3, $\mathfrak{sl}_2$. Recall that $\mathfrak{sl}_2$ is the Lie algebra generated by E, F, H over some field, whose multiplication is a Lie bracket that satisfies $[E,F] = H, [H,E] = 2E, [H,F] = -2F$. Then, informally, $U(\mathfrak{sl}_2)$ is simply an algebra of equivalence classes generated by E, F, H that is equipped with regular (symbolic) addition and multiplication, where we let the commutator equal the corresponding Lie bracket (e.g., we let $EF - FE = [E,F] = H$). Then, the associated quantized enveloping algebra $U_\hbar(\mathfrak{sl}_2)$ is generated over the completion ring $\mathbb{Q}[[\hbar]]$ by the generators E, F, H, K, K^{-1}, where these obey the same commutator relations as in the universal enveloping algebra, except we let $[E,F] = \frac{K-K^{-1}}{v-v^{-1}}$, where $K = e^{\frac{\hbar H}{2}}$ and $v = e^{\frac{\hbar}{2}}$. Note that if we let $\hbar \to 0$, l'Hôpital's rule shows that $[E,F] \to H$, so this just collapses into $U(\mathfrak{sl}_2)$.

A conjectured relation to the Habiro ring: let $C = (v - v^{-1})^2 FE + vK + v^{-1}K^{-1}$ and $q = e^\hbar$. Note that, if we expand q as a Taylor series on $\hbar$, we immediately see that q is an element of $\mathbb{Q}[[\hbar]]$. Similarly, we check that C is in $U_\hbar(\mathfrak{sl}_2)$. Thus, consider the algebra $\mathbb{Z}[q,q^{-1}][C^2]$, which is a $\mathbb{Z}[q,q^{-1}]$-subalgebra of $U_\hbar(\mathfrak{sl}_2)$. Then (bear with me here), we define one more thing, namely, let

$$\sigma_n(C) := \prod_{k=1}^{n} (C^2 - (q^k + q^{-k} + 2)),$$

where $\sigma_n(C)$ is in $\mathbb{Z}[q,q^{-1}][C^2]$. Then, consider the completion of the inverse system as shown below, whose maps are natural maps induced by the ideals:

$$\begin{tikzcd} & \lim_{\leftarrow n} \mathbb{Z}[q,q^{-1}][C^2]/\langle \sigma_n(C) \rangle \arrow[dl] \arrow[dr] & \\ \mathbb{Z}[q,q^{-1}][C^2]/\langle \sigma_i(C) \rangle & & \mathbb{Z}[q,q^{-1}][C^2]/\langle \sigma_j(C) \rangle \arrow[ll] \end{tikzcd} \qquad (j \geq i)$$

Then, there is a natural map $\phi : \lim_{\leftarrow n} \mathbb{Z}[q, q^{-1}][C^2]/\langle \sigma_n(C) \rangle \to \widehat{\mathbb{Z}[q]}$ induced by mapping $C \mapsto 2$. This is an important result; if this map ϕ can be shown to be an injection and $\lim_{\leftarrow n} \mathbb{Z}[q, q^{-1}][C^2]/\langle \sigma_n(C) \rangle$ is in the quantized enveloping algebra (as Habiro conjectures), then that will imply that the Habiro ring (or a subset) is isomorphic to a subset of the quantized enveloping algebra.

It was important for us to investigate the quantized enveloping algebra, since it is deeply connected to the idea of a Universal Knot Invariant, which I will get to shortly.

But first, we examined the idea of a knot or link invariant. This required taking a look at the Artin braid group. The Artin braid group B_n is the group of equivalence classes of braids on n strands, such that two braids are considered equivalent if they are equivalent up to isotopy. By Alexander's theorem, we have that any link or knot can be represented as a braid closure; furthermore, by Markov's theorem, we have that braid closures are invariant under conjugation (as well as a stabilization, but let us omit that for simplicity). Then, if we could find a class function that is defined on B_n, by Markov's theorem, that would be a knot invariant. By defining the action of the braid group generators using $R : V \otimes V \to V \otimes V$ for some vector space V, such that R satisfies the Yang-Baxter equation, we constructed a map (i.e. a representation) of B_n into $\mathrm{Aut}(V^{\otimes n})$ that respects the braid group structure:

$$\sigma_i \mapsto I \otimes ... \otimes I \otimes R_i \otimes I \otimes ... \otimes I$$

where σ_i is the ith braid group generator of B_n, and $I \otimes ... \otimes I \otimes R_i \otimes I \otimes ... \otimes I$ is an automorphism (we choose R_i to be invertible) of $V^{\otimes n}$ such that R_i acts on the tensor product of the ith and $(i+1)$th vector spaces $V_i \otimes V_{i+1}$; note that here, V_i and V_{i+1} represent the same vector space, and are only indexed to specify the action of σ_i. I is the identity map defined on V.

By doing so, we were able to construct a simple example of a (partial) knot invariant in the form of the character of that representation, since the character is invariant under conjugation of braids.

Having explored a simple example of a knot invariant, we then discussed Habiro's universal knot invariant. First, we looked at what Habiro calls the universal $\mathfrak{sl}_2$-invariant. For simplicity, we looked at the case of a tangle with one component (which Habiro calls a "bottom knot"). This invariant is constructed by considering the universal R-matrix associated with the Hopf algebra structure of $U_\hbar(\mathfrak{sl}_2)$, which Habiro (for the sake of notation) simplifies to $R = \sum \alpha \otimes \beta$. Then, by assigning the crossings of the tangle with the proper terms of R, then taking the product along the tangle and summing over the indices, we recover the invariant, which Habiro calls J_T for a tangle T. After introducing this, we computed the value for an unknot and checked that the invariant works. Habiro also gives the theorem that every such J_T for a bottom knot T can be uniquely associated with an element of the aforementioned completion $\lim_{\leftarrow n} \mathbb{Z}[q, q^{-1}][C^2]/\langle \sigma_n(C) \rangle$, and thus can be written as a sum

$$\sum_n a_n(q)\sigma_n(C)$$

where $\sigma_n(C)$ is the same as defined above [1]. Recall that this completion has a natural map to the Habiro ring; this provides a direct connection between the universal knot invariant and the Habiro ring.

Finally, our last topic that we began to cover was D-modules, which included a discussion on topics like sheaves, complex manifolds, and (co)homology. We began by learning about and specifically constructing examples of sheaves. Of importance was the structure sheaf $\mathcal{O}_X$ of a complex manifold X, which is a contravariant functor that maps every open subset of X to the ring of holomorphic functions on that set; we defined a function on a coordinate chart to be holomorphic if its composition with the coordinate map's inverse is holomorphic on its domain in $\mathbb{C}^n$, and we defined a holomorphic function on an open set of X to be one that is holomorphic on each chart, such that it glues together on overlaps. Then, it is, of course, natural to reverse the direction of the morphisms in the ring category and define them as restriction maps.

We defined D-modules in a sheaf-theoretic framework, namely, as a sheaf that assigns open subsets U to $\mathcal{D}(U)$-modules, where $\mathcal{D}(U)$ is a ring of differential operators on U, which can be described on each local coordinate chart as a ring generated by the aforementioned structure sheaf and partial derivatives in the local coordinates. We also described it in terms of vector bundles.

3 Future Plans

All this has only been the beginning of a longer process, that we intend to continue in the course of the next year. We will continue to explore the topic of D-modules and especially their relationship to the Habiro ring and knot invariants.

References

[1] Kazuo Habiro, *A unified Witten–Reshetikhin–Turaev invariant for integral homology spheres*, Inventiones Mathematicae **171** (2008), no. 1, 1–81.