Davis Math Lab Project Report, Spring 2026
The Plactic and Hypoplactic Monoid

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1 Introduction

The goal of this project is to study the plactic and hypoplactic monoid and implement them in SageMath. Both the plactic and hypoplactic monoid are built from words over an ordered alphabet, where equivalence relations are used to partition words into equivalence classes. These equivalence classes can be represented by combinatorial objects called tableaux, which gives a structured way to work with the elements in the monoids.

In the plactic monoid, words are represented by Young tableaux through an algorithm called RSK. In the hypoplactic monoid, words are represented by quasi-ribbon tableaux through Krob–Thibon insertion. Our project focuses on writing SageMath code that models these monoid structures.

2 Background

2.1 Words and Tableaux

Let A be a finite ordered alphabet, for example

$$A = \{1 < 2 < \cdots < n\}.$$

We consider finite words formed from this alphabet, such as

$$21235, \qquad 451284, \qquad 33712.$$

In this project, words are multiplied by concatenating them. For example, concatenating 5241 and 13 gives the word 524113. The empty word acts as the identity element because concatenating it with any word does not change the word.

A Young tableau is a filling of a Young diagram with entries from an ordered alphabet such that the entries weakly increase across rows and strictly increase down columns. For example,

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 4 \\ \cline{1-2} 5 \\ \cline{1-1} \end{array}$$

is a Young tableau.

2.2 The Plactic Monoid

The plactic monoid groups these words into equivalence classes using the Knuth relations. These relations are

$$(K1) \qquad yzx \equiv yxz \qquad \text{if } x < y \leq z,$$

and

$$(K2) \qquad xzy \equiv zxy \qquad \text{if } x \leq y < z.$$

Two words are called Knuth equivalent if one can be changed into the other by a sequence of elementary Knuth transformations. Each equivalence class of words represents one element of the plactic monoid.

The connection between words and tableaux comes from the Robinson–Schensted–Knuth correspondence, or RSK. RSK associates a word with a pair of tableaux: an insertion tableau and a recording tableau. For the plactic monoid, the more important object is the insertion tableau, denoted $P(w)$ [Ful97]. Two words represent the same element of the plactic monoid exactly when they have the same insertion tableau. For example,

$$\begin{array}{c} 34123 \\ 13243 \\ 13423 \end{array} \xrightarrow{\text{RSK insertion tableau}} \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 3 & 4 \\ \cline{1-2} \end{array}$$

so these three words are equivalent in the plactic monoid. The tableau on the right represents the whole equivalence class. Since multiplication of words is given by concatenation, this operation also gives the product in the plactic monoid.

2.3 The Hypoplactic Monoid

The hypoplactic monoid [Nov00] is similar to the plactic monoid, but its equivalence classes are represented by quasi-ribbon tableaux instead of Young tableaux. It is formed using the Knuth relations together with additional quartic relations:

$$(Q1) \qquad dbca \equiv bdac \qquad \text{if } a < b \leq c < d,$$

and

$$(Q2) \qquad cadb \equiv acbd \qquad \text{if } a \leq b < c \leq d.$$

A quasi-ribbon tableau is defined to be a filling of a connected ribbon shape, with no 2×2 block of boxes, whose rows weakly increase from left to right and whose columns strictly increase from top to bottom. For example,

$$\begin{array}{cccccc} \cline{1-3} \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{4} & & & \\ \cline{1-4} & & \multicolumn{1}{|c|}{5} & \multicolumn{1}{c|}{7} & & \\ \cline{3-6} & & & \multicolumn{1}{|c|}{8} & \multicolumn{1}{c|}{8} & \multicolumn{1}{c|}{9} \\ \cline{4-6} \end{array}$$

is a quasi-ribbon tableau.

The analogous insertion algorithm for the hypoplactic monoid is Krob–Thibon insertion, which takes a word and produces a quasi-ribbon tableau. Just as the RSK insertion tableau represents a plactic equivalence class, the quasi-ribbon tableau represents a hypoplactic equivalence class. Two words are equivalent in the hypoplactic monoid when Krob–Thibon insertion produces the same quasi-ribbon tableau. For example,

$$\begin{array}{c} 12548789 \\ 15284789 \end{array} \xrightarrow{\text{Krob–Thibon insertion}} \begin{array}{|c|c|c|ccc} \hline 1 & 2 & 4 & & & \\ \hline & & 5 & \multicolumn{1}{|c|}{7} & & \\ \cline{3-6} & & & \multicolumn{1}{|c|}{8} & \multicolumn{1}{c|}{8} & \multicolumn{1}{c|}{9} \\ \cline{4-6} \end{array}$$

so these two words are in the same hypoplactic equivalence class.

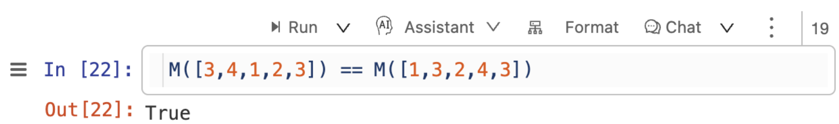
3 Progress

3.1 PlacticMonoid Class

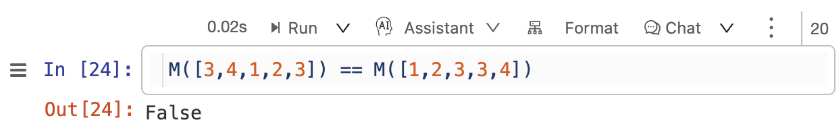
We have created the `PlacticMonoid` class in SageMath. The constructor `PlacticMonoid(n)` takes in a positive integer n to create a plactic monoid on the alphabet $\{1, 2, \ldots, n\}$. The elements of this class are constructed from words. We can take a word and return its corresponding Young tableau via RSK insertion.

```
In [5]:  M = PlacticMonoid(4)
         M([3,4,1,2,3]).to_tableau().pp()
Out[5]:  1  2  3
         3  4
```

We use the `to_tableau()` method to check if two words are Knuth equivalent.



```
In [22]:  M([3,4,1,2,3]) == M([1,3,2,4,3])
Out[22]:  True
```



```
In [24]:  M([3,4,1,2,3]) == M([1,2,3,3,4])
Out[24]:  False
```

This code has been pushed to ticket #44218 in the SageMath Github [Sch] and is currently under review by Martin Rubey.

3.2 HypoplacticMonoid Class

Similar to how the plactic monoid uses Young tableaux for insertion and equivalence, we had to first create the QuasiRibbonTableau class. Analogous to the RSK insertion, the QuasiRibbonTableau class has its own Knob–Thibon insertion algorithm.

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In [4]:
```
Q = QuasiRibbonTableau([[1, 2, 3], [4, 5]])

print("Original tableau:")
print(Q)

R = insert_letter(Q, 4)

print("After inserting 4:")
print(R)

R
```

Out[4]:
```
Original tableau:
1  2  3
      4  5
After inserting 4:
1  2  3
      4  4
         5

[[1, 2, 3], [None, None, 4, 4], [None, None, None, 5]]
```

The `HypoplacticMonoid` inherits from the class `PlacticMonoid`, and is constructed similarly on an alphabet $\{1, 2, \ldots, n\}$. Similar to the `PlacticMonoid` class, it also has `_eq_()` and `_mul_()` functions, but now under quartic relations and the Knob Thibon insertion.

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In [8]:
```
H = HypoplacticMonoid(9)
H([1,2,5,4,8,7,8,9]) == H([1,5,2,8,4,7,8,9])
```

Out[8]:
```
True
```

The `HypoplacticMonoid` class can be found on ticket #44219 in the SageMath Github [Joh].

4 Future Plans

The SageMath community has recognized a need to include the plactic and hypoplatic monoids, yet unimplemented, for computational applications to combinatorics and representation theory. The code for the plactic monoid is already on github and under review for SageMath. We will submit our implementations of the hypoplactic monoid and the also unimplemented class of quasiribbon tableaux to SageMath by the end of June, intending our code to become an official library after review by SageMath's maintainers and adjustments according to their comments. Afterwards the code will serve to perform computations and test conjectures in algebra and combinatorics. The code for the plactic monoid has already sparked the interest of Martin Rubey, who is currently reviewing the code and building on it for faster methods.

Our implementation has two limitations: First, it concerns mainly the basic definitions and properties of these monoids; there are further methods that can be integrated for algebraic, combinatorial, geometric, and probabilistic applications as described in, for example, [Ful97] and [Nov00]. Second, the algorithms for computing equivalence classes of plactic or hypoplactic words and generating all quasiribbon tableaux of a given shape are computationally inefficient. The first problem presents many tasks for future developers; the second problem cannot be resolved without answering the theoretical question as to whether there are superior algorithms.

References

[Ful97] William Fulton. *Young tableaux: With applications to representation theory and geometry*, volume 35 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1997.

[Joh] Implementation of the hypoplactic monoid issue #42219 sagemath/sage. `https://github.com/sagemath/sage/issues/42219`. Accessed: 2025-06-12.

[Nov00] Jean-Christophe Novelli. On the hypoplactic monoid. volume 217, pages 315–336. 2000. Formal power series and algebraic combinatorics (Vienna, 1997).

[Sch] Implementation of the plactic monoid issue #42218 sagemath/sage. `https://github.com/sagemath/sage/issues/42218`. Accessed: 2025-06-12.