# Decompositions of Tetrahedra into Height Functions

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**Abstract**

In short, we call a polyhedron $P \subset \mathbb{R}^3$ a height function if there exists some face $F$ of $P$ such that $P$ lays entirely within $F$ when orthogonally projection onto the affine plane containing $F$. Because tetrahedra come in various configurations, we propose a dihedral angle classification of tetrahedra, and prove that a tetrahedron $T$ is is either a height function already, or, at most one planar bisection of $T$ is needed to produce two height function tetrahedra.

KEYWORDS: Tetrahedron, Height Function, 3D-Printing,

# 1 Introduction

When 3-D printing objects with features that are both geometrically simple or complex, a bottom-up process is used to print each layer at a time. In doing so, the printing process is executed in a way that, in theory, ensures the structural stability required for printing layers that lie higher from the **build plate**. While this process is relatively simple for layers that lie directly above the first layer of the print, it becomes much more complicated for features of the object that protrude past the first layer.

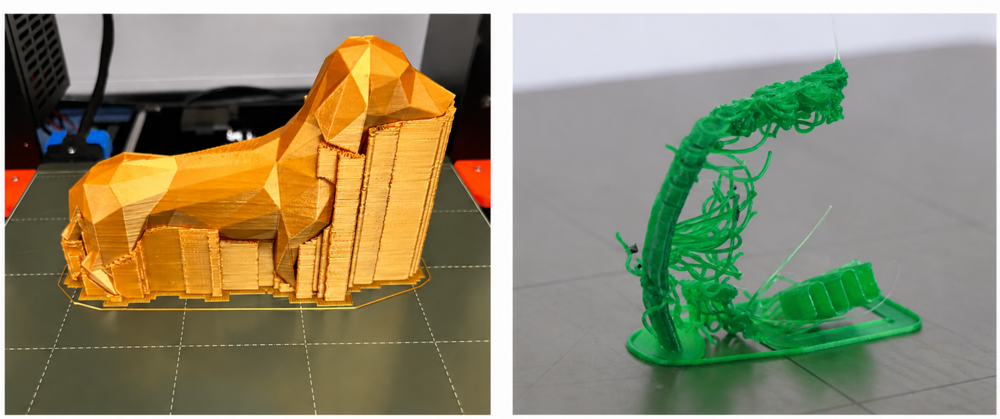

Figure 1: Example of successful support generation for a low-poly dachshund model [3] alongside an example of failed support structures during printing [4].

In particular, additional **supports** must be printed to ensure that the overhanging features do not collapse during the printing process. Though the use of supports can be, at times, effective for printing overhanging features, they open up the possibility for printing failures as seen in figure 1. For objects that have a significantly large amount of overhanging features due to their complexity, this issue becomes more likely.

As a solution, methods of object decomposition can be employed to reduce the number of supports required to print objects with overhanging features. However, any amount of decomposition comes at the cost of increasing the total number of parts, which, in turn, may increase the total points of failure for objects being used in practical applications involving any amount of physical stress. Thus, it is within our best interest to search for decompositions that minimize both the total number of objects after decomposition and the number of supports required to print all pieces.

In this paper, we study tetrahedra in $\mathbb{R}^3$, with particular emphasis on the relationship between their projective properties and geometric decompositions. Our primary objective is to understand when a tetrahedron admits a height function structure and how planar bisections can be used to produce such structures. By combining geometric, combinatorial, and projective techniques, we obtain a classification of tetrahedra and a complete description of the subdivisions that arise from planar cuts.

In Section 2, we introduce relevant terminology with heavy emphasis on developing mathematical descriptions of supports. In particular, we define height function polyhedra and develop the geometric intuition necessary to study them in the context of tetrahedra. We also introduce planar bisections and discuss the special case of tetrahedral subdivisions, which form the principal class of decompositions considered in this work. These concepts establish the framework upon which the later classification results are built.

In Section 3, we develop a classification of tetrahedra based on the number and arrangement of their obtuse dihedral angles. This classification significantly reduces the complexity of the problem by organizing all tetrahedra into a finite collection of geometric types. We then establish a height function criterion relating the dihedral angle structure of a tetrahedron to its projective properties.

In Section 4, we prove that every tetrahedron that is not already a height function can be transformed into one through at most a single tetrahedral subdivision.

# 2 Polyhedrons, Height Functions, and Planar Bisections

Generally speaking, representations of polyhedra appear in a wide range of mathematical contexts, including but not limited to combinatorics, simplicial geometry, and of course, convex geometry. In the convex setting, we formally define a polyhedron $P$ as a convex set in $\mathbb{R}^n$ obtained as the convex hull of finitely many points with finitely many rays. Within this paper, we specifically focus on tetrahedra, that is, polyhedra formed by four affinely independent points in $\mathbb{R}^3$. More concretely, for the entirety of this paper, unless otherwise specified, let us define any tetrahedron $T$ as

$$T = \operatorname{conv}\{v_0, v_1, v_2, v_3\}, \quad v_i \in \mathbb{R}^3$$

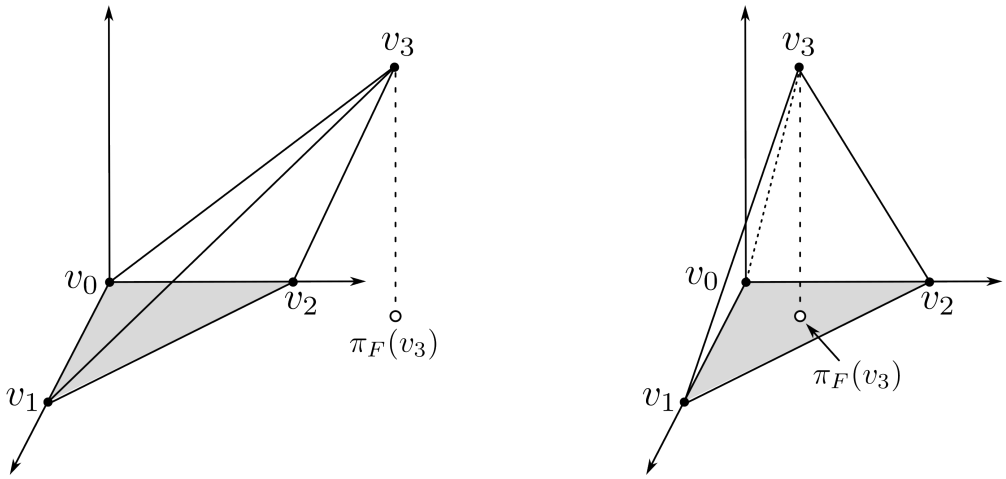


Figure 2: (*Left*) A non height function tetrahedron $T \subset \mathbb{R}^3$. (*Right*) A height function tetrahedron $T \subset \mathbb{R}^3$ with a foundation $F$, highlighted in gray.

Given that we aim to analyze the concept of 3-Dimensional supports in a mathematical setting relating to $T$, we would naturally like to focus on what necessarily determines a feature to be "overhanging" in $\mathbb{R}^3$ relative to some face $F$ of $T$. Conveniently, we may use the orthogonal projection onto the affine plane containing the face $F$ to formalize this notion. Intuitively, a point of $T$ may be regarded as non-overhanging relative to a face $F$ whenever its orthogonal projection onto the affine plane containing $F$ lies within $F$ itself. This motivates the following definition, which captures the idea that every point of the tetrahedron is vertically supported by the face F.

**Definition 2.1.** A polyhedron $P$ is called a *height function* if there exists a face $F \in \mathcal{F}(P)$ such that the orthogonal projection $\pi_F \colon \mathbb{R}^3 \to \operatorname{aff}(F)$ satisfies

$$\pi_F(P) \subseteq F.$$

We call the face $F$ the *foundation* of $P$.

As we will discuss later in section 4, it may be the case that a tetrahedron is not naturally a height function over any of its faces. In that case, we aim to decompose $T$ in the interest of producing sub-tetrahedron that are height functions over their respective foundation faces. While there exists a wide variety of decomposition methods, we opt for a simple yet effective method that complies with the geometric constraints imposed by the convexity of $T$.

**Definition 2.2.** A planar bisection of a polyhedron $P \subset \mathbb{R}^3$ is a decomposition

$$P = P_1 \cup P_2 \tag{2.1}$$

such that there exists an affine plane $\Pi \subset \mathbb{R}^3$ satisfying

$$P_1 \cap P_2 = P \cap \Pi \tag{2.2}$$

where $P_1$ and $P_2$ are convex polyhedra contained in the two closed half-spaces[1] determined by $\Pi$.

[1] In the proof for proposition 2.4, we explicitly use half-spaces. However, for future proofs, we will directly define the decomposition, while implying the use of half-spaces.

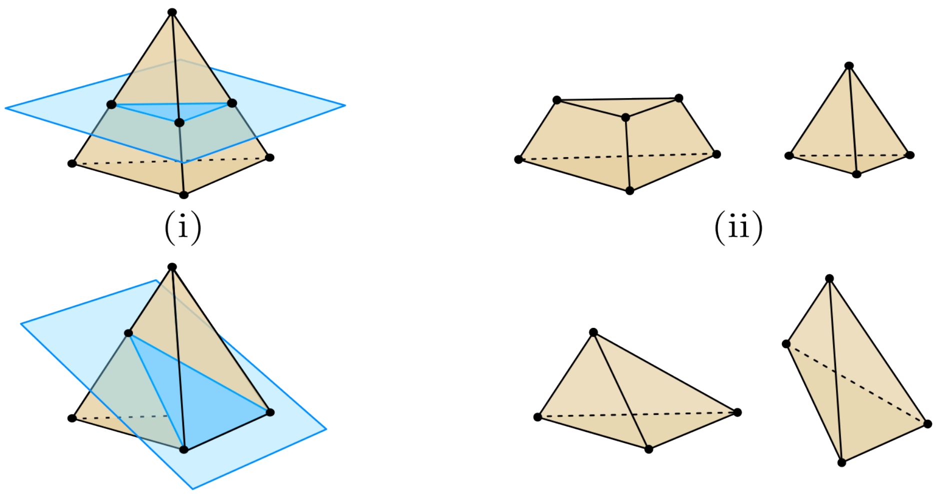


Figure 3: (i) Non-tetrahedral subdivision planar bisection of $T$ (ii) Resulting sub-polyhedra of non-tetrahedral subdivision planar bisection (iii) Tetrahedral subdivision of $T$ (iv) Resulting sub-tetrahedra of tetrahedral subdivision.

Intuitively, we may view the affine plane $\Pi$ as a bisecting mechanism whose intersection with $P$ defines a common face to be shared by the resulting polyhedra $P_1$ and $P_2$. Note however, that the decomposition defined by (2.1) and (2.2) imposes no restrictions on the combinatorial or geometric type of the resulting polyhedra $P_1$ and $P_2$. In particular, if the affine plane $\Pi$ coincides with a face $F$ of $P$, then the induced decomposition fails to produce two 3-dimensional sub-polyhedra of $P$. For the purpose of this study, we regard any such planar bisection as *degenerate*.

The existence of degenerate bisections can be found in all types of polyhedra, including tetrahedra, hinting at an important dichotomy present in the family of all affine planes $\Pi$ that induce planar bisections on polyhedra. In the context of tetrahedra, we focus on a very specific sub-family of affine planes that induce planar bisections of a more favorable type for our study.

**Definition 2.3.** Let $T \subset \mathbb{R}^3$ be a tetrahedron. A *tetrahedral subdivision* of $T$ is a planar bisection

$$T = T_1 \cup T_2$$

such that $T_1 \cap T_2 = T \cap \Pi$ for some affine plane $\Pi \subset \mathbb{R}^3$, where $T_1$ and $T_2$ are distinct 3-dimensional tetrahedra.

Since the primary focus of this paper is to analyze height function tetrahedra, we adopt an informal definition of a specialized tetrahedral subdivision. We call a tetrahedral subdivision a *height function tetrahedral subdivision* if the resulting sub-tetrahedra $T_1$ and $T_2$ are both height functions.

As a conclusion to this section, let us move to formally classify the conditions for which any affine plane $\Pi$ must satisfy to induce a tetrahedral subdivision on $T$.

**Proposition 2.4.** *Let $T \subset \mathbb{R}^3$ be a tetrahedron, and let $\Pi$ be an affine plane. Then $\Pi$ induces a tetrahedral subdivision of $T$ if and only if $\Pi$ contains an edge $e \in E(T)$ and intersects the opposite edge $e' \in E(T)$ at exactly one point $p \in \operatorname{relint}(e')$.*

*Proof.* ($\Rightarrow$) Let $T = \operatorname{conv}\{v_0, v_1, v_2, v_3\} \subset \mathbb{R}^3$ be a tetrahedron, and assume $\Pi$ induces a tetrahedral subdivision of $T$. By definition,

$$T_0 = T \cap H^- \quad \text{and} \quad T_1 = T \cap H^+$$

are both tetrahedra where $H^-$ and $H^+$ are the closed half-spaces determined by $\Pi$. Thus, the face they share, defined by $T \cap \Pi$ is triangular and can be described as,

$$T \cap \Pi = \operatorname{conv}\{p_1, p_2, p_3\} \quad p_i \in \mathbb{R}^3$$

Because $T_0$ and $T_1$ are both tetrahedra we may express them in terms of their 4 vertices, 3 of which they have in common through $T \cap \Pi$,

$$T_0 = \operatorname{conv}\{v_i, p_1, p_2, p_3\} \quad \text{and} \quad T_1 = \operatorname{conv}\{v_j, p_1, p_2, p_3\}$$

Now, notice that, for any non-degenerate tetrahedral subdivision of $T$, every vertex of $T$ must be contained in at least $T_0$ or $T_1$. In other words,

$$\{v_0, v_1, v_2, v_3\} \subset \{v_i, v_j, p_1, p_2, p_3\}$$

Moreover, $T_0$ and $T_1$ are distinct tetrahedra, meaning they cannot both contain all 4 vertices of $T$. As such, there exists 2 vertices of $T$, without loss of generality call them $v_0$ and $v_1$, such that

$$v_0 \in T_0 \setminus T_1 \quad \text{and} \quad v_1 \in T_1 \setminus T_0$$

Since $T_0$ and $T_1$ only differ by one vertex, it follows that $v_i = v_0$ and $v_j = v_1$. This then implies that,

$$\{v_2, v_3\} \subset \{p_1, p_2, p_3\}$$

After possibly relabeling the indices we can assume $p_2 = v_2$ and $p_3 = v_3$. Thus,

$$T_0 = \operatorname{conv}\{v_0, v_2, v_3, p_1\} \quad \text{and} \quad T_1 = \operatorname{conv}\{v_1, v_2, v_3, p_1\}$$

Because $v_2 = p_2 \in \Pi$ and $v_3 = p_3 \in \Pi$, the segment $[v_2, v_3] = [p_2, p_3] \subset \Pi$ is contained within $\Pi$. Thus, the edge $[v_2, v_3] \in E(T)$ is contained within $\Pi$. Next, observe that $p_1$ cannot be a vertex in $T$, otherwise $\Pi$ would contain the face of $T$ given by $\operatorname{conv}\{p_1, v_2, v_3\}$ contradicting the assumption that $\Pi$ induces a non-degenerate subdivision. Hence, $p_1$ is in the interior of some edge $e \in E(T)$. Since $p_1 \in \Pi$, the endpoints of $e$ lie in different half-spaces. However, the only edge with this property is $[v_0, v_1]$. Therefore, $p_1 \in \operatorname{relint}([v_0, v_1])$. Since $[v_2, v_3]$ and $[v_0, v_1]$ are opposite edges, this proves the claim.

($\Leftarrow$) Assume the affine plane $\Pi$ contains an edge $e \in E(T)$ and intersects the opposite edge $e' \in E(T)$ at exactly one point $p \in \operatorname{relint}(e')$. Without loss of generality, let

$$e = [v_2, v_3], \quad e' = [v_0, v_1]$$

Then

$$\Pi = \text{aff}\{v_2, v_3, p\} \quad \text{and} \quad T \cap \Pi = \text{conv}\{v_2, v_3, p\}$$

Since $p \in \text{relint}([v_0, v_1])$, the plane $\Pi$ separates $v_0$ and $v_1$. Thus, we can choose $H^-$ and $H^+$ to be the closed half-spaces determined by $\Pi$, such that $v_0 \in H^-$ and $v_1 \in H^+$.

Then let us define

$$P_0 = T \cap H^-, \quad P_1 = T \cap H^+$$

Since $P_0 = T \cap H^-$ is the intersection of a tetrahedron with a half-space, its vertices are among the vertices of $T$ lying in $H^-$, together with the intersection of points of $\Pi$ with edges of $T$ joining opposite half-spaces. Because $v_0, v_2, v_3 \in H^-$, $v_1, v_2, v_3 \in H^+$, and $\Pi$ intersects the edge $[v_0, v_1]$ at the point $p$, it follows that the only vertices of $P_0$ are $v_0, v_2, v_3$, and $p$. Similarly, the only vertices of $P_1$ are $v_1, v_2, v_3$, and $p$. Therefore,

$$P_0 = \text{conv}\{v_0, v_2, v_3, p\}, \quad P_1 = \text{conv}\{v_1, v_2, v_3, p\}$$

Furthermore, $p \notin \text{aff}\{v_0, v_2, v_3\}$ and $p \notin \text{aff}\{v_1, v_2, v_3\}$. Both vertex sets consist of four affinely independent points, thus, both $P_0$ and $P_1$ are tetrahedra with

$$T = P_0 \cup P_1$$

Therefore, $\Pi$ induces a tetrahedral subdivision of $T$. $\square$

# 3 Classifications and The Dihedral Criterion

Over 2300 years ago, Euclid of Alexandria, the father of Euclidean Geometry, classified triangles in $\mathbb{R}^2$ by the lengths of their edges and the interior angles measured between their edges. In particular, Euclid formalized the terms "obtuse," "acute," and "right" to describe both the types of angles that may occur and the subsequent types of triangles. By formally defining these terms, Euclid provided mathematicians with the tools necessary to, when possible, identify the underlying simplified structure of more complex geometric problems.

For tetrahedra in $\mathbb{R}^3$ analyzing underlying structure becomes much more difficult, as the number of empirical quantities determining particular properties of a tetrahedra vastly increases. To avoid an over complication of scope, we propose a dihedral angle classification that aims to collapse the infinite number of tetrahedra that may occur, while preserving a level of freedom[2] in argument.

**Definition 3.1.** A tetrahedron $T$ is said to be *Class* $n$ if it has exactly $n$ obtuse dihedral angles. A *subclass* of class-$n$ tetrahedra is an equivalence class of configurations of $n$ obtuse edges under relabeling of vertices.

While not geometrically exact representations, for the purpose of visual simplicity, we switch to using the full graph $K_4$ for illustrating unique subclasses. Subclasses are distinguished by letter labels according to the adjacency structure of these obtuse edges. Configurations in which all obtuse dihedral angles are incident to a single vertex are labeled (a),

[2]Within class 1 and class 2 subclass (a) or (b) there exists at least one edge for which the dihedral angle may be acute or right.

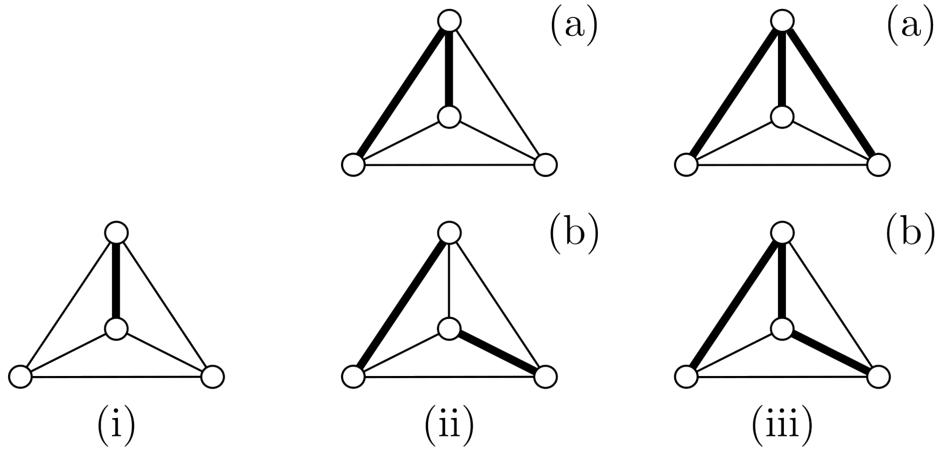


Figure 4: Classes of Class $n$ tetrahedra. (i) Class 1, (ii) Class 2 subclasses, (iii) Class 3 subclasses. Thickened edges indicate where obtuse dihedral angles occur.

while configurations not sharing a common vertex are labeled (b). Since class-1 tetrahedra admit only one subclass, no subclass identifier is assigned.

From figure 3, two immediate observations follow; Class 3 does not contain a subclass corresponding to a 3-Cycle, and for $n > 3$ no such Class $n$ tetrahedron exists. The former can be verified by expressing the supposed face $F$ containing three dihedral obtuse angles as the intersection of three half-spaces. In this case, the orthogonal projection of the opposite vertex would necessarily lie outside all three associated half-planes simultaneously, which is impossible. The latter follows from a theorem resulting from an open problem initially proposed by Klamkin and Pook [1], and later proven by Leng [2].

**Theorem 3.2.** *There exists at least $n$ acute dihedral angles in any $n$-simplex, and there exists an $n$-simplex which has only $n$ acute dihedral angles.*

The quantification of the minimal number of acute dihedral angles in a tetrahedron imposes a corresponding upper bound on the number of obtuse dihedral angles, thereby allowing the use of counting arguments in place of explicit geometric proofs. With this established, we now move to discuss a subtle, yet important property pertaining to the projective nature of height function tetrahedra. In particular, we introduce a tool for determining the existence of a foundation face $F$ of $T$.

**Lemma 3.3** (Height-Dihedral Criterion)**.** *Let $T \subset \mathbb{R}^3$ be a tetrahedron and $F$ be a face of $T$. Then $T$ is a height function relative to the foundation $F$ if and only if the dihedral angles $\theta_1, \theta_2, \theta_3$, along the edges of $F$ are all not obtuse.*

*Proof.* ($\Rightarrow$) Let $T$ be a height function relative to the foundation $F$. After possibly relabeling the vertices of $T$ we may assume $F = \operatorname{conv}\{v_0, v_1, v_2\}$. Notice that applying isometries and scaling to $\mathbb{R}^3$ does not alter the dihedral angles of $T$, as such, we will apply translations, rotations and reflections to $T$ in order to produce a simpler environment for studying $\theta_1, \theta_2$ and $\theta_3$. After applying a translation, assume $v_0 = 0 \in \mathbb{R}^3$. Next, we apply a rigid rotation of $\mathbb{R}^3$ such that the affine hull of $F$ coincides with the $xy$-plane and the edge $[v_0, v_1]$ lies along the positive $x$-axis. In particular, this means that $v_1, v_2 \in \mathbb{R}^3$ have zero $z$ component. Hence, we may write

$$v_1 = (x_1, 0, 0), \quad v_2 = (x_2, y_2, 0)$$

with $x_1 > 0$. If necessary, reflect $T$ across the $xz$ plane to ensure $y_2 > 0$, and scale $T$ so that

$$v_3 = (x_3, y_3, 1)$$

Considering the face $K = \operatorname{conv}\{v_0, v_1, v_3\}$ we move to demonstrate that $\theta_1$, the dihedral angle that occurs at the edge $[v_0, v_1]$ shared by $F$ and $K$ must be non-obtuse. Let $P_1 = \operatorname{aff}\{v_0, v_1, v_2\}$ be the plane containing $F$, and $P_2 = \operatorname{aff}\{v_0, v_1, v_3\}$ be the plane containing $K$. Since $v_0 = 0$ these planes simplify as,

$$P_1 = \operatorname{span}\{v_1, v_2\} \quad \text{and} \quad P_2 = \operatorname{span}\{v_1, v_3\}$$

Because $v_1, v_2$ are linearly independent vectors in the $xy$ plane we conclude that $P_1$ is the $xy$ plane, as such, the vector $n = (0, 0, 1)$ is normal to $P_1$. Also,

$$m = v_1 \times v_3 = (x_1, 0, 0) \times (x_3, y_3, 1) = (0, -x_1, x_1 y_3)$$

is normal to $P_2$. Then the dot product between both normal vectors is

$$\begin{aligned} m \cdot n &= (0, -x_1, x_1 y_3) \cdot (0, 0, 1) \\ &= x_1 y_3 \end{aligned}$$

Because $x_1 > 0$, it remains to show that $y_3 \geq 0$. Since $F$ is a foundation of $T$,

$$\pi_F(v_3) = (x_3, y_3, 0) \in F$$

Then by the convexity of $F$, there exists $r, s, t \geq 0$ with $r + s + t = 1$ such that

$$(x_3, y_3, 0) = r v_0 + s v_1 + t v_2 = s v_1 + t v_2$$

Hence

$$(x_3, y_3, 0) = (s x_1 + t x_2, t y_2, 0) \implies y_3 = t y_2$$

Since $y_2 > 0$ and $t \geq 0$, we get that $y_3 \geq 0$, and therefore

$$n \cdot m = x_1 y_3 \geq 0$$

Due to the dot product angle formula, we have that $\cos(\theta_1) \geq 0$, and subsequently $\theta_1 \leq \frac{\pi}{2}$. Because the initial normalization and coordinate transformations may be applied relative to any edge of $F$, the preceding argument applies identically to the remaining edges of $F$. Therefore, the remaining angles $\theta_2, \theta_3$ are non-obtuse.

($\Leftarrow$) Suppose the dihedral angles of the edges of $F = \operatorname{conv}\{v_0, v_1, v_2\}$, given by, $\theta_1, \theta_2, \theta_3$, are all non-obtuse. To show $F$ is a foundation of $T$ we must show that $\pi_F(x) \in F$ for all $x \in T$. In fact, it is sufficient to show that $\pi_F(v_3) \in F$. Indeed, if $\pi_F(v_3) \in F = \operatorname{conv}\{v_0, v_1, v_2\}$ then,

$$\pi_F(v_3) = \sigma_0 v_0 + \sigma_1 v_1 + \sigma_2 v_2, \quad \text{with } \sigma_i \geq 0, \text{ and } \sum_{i=0}^{2} \sigma_i = 1$$

By the convexity of $T$, we may express any $x \in T$ as a convex combination:

$$x = \lambda_0 v_0 + \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3, \quad \text{with } \lambda_i \geq 0, \text{ and } \sum_{i=0}^{3} \lambda_i = 1$$

Then the orthogonal projection of $x$ onto the plane $\operatorname{aff}(F)$ is given by,

$$\begin{aligned}\pi_F(x) &= \lambda_0 \pi_F(v_0) + \lambda_1 \pi_F(v_1) + \lambda_2 \pi_F(v_2) + \lambda_3 \pi_F(v_3) \\ &= \lambda_0 v_0 + \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 \pi_F(v_3) \\ &= (\lambda_0 + \lambda_3 \sigma_0) v_0 + (\lambda_1 + \lambda_3 \sigma_1) v_1 + (\lambda_2 + \lambda_3 \sigma_2) v_2\end{aligned}$$

Observe that,

$$\lambda_i + \lambda_3 \sigma_i \geq 0 \quad \text{and} \quad \sum_{i=0}^{2} (\lambda_i + \lambda_3 \sigma_i) = \sum_{i=0}^{3} \lambda_i = 1$$

Therefore $\pi_F(x) \in \operatorname{conv}\{v_0, v_1, v_2\} = F$.

With this in mind, we aim to show that $\pi_F(v_3) \in F$. Note that applying isometries and scaling $\mathbb{R}^3$ does not change whether or not $F \subset T$ is a foundation. As such, we will apply the exact same translation, rotations, reflections and scaling to $T$ as we did above.

We start by extending the edge $[v_0, v_1]$ into a line $L = \operatorname{span}\{v_1\}$ (since $v_0 = 0$ and $v_1 = (x_1, 0, 0)$, $L$ is simply the $x$ axis). Observe that $L$ partitions the plane containing $F$ (the $xy$ plane) into two half spaces, namely

$$H_1^+ = \{(x, y) : y \geq 0\} \quad \text{and} \quad H_1^- = \{(x, y) : y < 0\}$$

By our configuration of $T$, $v_2 = (x_2, y_2, 0)$ with $y_2 > 0$. Hence, $v_2 \in H_1^+$ and $F = \operatorname{conv}\{v_0, v_1, v_2\} \subset H_1^+$.

Note that the dihedral angle $\theta_1$ of the edge $[v_0, v_1]$ is given by the angle between the planes $\operatorname{aff}\{v_0, v_1, v_2\}$ and $\operatorname{aff}\{v_0, v_1, v_3\}$. Another way to express $\theta_1$ is as the angle between two vectors $n \in \operatorname{aff}\{v_0, v_1, v_2\}, m \in \operatorname{aff}\{v_0, v_1, v_3\}$ that are both orthogonal to the intersection line of the planes. It will soon be clear why we wish to express $\theta_1$ in this way, but first we aim to construct these vectors $n \in \operatorname{aff}\{v_0, v_1, v_2\}, m \in \operatorname{aff}\{v_0, v_1, v_3\}$ that are both orthogonal to $L = \operatorname{aff}\{v_0, v_1, v_2\} \cap \operatorname{aff}\{v_0, v_1, v_3\}$.

Define $n$ as follows:

$$n = (0, 1, 0) \in \operatorname{aff}\{v_0, v_1, v_2\}$$

Clearly, $n \perp L$ because $L$ is the $x$ axis.

Next, let $\pi_L(v_3) = (x_3, 0, 0) \in L$, the orthogonal projection of $v_3$ onto the line $L$. By the properties of orthogonal projection,

$$v_3 - \pi_L(v_3) \perp L.$$

Moreover, $\pi_L(v_3) \in L$ means that $\pi_L(v_3) = t v_1$ for some $t \in \mathbb{R}$. Then,

$$v_3 - \pi_L(v_3) = v_3 - t v_1 \in \operatorname{span}\{v_1, v_3\} = \operatorname{aff}\{v_0, v_1, v_3\}$$

So, if we let $m = v_3 - \pi_L(v_3)$ we have our desired vectors. Thus, $\theta_1$ is given by the angle between $n$ and $m$. Because $\theta_1$ is non-obtuse, it follows that,

$$0 \leq n \cdot m = (0, 1, 0) \cdot (0, y_3, 1) = y_3.$$

Therefore, $\pi_F(v_3) = (x_3, y_3, 0) \in H_1^+$. Repeatedly applying this argument to the remaining two edges of $F$ yields $\pi_F(v_3) \in H_1^+ \cap H_2^+ \cap H_3^+ = F$. Therefore $F \subset T$ is a foundation and $T$ is a height function.

□

# 4 Existence of Height Function Decompositions

The dihedral angle classification and Lemma 3.3 proved in the previous section allows us to prove the following primary result.

**Theorem 4.1.** *Let $T$ be a tetrahedron. Either $T$ is a height function, or, at most one tetrahedral subdivision of $T$ is needed to produce two height function tetrahedra $T_0$ and $T_1$.*

*Proof.* As shown before, all terahedra fall within 4 distinct classes, some with multiple subclasses. As such, our strategy is to produce a proof for each class of tetrahedra. As it turns out, there are only 2 kinds of tetrahedra that require planar bisections to be decomposed into height functions, while all other kinds of thetrahedra are already height functions themselves. Let us first examine the fairly trivial cases of $T$ for which the initial configuration of $T$ is already a height function.

**Class 0 tetrahedron:** Every class 0 tetrahedron $T$ is a height function over all of its faces and no planar bisection is required. This result can be obtained by applying Lemma 3.3 to any face of $T$.

**Class 1 tetrahedron:** Let $e_0$ be the only edge with an obtuse dihedral angle. Now, consider the face $F$ with edges $e_1, e_2, e_3$ such that $e_i \neq e_0$ for all $i = 1, 2, 3$. Because $e_0$ is the only edge with an obtuse dihedral angle, it must follow that $e_1, e_2, e_3$ all have non-obtuse dihedral angles. Then by Lemma 3.3, $T$ is a height function relative to $F$, and no planar bisection is required.

**Class 2 Subclass (a):** Let $e_0$ and $e_1$ be the only edges of $T$ with obtuse dihedral angles. Since $T$ is of Class 2 Subclass (a), $e_0$ and $e_1$ share a common vertex, without loss of generality we can assume this vertex is $v_0$. Then consider the face $F = \text{conv}\{v_1, v_2, v_3\}$ formed by the remaining three vertices. None of the three edges of $F$ contain $v_0$, hence every edge of $F$ must have a non-obtuse dihedral angle. Applying Lemma 3.3 shows that $F$ is a foundation. Therefore, $T$ is a height function and no planar bisection is required.

**Class 2 Subclass (b):** All Class 2 (b) tetrahedra are arranged so that the edges with obtuse dihedral angles are opposite from one another. After possibly relabeling the vertices of $T$, we can assume $[v_0, v_1], [v_2, v_3]$ are the two opposite edges with obtuse dihedral angles. The configuration of these edges requires that every face of $T$ contains one of these edges. Since these edges have obtuse dihedral angles, Lemma 3.3 shows that no face of $T$ is a foundation, hence no Class 2 (b) tetrahedron is a height function.

Now, we proceed by defining a decomposition of $T$ into height function tetrahedra. For each $p \in \mathrm{relint}([v_0, v_1])$, define the affine plane $\Pi = \mathrm{aff}\,\{v_2, v_3, p\}$, and its' induced tetrahedral subdivision[3]

$$T = T_0^p \cup T_1^p \quad \text{ and } \quad T_0^p \cap T_1^p = \mathrm{conv}\,\{v_2, v_3, p\}$$

where

$$T_0^p = \mathrm{conv}\,\{v_0, v_2, v_3, p\} \quad T_1^p = \mathrm{conv}\,\{v_1, v_2, v_3, p\}$$

Next, we introduce the notion of angle functions which are the main mechanism throughout this proof and will remain prevalent throughout the rest of the paper. Define the following functions,

$$\alpha_{T_0^p}(v_2, v_3) : \mathrm{relint}([v_0, v_1]) \to [0, 2\pi], \quad \alpha_{T_1^p}(v_2, v_3) : \mathrm{relint}([v_0, v_1]) \to [0, 2\pi]$$

that map $p$ to the dihedral angle of $[v_2, v_3]$ when considered as an edge of $T_0^p$ and $T_1^p$ respectively. The dihedral angle function is, by nature, continuous, bounded, and monotonic relative to the natural ordering of the interval $[v_0, v_1]$ where $v_0 < v_1$. Note that, by definition, $T_0^{v_0}$ and $T_1^{v_1}$ are degenerate decompositions. As such, both dihedral angle functions are, by default, undefined at the vertex endpoints $v_0$ and $v_1$, hence the domain being restricted to the relative interior of the interval. Though we do not consider these as valid decompositions, it will still be useful to make sense of the angle functions evaluated at these inputs. We will extend the domain of $\alpha_{T_0^p}(v_2, v_3)$ by defining,

$$\alpha_{T_0^{v_0}}(v_2, v_3) = \lim_{p \to v_0} \alpha_{T_0^p}(v_2, v_3)$$

The limit on the right is then monotonic and bounded, so the Monotone Convergence Theorem implies that it converges. Furthermore, the continuity of the angle function ensures this extension is also continuous on the expanded domain. We can perform the same extension to define $\alpha_{T_1^p}(v_2, v_3)$ on $v_1$. For the remainder of the paper, all angle functions will be extended this way and unambiguously defined as follows:

$$\alpha_{T_0^p}(v_2, v_3) : [v_0, v_1] \to [0, 2\pi], \quad \alpha_{T_1^p}(v_2, v_3) : [v_0, v_1] \to [0, 2\pi]$$

Now, proceeding with the argument, because $T_0^p$ and $T_1^p$ share the edge $[v_2, v_3]$, the affine plane $\mathrm{aff}\,\{v_2, v_3, p\}$ used to induce a tetrahedral subdivision of $T$ splits the dihedral angle of $[v_2, v_3] \subset T$ and imposes the following angle relation:

$$\alpha_T(v_2, v_3) = \alpha_{T_0^p}(v_2, v_3) + \alpha_{T_1^p}(v_2, v_3) \quad \text{ for all } p \in \mathrm{relint}([v_0, v_1]) \tag{4.1}$$

This relation and the fact that $T_1^{v_0} = T$ lets us determine $\alpha_{T_0^p}(v_2, v_3)$ at the endpoints.

$$\alpha_{T_0^{v_0}}(v_2, v_3) = \lim_{p \to v_0} \alpha_{T_0^p}(v_2, v_3) = \lim_{p \to v_0} (\alpha_T(v_2, v_3) - \alpha_{T_1^p}(v_2, v_3)) = 0 < \frac{\pi}{2}$$

[3]Although arbitrary, the choice of edges with obtuse dihedral angles is done to simplify notation. Indeed, observe that for all decompositions of $T$ it is always the case that $T_0^p$ contains $v_0$ and $T_1^p$ contains $v_1$.

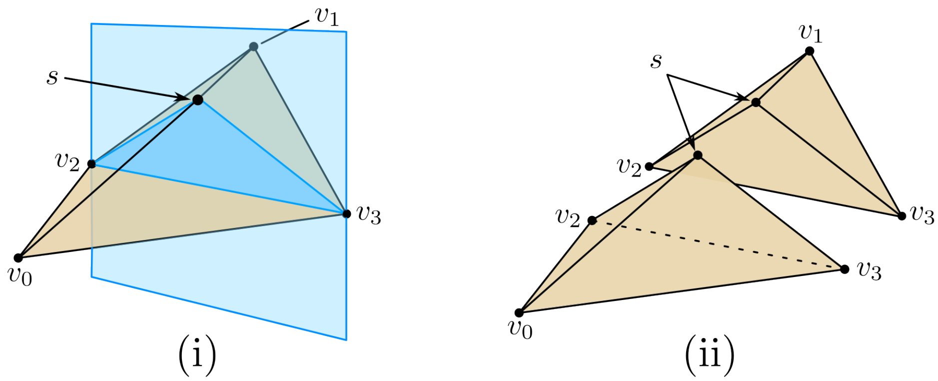


Figure 5: Tetrahedral Subdivision of a Class 2 Subclass (b) tetrahedron (i) Original tetrahedron $T$ with obtuse dihedral angles occurring at edges $[v_0, v_1]$ and $[v_2, v_3]$, and proposed planar bisection. (ii) Two height function tetrahedra $T_0^s = \operatorname{conv}\{v_0, v_1, v_2, s\}$ and $T_1^s = \operatorname{conv}\{v_1, v_2, v_3, s\}$ formed after bisecting.

Then since $T_0^{v_1} = T$,

$$\alpha_{T_0^{v_1}}(v_2, v_3) = \alpha_T(v_2, v_3) > \frac{\pi}{2}$$

The continuity of $\alpha_{T_0^p}(v_2, v_3)$ along with the Intermediate Value Theorem yields some $s \in \operatorname{relint}([v_0, v_1])$ such that $\alpha_{T_0^s}(v_2, v_3) = \frac{\pi}{2}$. Then by the angle relation described in equation 4.1, it follows that $\alpha_{T_1^s}(v_2, v_3) < \frac{\pi}{2}$. Let us consider the tetrahedral subdivision $T = T_0^s \cup T_1^s$. First, the face $F = \operatorname{conv}\{v_0, v_2, v_3\} \subset T_0^s$ has edges $[v_0, v_2], [v_0, v_3]$ and $[v_2, v_3]$. By the choice of $s$ we have $\alpha_{T_0^s}(v_2, v_3) = \frac{\pi}{2}$. Notice that the dihedral angles of other 2 edges remain unchanged by the decomposition. So, by hypothesis, the other 2 edges have non-obtuse dihedral angles in $T_0^s$. An application of Lemma 3.3 shows that F is a foundation of $T_0^s$. Next, consider the face $K = \operatorname{conv}\{v_1, v_2, v_3\} \subset T_1^s$ with edges $[v_1, v_2], [v_1, v_3]$ and $[v_2, v_3]$. We know that $\alpha_{T_1^s}(v_2, v_3) < \frac{\pi}{2}$ and the dihedral angles of the other 2 edges are unchanged, so they must have non-obtuse dihedral angles in $T_1^s$. Thus, by Lemma 3.3, $K$ is a foundation for $T_1^s$. Therefore, $T_0^s$ and $T_1^s$ are both height function tetrahedra relative to their respective foundation faces.

**Class 3 Subclass (a):** Let $e_0, e_1, e_2$ be the only 3 edges with obtuse dihedral angles. Since $T$ is of Class 3 (a), it must be the case that all these edges meet at a single vertex, after possibly relabeling, assume that this vertex is $v_0$. Then consider the face $F = \operatorname{conv}\{v_1, v_2, v_3\}$ formed by the remaining three vertices. None of the three edges of $F$ contain $v_0$, hence every edge of $F$ must have a non-obtuse dihedral angle. Applying Lemma 3.3 shows that $F$ is a foundation. Therefore, $T$ is a height function and no planar bisection is required.

**Class 3 Subclass (b):** Class 3 subclass (b) tetrahedra have 3 edges with obtuse dihedral angles that do not meet at a common vertex. After possibly relabeling vertices, we may assume that $[v_0, v_1], [v_1, v_2]$ and $[v_2, v_3]$ are the edges where obtuse dihedral angles occur. By Theorem 3.2, the remaining edges $[v_0, v_2], [v_0, v_3], [v_1, v_3]$ all have acute dihedral angles. For each point $p \in \mathrm{relint}([v_0, v_1])$ recall the decomposition defined above:

$$T_0^p = \mathrm{conv}\,\{v_0, v_2, v_3, p\} \quad T_1^p = \mathrm{conv}\,\{v_1, v_2, v_3, p\}$$

$$T = T_0^p \cup T_1^p \text{ and } T_0^p \cap T_1^p = \mathrm{conv}\,\{v_2, v_3, p\}$$

Here we will introduce an angle function similar to the one defined in the proof of Class 2 (b) tetrahedra yet better suited to deal with Class 3 (b) tetrahedra (recall these are extended continuously to be defined on endpoints):

$$\alpha_{T_0^p}(v_2, p) : [v_0, v_1] \to [0, \pi], \quad \alpha_{T_1^p}(v_2, p) : [v_0, v_1] \to [0, \pi]$$

$$\alpha_{T_0^p}(v_3, p) : [v_0, v_1] \to [0, \pi], \quad \alpha_{T_1^p}(v_3, p) : [v_0, v_1] \to [0, \pi]$$

In contrast to the angle functions described in class 2 subclass (b), these functions measure dihedral angles along the edges contained in the interior of the shared face $T_0^p \cap T_1^p$ when considered as edges of the respective tetrahedra. These edges arise from the induced intersection structure and therefore depend continuously on $p$, rather than being fixed edges of $T$. Nevertheless, the newly defined functions remain continuous, bounded, and monotonic relative to the natural order of the interval $[v_0, v_1]$.

Since the segments $[v_2, p]$ and $[v_3, p]$ are not edges of the original tetrahedron $T$, the corresponding dihedral angles are computed relative to the affine planes containing the original faces $F_1 = \mathrm{conv}\,\{v_0, v_1, v_3\}$ and $F_2 = \mathrm{conv}\,\{v_0, v_1, v_2\}$ of $T$. Thus, it follows that the angle functions are constrained by a supplementary relationship:

$$\alpha_{T_0^p}(v_2, p) + \alpha_{T_1^p}(v_2, p) = \pi, \quad \alpha_{T_0^p}(v_3, p) + \alpha_{T_1^p}(v_3, p) = \pi \quad \forall p \in \mathrm{relint}([v_0, v_1]) \tag{4.2}$$

Now, because $T_0^{v_1} = T$ we know that

$$\alpha_{T_0^{v_1}}(v_3, v_1) = \alpha_T(v_1, v_3) < \frac{\pi}{2}$$

Similarly, since $T_1^{v_0} = T$, the angle relations above show that,

$$\alpha_{T_0^{v_0}}(v_3, v_0) = \lim_{p \to v_0} \alpha_{T_0^p}(v_3, p) = \lim_{p \to v_0} (\pi - \alpha_{T_1^p}(v_3, p)) = \pi - \alpha_T(v_3, v_0) > \frac{\pi}{2}$$

Then by the Intermediate Value Theorem, there exists some $r \in \mathrm{relint}([v_0, v_1])$ such that

$$\alpha_{T_0^r}(v_3, r) = \frac{\pi}{2}$$

Then by the supplementary relationship described in equation 4.2, it follows that

$$\alpha_{T_1^r}(v_3, r) = \frac{\pi}{2}$$

We will return to this selection of $r$ later.

Now, consider the new edge $[v_2, p]$. We claim that, for all $p \in \mathrm{relint}([v_0, v_1])$, the angle functions satisfy

$$\alpha_{T_1^p}(v_2, p) < \frac{\pi}{2}, \quad \alpha_{T_0^p}(v_2, p) > \frac{\pi}{2}$$

Indeed observe that since $T_0^{v_1} = T$ and $T_1^{v_0} = T$, we have that that

$$\alpha_{T_0^{v_0}}(v_2, v_1) = \alpha_T(v_1, v_2) > \frac{\pi}{2}, \quad \alpha_{T_1^{v_1}}(v_2, v_0) = \alpha_T(v_0, v_2) < \frac{\pi}{2}$$

Applying the supplementary angle relation again yields

$$\alpha_{T_0^{v_0}}(v_2, v_0) = \pi - \alpha_{T_1^{v_0}}(v_2, v_0) > \frac{\pi}{2}$$

Thus, since $\alpha_{T_0^{v_0}}(v_2, v_0)$ and $\alpha_{T_0^{v_1}}(v_2, v_1)$ are greater than $\frac{\pi}{2}$ and $\alpha_{T_0^p}(v_2, p)$ is non-constant, monotonicity guarantees that for all $p \in \operatorname{relint}([v_0, v_1])$,

$$\alpha_{T_0^p}(v_2, p) > \frac{\pi}{2}$$

and consequently, by the supplementary angle relationship

$$\alpha_{T_1^p}(v_3, p) < \frac{\pi}{2}$$

as desired.

Now, from before, fix $r \in \operatorname{relint}([v_0, v_1])$ so that $\Pi = \operatorname{aff}\{v_2, v_3, r\}$ induces a tetrahedral subdivision $T = T_0^r \cup T_1^r$. We first show $T_0^r$ is a height function. Consider the face $F = \operatorname{conv}\{v_2, v_3, r\} \subset T_0^r$. We know that by our selection $r$, we have $\alpha_{T_0^r}(v_2, r) > \frac{\pi}{2}$ and $\alpha_{T_0^r}(r, v_3) = \frac{\pi}{2}$. It is impossible for all three edges of a face to have non-acute dihedral angles, as such, it must be the case that $\alpha_{T_0^r}(v_2, v_3) < \frac{\pi}{2}$. Then instead of $F$, consider the face $F' = \operatorname{conv}\{v_0, v_2, v_3\}$ for which $[v_2, v_3]$ is an edge of. By hypothesis, the dihedral angles of $[v_0, v_2]$ and $[v_0, v_3]$ are acute in $T$ and remain unchanged by the decomposition, and thus must also be acute in $T_0^r$. Therefore, an application of Lemma 3.3 shows that $F' = \operatorname{conv}\{v_0, v_2, v_3\} \subset T_0^r$ is a foundation, and $T_0^r$ is a height function.

Next, we show $T_1^r$ is a height function. Consider the edges $[v_1, v_2], [v_1, r], [v_3, r] \in E(T_1^r)$. By our choice of $r$ we have $\alpha_{T_1^r}(v_3, r) = \frac{\pi}{2}$. Moreover, the other 2 edges inherit obtuse angles from $T$. Hence, all three edges have non-acute dihedral angles, so by Theorem 3.2 we know the 3 remaining edges must have acute dihedral angles in $T_1^r$. In particular, $[v_2, v_3]$ and $[v_2, r]$ have acute dihedral angles. Moreover, $[v_3, r]$ is non-obtuse, thus an application of Lemma 3.3 to the face $K = \operatorname{conv}\{v_2, v_3, r\} \subset T_1^r$ shows that $K$ is a foundation and $T_1^r$ is a height function. Therefore $T_0^r$ and $T_1^r$ are both height functions. $\square$

While developing a proof for Theorem 4.1, a primary motivator for the selection of edges to perform a planar bisection along was based on the need to reduce the number of obtuse dihedral angles present with the expectation of then being able to directly apply Lemma 3.3 to the subsequent sub-tetrahedra. Observe that, for both class 3 subclass (b) and class (2) subclass (b), the affine plane used to induce a tetrahedral subdivision contains an edge whose dihedral angle is obtuse, and passes through a single point contained within the opposite edge whose dihedral angle is also obtuse. Naturally, one may immediately question whether such a phenomenon is purely coincidence or rather a result of underlying structure. After investigating other potential cuts, we find that the latter is in fact true.

# 5 Conclusion and Conjecture

In Section 2, we established the notion of a tetrahedral subdivision and for the remainder of the paper, we restricted our focus to planar bisections of this form. The driving reason behind this specification was ultimately to simplify our task by avoiding complex polyhedra. Although this restriction does not capture the theory of height functions in full generality, for the context of this paper, no significant sacrifices are made by narrowing our scope. In theory, decomposing more complex polyhedra will likely require a broader class of subdivisions. However, in the case tetrahedra, we propose the following conjecture without proof.

**Proposition 5.1.** *Let $T$ be a tetrahedron of Class 2 (b) or Class 3 (b), and let the plane $\Pi$ induce a planar bisection $T = P_0 \cup P_1$, where $P_0$ and $P_1$ are both height function polyhedra. Then this planar bisection must be a tetrahedral subdivision.*

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