Davis Math Lab Project Report, Spring 2026
Low-Degree Approximation of Algorithms

Avery Li, Phoebe McDonald, Scott Ruth
Wenzhou Wu (Graduate Mentor)
Alex Wein (Faculty Mentor)

June 16, 2026

Contents

1 Introduction: The Planted Spike Model and the BBP Phase Transition

1.1 Models

The Wigner Matrix

- Let $W \in \mathbb{R}^{n \times n}$ be a symmetric, standard Gaussian Wigner matrix.
- Off-diagonal entries $W_{ij} = W_{ji}$ are independent $\mathcal{N}(0,1)$, and diagonal entries W_{ii} are independent $\mathcal{N}(0,2)$.
- **Wigner's Semicircle Law:** As $n \to \infty$, the empirical spectral distribution of the eigenvalues of $\frac{1}{\sqrt{n}}W$ converges almost surely to the semicircle distribution bounded by $[-2,2]$:
$$dSC(x) = \frac{1}{2\pi}\sqrt{4-x^2}\mathbf{1}_{[-2,2]}(x)\mathrm{d}x$$

The Planted Spike Model

- Now we introduce a rank-1 deterministic perturbation:
$$W' = \frac{1}{\sqrt{n}}W + \beta a a^T$$
where $\|a\|_2 = 1$ is a unit vector "spike" and $\beta > 0$ is the signal strength.

1.2 Existing Knowledge: BBP Phase Transition

- **The BBP Phase Transition:** As $n \to \infty$, if $\beta > 1$, the largest eigenvalue of W' escapes the noise $[-2,2]$ and converges almost surely to:
$$\lambda_{\max}\left(\frac{1}{\sqrt{n}}W + \beta a a^T\right) \to \beta + \frac{1}{\beta}$$

1.3 Our Problem

- We would like to introduce more precise probability bounds in a non-asymptotic way ($n \not\to \infty$) to support an algorithm for differentiating the Spiked and Unspiked models.

2 Weyl's Inequality for Bounding Intermediate Eigenvalues

Let $\frac{1}{\sqrt{n}}W$ be an $n \times n$ Wigner noise matrix and βaa^T be a rank-1 spike matrix constructed from a unit vector $a \in \mathbb{R}^n$, with signal strength $\beta > 0$. The Spiked Wigner matrix is defined as the sum of these matrices $\frac{1}{\sqrt{n}}W + \beta aa^T$.

$\frac{1}{\sqrt{n}}W$ is real and symmetric by definition, and βaa^T is real and symmetric as $(\beta aa^T)^T = \beta(a^T)^T a^T = \beta aa^T$. This means that they are Hermitian matrices. Hence, their eigenvalues are real and can be ordered in descending order: $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

Because βaa^T is a rank-1 square matrix, it has exactly one non-zero eigenvalue. Thus, we have $\lambda_1(\beta aa^T) = \mathrm{tr}(\beta aa^T)$ and:

$$\lambda_2(\beta aa^T) = \lambda_3(\beta aa^T) = \cdots = \lambda_n(\beta aa^T) = 0$$

Applying Weyl's inequality on our Hermitian matrices $\frac{1}{\sqrt{n}}W$ and βaa^T, we get the following inequality:

$$\lambda_{i+j-1}\left(\frac{1}{\sqrt{n}}W + \beta aa^T\right) \leq \lambda_i\left(\frac{1}{\sqrt{n}}W\right) + \lambda_j(\beta aa^T) \leq \lambda_{i+j-n}\left(\frac{1}{\sqrt{n}}W + \beta aa^T\right)$$

To bound $\lambda_n\left(\frac{1}{\sqrt{n}}W + \beta aa^T\right)$ from below, we use the right-hand side of Weyl's inequality. Evaluating this at indices $i = n$ and $j = n$:

$$\begin{aligned}
&\lambda_n\left(\frac{1}{\sqrt{n}}W\right) + \lambda_n(\beta aa^T) \leq \lambda_{n+n-n}\left(\frac{1}{\sqrt{n}}W + \beta aa^T\right) \\
\Rightarrow \quad &\lambda_n\left(\frac{1}{\sqrt{n}}W\right) + 0 \leq \lambda_n\left(\frac{1}{\sqrt{n}}W + \beta aa^T\right) \\
\Rightarrow \quad &\lambda_n\left(\frac{1}{\sqrt{n}}W\right) \leq \lambda_n\left(\frac{1}{\sqrt{n}}W + \beta aa^T\right)
\end{aligned}$$

Combining these bounds, we get:

$$\lambda_n\left(\frac{1}{\sqrt{n}}W\right) \leq \lambda_n\left(\frac{1}{\sqrt{n}}W + \beta aa^T\right) \leq \lambda_2\left(\frac{1}{\sqrt{n}}W + \beta aa^T\right) \leq \lambda_1\left(\frac{1}{\sqrt{n}}W\right)$$

Therefore, to bound the eigenvalues of the Spiked Wigner matrix, it suffices to bound the extreme eigenvalues of the pure Wigner noise matrix, $\lambda_1\left(\frac{1}{\sqrt{n}}W\right)$ and $\lambda_n\left(\frac{1}{\sqrt{n}}W\right)$ above and below respectively.

3 The Lipschitz Condition

In order to use Gaussian Concentration (Theorem 8.9 of [2]), we must first show $\lambda_{\max}(W + \beta aa^T)$ is a $\sqrt{2/n}$-Lipschitz function.

Let $g \in \mathbb{R}^n \sim \mathcal{N}(0, I_n)$ be the vector with the diagonal and upper triangular entries of W.

Then $f(g) = \lambda_{\max}\left(\frac{1}{\sqrt{n}}W + \beta aa^T\right)$.
Let g_1, g_2 be the vectors corresponding to the independent entries of W_1, W_2.
Then $|f(g_1) - f(g_2)| = \left|\lambda_{\max}\left(\frac{1}{\sqrt{n}}W_1 + \beta aa^T\right) - \lambda_{\max}\left(\frac{1}{\sqrt{n}}W_2 + \beta aa^T\right)\right|$.

$$\begin{aligned}
\text{Using Weyl's Inequality, } &\leq \left|\lambda_{\max}\left(\frac{1}{\sqrt{n}}W_1 - \frac{1}{\sqrt{n}}W_2\right)\right| \\
&\leq \left\|\frac{1}{\sqrt{n}}W_1 - \frac{1}{\sqrt{n}}W_2\right\|_2 \quad \text{(by the spectral norm)} \\
&= \frac{1}{\sqrt{n}}\|W_1 - W_2\|_2 \\
&\leq \frac{1}{\sqrt{n}}\|W_1 - W_2\|_F \quad \text{(by the Frobenius norm)}
\end{aligned}$$

Note that:

$$\|W_1 - W_2\|_F^2 = \sum_{i=1}^{n}(W_{ii_1} - W_{ii_2})^2 + 2\sum_{i<j}(W_{ij_1} - W_{ij_2})^2$$

$$\|g_1 - g_2\|_2^2 = \sum_{i=1}^{n}(W_{ii_1} - W_{ii_2})^2 + \sum_{i<j}(W_{ij_1} - W_{ij_2})^2$$

So, $\|W_1 - W_2\|_F^2 \leq 2\|g_1 - g_2\|_2^2$

$$\Rightarrow \|W_1 - W_2\|_F \leq \sqrt{2}\|g_1 - g_2\|_2$$

$$\Rightarrow |f(g_1) - f(g_2)| \leq \frac{\sqrt{2}}{\sqrt{n}}\|g_1 - g_2\|_2$$

So, f is $\sqrt{2/n}$-Lipschitz continuous and so, we can apply theorem 8.9 (Gaussian Concentration) to the $\lambda_{\max}$ function.

4 Gaussian Concentration

From Section 3 we know that $\lambda_{\max}(W + \beta aa^T)$ is a $\sqrt{2/n}$-Lipschitz function. By Gaussian Concentration [2] we have:

$$\mathbb{P}(|\lambda_{\max}(W + \beta aa^T) - \mathbb{E}\lambda_{\max}(W + \beta aa^T)| \geq t) \leq 2\exp\left(-\frac{t^2 n}{4}\right).$$

We first choose $t = 16\ln^2(n)\sqrt{\frac{(\beta-1)^2}{\beta^2+1} \cdot \frac{1}{n}}$, then, $t^2 = 16^2\ln^4(n)\left(\frac{(\beta-1)^2}{\beta^2+1}\right)\left(\frac{1}{n}\right)$.
Plugging this into the above equation yields:

$$\begin{aligned}
&\mathbb{P}(|\lambda_{\max}(W + \beta aa^T) - \mathbb{E}\lambda_{\max}(W + \beta aa^T)| \geq t) \\
&\leq 2\exp\left(-\frac{16^2\ln^4(n)\frac{(\beta-1)^2}{\beta^2+1} \cdot \frac{n}{n}}{4}\right) \\
&\leq 2\exp\left(-64\ln^4(n)\left(\frac{(\beta-1)^2}{\beta^2+1}\right)\right).
\end{aligned}$$

To establish a gap between the largest eigenvalue and second largest eigenvalue, it now suffices to show that $\mathbb{E}(\lambda_{\max}(W + \beta aa^T)) = \beta + \frac{1}{\beta}$, as we have already shown that all eigenvalues except $\lambda_{\max}$ are concentrated within the range $(-2-\epsilon, 2+\epsilon)$ as shown in 2. There are several methods which we attempted to use to show this, using concentration inequalities and the Trace Method. Concentration inequalities such Slepian's Inequality did not apply well to this problem, as the random variables selected to apply these inequalities are required to be centered, a property which is not inherent to W'. Other concentration inequalities which we tried to adapt included Matrix Chernoff Bounds, which we believe could work but did not come up with a formulation which worked. The final method that we attempted was the Trace Method, described in the following section.

5 The Trace Method

The following section is an incomplete proof containing ideas which follow closely the exposition in Kemp's lecture notes on random matrix theory [3]. It is recommended that one understands concepts such as matrix multiplication as walks over a graph and Dyck paths, the latter of which is explained well in the notes mentioned previously.

Consider a matrix M for which we want a bound for $\mathbb{E}\lambda_{\max}(M)$. Let us order the eigenvalues λ_i from smallest to largest, note that $\lambda_1 + \lambda_2 + \cdots + \lambda_{\max} = \mathrm{tr}(M)$. Then, for positive integer k,

$$(\lambda_1 + \cdots + \lambda_{\max})^{2k} = \mathrm{tr}(M)^{2k} = \mathrm{tr}(M^{2k})$$

and for large k, $\mathbb{E}[\lambda_{\max}^{2k}] = \mathbb{E}[\mathrm{tr}(M^{2k})]$. This motivates us to find a method to calculate $\mathbb{E}[\mathrm{tr}(M^{2k})]$.

Let $M = W'$ and $A = aa^T$, then we have $\frac{1}{n}\mathrm{tr}(M^{2k}) = \frac{1}{n}\mathrm{tr}((\frac{1}{\sqrt{n}}W + \beta A)^{2k}) = \frac{1}{n}\lambda_{\max}^{2k}$ for large k. Expanding the exponent yields a sum with terms of the form

$$\left(\frac{1}{\sqrt{n}}\right)^B \mathrm{tr}(W^{b_1}A^{c_1}W^{b_2}A^{c_2}\cdots W^{b_\ell}A^{c_\ell})$$

where $P = \{b_1, c_1, b_2, c_2, \ldots, b_\ell, c_\ell\}$ is a integer partition of $2k$ with $B = \sum_{i=1}^{\ell} b_i$ and $C = \sum_{i=1}^{\ell} c_i$. Note that we can always write a term in this form, even if there are A's on the left or W's on the right using the cyclic property of the trace, however, this will need to be accounted for later. We know $a^T a = 1 \Rightarrow A^x = A$, so we can simplify further to

$$\begin{aligned}
\mathbb{E}\left[\left(\frac{1}{\sqrt{n}}\right)^B \mathrm{tr}(W^{b_1}A^{c_1}W^{b_2}A^{c_2}\cdots W^{b_\ell}A^{c_\ell})\right] &= \mathbb{E}\left[\frac{1}{n}n^{-B/2}\beta^C \prod_{i=1}^{\ell}\mathrm{tr}(a^T W^{b_i} a)\right] \\
&= n^{-B/2-1}\beta^C \mathbb{E}\left[\prod_{i=1}^{\ell}\mathrm{tr}(a^T W^{b_i} a)\right] \\
&= n^{-B/2-1}\beta^C \mathbb{E}\left[\prod_{i=1}^{\ell} a^T W^{b_i} a\right].
\end{aligned}$$

We know that W and a are spherically symmetric, so $a^T W a$ has the same distribution as if we fixed $a = e_1$, giving the formula

$$n^{-B/2-1}\beta^C \mathbb{E}\left[\prod_{i=1}^{\ell} e_1^T (W^{b_i}) e_1\right] = n^{-B/2-1}\beta^C \mathbb{E}\left[\prod_{i=1}^{\ell} (W^{b_i})_{1,1}\right].$$

Focusing on the expected value term and expanding the $(W^{b_i})_{1,1}$ term gives

$$\prod_{i=1}^{\ell} (W^{b_i})_{1,1} = \prod_{i=1}^{\ell} \left(\sum_{1 \leq (j_i)_2, (j_i)_3, \cdots (j_i)_{b_i} \leq n} W_{1,(j_i)_2} W_{2,(j_i)_3} \cdots W_{(j_i)_{b_i},1} \right).$$

A natural way to index all such combinations to obtain the inner sum is walks of length b_i on n vertices starting at 1 and ending at 1. More formally, let $j_i \in [n]^{b_i}$ be b_i-indexes and $\mathbf{j} = (j_1, j_2, \ldots, j_\ell)$. Define the graph $G_{P,\mathbf{j}} = (V, E)$ as follows:

$$V = \{(j_i)_k : 1 \leq i \leq \ell, 1 \leq k \leq b_i\}$$
$$E = \{\{1, (j_i)_2\}, \{(j_i)_2, (j_i)_3\}, \ldots, \{(j_i)_{b_i}, 1\} : 1 \leq i \leq \ell\}$$

Let $w_{P,\mathbf{j}}$ denote our walk and $w_{P,\mathbf{j}}(e)$ denote how many times an edge e is used in the walk. Note that V includes only the vertices which are walked over and the walk starts at 1 and ends at 1, so $w_{P,\mathbf{j}}$ is a closed walk which covers $G_{P,\mathbf{j}}$, but that there are also multiple such walks which can cover $G_{P,\mathbf{j}}$. There are two such edges which can exists in these walks, self-edges $W_{x,x}$ and connecting-edges $W_{x,y},\ x \neq y$. Let the self-edges and connecting-edges be

$$E^S_{P,\mathbf{j}} = \{\{x, x\} \in E_{P,\mathbf{j}}\}$$
$$E^C_{P,\mathbf{j}} = \{\{x, y\} \in E_{P,\mathbf{j}} : x \neq y\}$$

respectively. The entries of W are defined by independent random variables, so we can write the expected value of our walk as

$$\prod(G_{P,\mathbf{j}}, w_{P,\mathbf{j}}) = \prod_{e_s \in E^S_{P,\mathbf{j}}} \mathbb{E}[(W_{1,1})^{w_{P,\mathbf{j}}(e_s)}] \cdot \prod_{e_r \in E^C_{P,\mathbf{j}}} \mathbb{E}[(W_{1,2})^{w_{P,\mathbf{j}}(e_c)}].$$

Given a partition P, for any $\mathbf{j}$ we have that the total sum of the weights of the dges using in the walk $w_{P,\mathbf{j}}$ is B. Let $\mathcal{G}_P$ denote the set of all pairs (G, w) where $G = (V, E)$ is a connected digraph with at most B vertices and w is a closed walk covering G, $|w| = B$. Additionally, w must start at 1 and return to 1 at times $b_1, b_1 + b_2, \ldots .b_1 + \cdots + b_m$. Informally, $\mathcal{G}_B$ is the set of all pairs (G, w) with $|G| \leq B$, and w is a walk which starts at 1 and returns to 1 at times defined by the partition P.

By observation (and common graph traversal representations of matrix multiplication), $\mathcal{G}_B$ describes the sets of sequences of indices which produce the term

$$\mathbb{E}\left[\prod_{i=1}^{\ell} (W^{b_i})_{1,1}\right],$$

Therefore, we can rewrite a single term of our expansion of $\frac{1}{n}\mathrm{tr}((\frac{1}{\sqrt{n}}W + \beta A)^{2k})$ as

$$\frac{1}{n}\mathbb{E}[\mathrm{tr}(W^{b_1}A^{c_1}\cdots W^{b_\ell}A^{c_\ell})] = \sum_{(G,w)\in\mathcal{G}_B}\prod(G,w)\cdot\frac{\#\{\mathbf{j} : (G_{P,\mathbf{j}}, w_{P,\mathbf{j}}) = (G,w)\}}{n^{B/2}}$$

$$\mathbb{E}[\mathrm{tr}(W^{b_1}A^{c_1}\cdots W^{b_\ell}A^{c_\ell})] = \sum_{(G,w)\in\mathcal{G}_B}\prod(G,w)\cdot\frac{\#\{\mathbf{j} : (G_{P,\mathbf{j}}, w_{P,\mathbf{j}}) = (G,w)\}}{n^{B/2+1}}.$$

Effectively, we are counting the number of ways we can label the vertices while keeping the same value of $\prod(G,w)$ by iterating over all possible walks. Counting the number of such $\mathbf{j}$ is relatively simple, as is it just

$$n(n-1)(n-2)\cdots(n-|G|+1)$$

and because $|G| \leq B/2+1$, our summation simplifies to

$$\sum_{(G,w)\in\mathcal{G}_B}\prod(G,w)\cdot\frac{n(n-1)(n-2)\cdots(n-|G|+1)}{n^{B/2+1}},$$

because any $\mathbf{j}$ with less than $|G|$ vertices will have their terms tend to 0 for large n. There are also additional restrictions on w and G for non-zero terms. For $\prod(G,w)$ to be non-zero the $w(e)$ must be even for each edge in w because $W_{1,1} = \mathcal{N}(0,2), W_{1,2} = \mathcal{N}(0,1)$. Also note that if there are any self-loops, edges with weight $w(e) \geq 3$, $|G| \leq B/2$. Another subtlety is that after returning back to 1, any vertex which was previously visited cannot be visited again aside from 1, because then $n(n-1)(n-2)\cdots(n-|G|+1) = O(n^{B/2})$, and the right term will become 0. This is best understood as each b_i-index $j_i \in \mathbf{j}$ representing a tree (because each edge is traversed twice and no edges have weight $w(e) \geq 3$ as above), and each tree represented by j_i cannot share a vertex with any other tree except for at the vertex labeled 1.

Therefore, we can consider all graphs represented by non-overlapping trees with size b_i each, over n total labels. Let $\mathcal{G}_B^{B/2+1}$ be the set of pairs (G,w) such that G has exactly $B/2+1$ vertices, contains no self-edges, and the walk w crosses every edge exactly 2 times. Note that $\mathcal{G}_B^{B/2+1} \subseteq \mathcal{G}_B$. Then, for large n, condensing the summation to non-zero terms gives

$$\begin{aligned}
&\sum_{(G,w)\in\mathcal{G}_B^{B/2+1}}\prod(G,w)\cdot\frac{n(n-1)(n-2)\cdots(n-|G|+1)}{n^{B/2+1}} + O(n^{-1})\\
&= \sum_{(G,w)\in\mathcal{G}_B^{B/2+1}}\prod(G,w)\cdot\frac{n(n-1)(n-2)\cdots(n-(B/2+1)+1)}{n^{B/2+1}}\\
&= \sum_{(G,w)\in\mathcal{G}_B^{B/2+1}}\prod(G,w)\cdot\frac{n^{B/2+1}}{n^{B/2+1}}\\
&= \sum_{(G,w)\in\mathcal{G}_B^{B/2+1}}\prod(G,w).
\end{aligned}$$

We know that there are no self edges and $w(e) = 2$, so

$$\begin{aligned}
\prod(G, w) &= \prod_{e_c \in E^C} \mathbb{E}[(W_{1,2})^{w(e_c)}] \\
&= \prod_{e_c \in E^C} \mathbb{E}[(W_{1,2})^2] \\
&= \prod_{e_c \in E^C} \mathrm{Var}(W_{1,2}) \\
&= \prod_{e_c \in E^C} 1 \\
&= 1.
\end{aligned}$$

Then the final value of our single term is $\sum_{(G,w) \in \mathcal{G}_B^{B/2+1}} 1 = |\mathcal{G}_B^{B/2+1}|$.

There is a known bijection between $\mathcal{G}_B^{B/2+1}$ and Dyck paths as presented in [3], and it is also known that the number of Dyck paths is equal to $\# D_B^{B/2+1} = C_{B/2}$ where $C_{B/2}$ is the $B/2$-th Catalan number. Therefore,

$$\frac{1}{n}\mathbb{E}\left[\mathrm{tr}\left(\prod_{i=1}^{\ell} W^{b_i} A^{c_i}\right)\right] = C_{B/2}$$

It now suffices to find a way to enumerate over all partitions P (carefully counting partition equal under cyclic permutations) to find

$$\begin{aligned}
\frac{1}{n}\mathbb{E}[\lambda_{\max}^{2k}] &= \frac{1}{n}\mathbb{E}[\mathrm{tr}(A^{2k})] \\
&= \sum_P \left(\frac{1}{n}\mathbb{E}\left[\mathrm{tr}\left(\prod_{i=1}^{\ell} W^{b_i} A^{c_i}\right)\right]\right) \\
&= \sum_P C_{B/2},
\end{aligned}$$

from which $\mathbb{E}[\lambda_{\max}]$ can be found. We did not find a way of enumerating the partitions nor a way to separate the dominating terms, and will leave the final formula here.

6 Bootstrapping Existing Results

Currently, there are results showing that

$$\lambda_{\max}\left(\frac{1}{\sqrt{n}}W + \beta a a^T\right) \to \beta + \frac{1}{\beta}$$

almost surely whenever $\beta > 1$ [1]. This allows us to show our desired result, but for all $n \geq N$, for an unspecified N.

Proof. We know that $\lambda_{\max} \xrightarrow{\text{a.s.}} \beta + \frac{1}{\beta}$, so $\lambda_{\max} \xrightarrow{\text{p}} \beta + \frac{1}{\beta}$. Therefore, for any $\epsilon > 0$, there exists n_0 such that whenever $n \geq n_0$,

$$\mathbb{P}(|\lambda_{\max}(W + \beta aa^T) - \beta + \frac{1}{\beta}| \geq \frac{\epsilon}{3}) < 0.01.$$

Based on the result from the previous section, we can also find n_1 such that whenever $n \geq n_1$,

$$\begin{aligned} &\mathbb{P}(\lambda_{\max}(W + \beta aa^T) - \mathbb{E}\lambda_{\max}(W + \beta aa^T)| \geq t) \\ &\leq 2 \exp\left(-64 \ln^4(n) \left(\frac{(\beta-1)^2}{\beta^2+1}\right)\right) < 0.99. \end{aligned}$$

We can also find n_2 such that whenever $n \geq n_2$,

$$t = 16 \ln^2(n) \sqrt{\frac{(\beta-1)^2}{\beta^2+1} \cdot \frac{1}{n}} < \frac{\epsilon}{3}$$

Thus, for $N = \max(n_0, n_1, n_2)$, whenever $n \geq N$,

$$\begin{aligned} &(\mathbb{E}\lambda_{\max}(W + \beta aa^T) - t, \mathbb{E}\lambda_{\max}(W + \beta aa^T) + t) \cap (\beta + \frac{1}{\beta} - \frac{\epsilon}{3}, \beta + \frac{1}{\beta} + \frac{\epsilon}{3}) \neq \emptyset \\ &\Rightarrow |\mathbb{E}\lambda_{\max}(W + \beta aa^T) - \beta + \frac{1}{\beta}| \leq t + \frac{\epsilon}{3} \end{aligned}$$

Now, we can use the triangle inequality to expand $\left|\lambda_{\max}(W + \beta aa^T) - \left(\beta + \frac{1}{\beta}\right)\right|$:

$$\begin{aligned} \mathbb{P}\left(\left|\lambda_{\max}(W + \beta aa^T) - \left(\beta + \frac{1}{\beta}\right)\right| \geq \epsilon\right) &\leq \mathbb{P}\left(\left|\lambda_{\max}(W + \beta aa^T) - \left(\beta + \frac{1}{\beta}\right)\right| \geq 2t + \frac{\epsilon}{3}\right) \\ &\leq (\mathbb{P}(|\lambda_{\max}(W + \beta aa^T) - \mathbb{E}\lambda_{\max}(W + \beta aa^T)| \\ &\quad + |\mathbb{E}\lambda_{\max}(W + \beta aa^T) - \left(\beta + \frac{1}{\beta}\right)| \geq 2t + \frac{\epsilon}{3})) \\ &\leq \mathbb{P}(|\lambda_{\max}(W + \beta aa^T) - \mathbb{E}\lambda_{\max}(W + \beta aa^T)| + t + \frac{\epsilon}{3} \geq 2t + \frac{\epsilon}{3}) \\ &= \mathbb{P}(|\lambda_{\max}(W + \beta aa^T) - \mathbb{E}\lambda_{\max}(W + \beta aa^T)| \geq t) \\ &\leq 2 \exp\left(-64 \ln^4(n) \left(\frac{(\beta-1)^2}{\beta^2+1}\right)\right). \end{aligned}$$

Therefore, there exists an N such that $\forall n \geq N$,

$$\mathbb{P}\left(\left|\lambda_{\max}(W + \beta aa^T) - \left(\beta + \frac{1}{\beta}\right)\right| \geq \epsilon\right) \leq 2 \exp\left(-64 \ln^4(n) \left(\frac{(\beta-1)^2}{\beta^2+1}\right)\right).$$

□

7 The Stieltjes Transform Method

The following section is entirely separate from Gaussian Concentration and outlines what a proof employing the Stieltjes transform may look like. A useful reference for intuition is Tao's exposition of the semicircle law [4].

Let W_n be a Wigner matrix as defined in 1.1. By the definition of the Stieltjes Transform,

$$\begin{aligned}\int_{\mathbb{R}} \frac{1}{x-z} d\mu_{W_n}(x) &= \frac{1}{n}\sum_{i=1}^{n} \frac{1}{\lambda_i - z} \\ &= \frac{1}{n}\mathrm{tr}(\frac{1}{\sqrt{n}}W_n - zI)^{-1} \\ &= s_W(z),\end{aligned}$$

for $z \notin \mathrm{supp}(\lambda_{W_n})$.

We know that as $n \to \infty$ the ordered eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n \in [-2, 2]$. More formally, by showing that λ_{min} and λ_{max} are both $\sqrt{2/n}$-Lipschitz and using Gaussian Concentration, $\lambda_1, \lambda_2, \ldots, \lambda_n \in [-2-\epsilon_1, 2+\epsilon_1]$ for a large enough n which can be specified. This is important as we will eventually need to show that for all $z \in [\beta+\frac{1}{\beta}-\epsilon_2, \beta+\frac{1}{\beta}+\epsilon_2] \subseteq \mathbb{R}\backslash[-2-\epsilon_1, 2+\epsilon_1]$, our concentration bounds will hold.

We want to find $\lambda_{\max}(W')$, which we can rewrite as

$$\lambda_{\max}(\frac{1}{\sqrt{n}}W + \beta aa^T).$$

For any maximum eigenvalue $\lambda_{\max}(W') = z$ of W', we must have

$$0 = \det\left(zI - \left(\frac{1}{\sqrt{n}}W + \beta aa^T\right)\right) = \det\left(\left(zI - \frac{1}{\sqrt{n}}W\right) - \beta aa^T\right),$$

or equivalently, $\left(zI - \frac{1}{\sqrt{n}}W\right) - \beta aa^T$ is not invertible. Let $A = zI - \frac{1}{n}W$, $u = -\beta a$, $v = a$, by the Sherman–Morrison formula [5] we have

$$\begin{aligned}1 - \beta a^T(zI - \frac{1}{\sqrt{n}}W)^{-1}a &= 0 \\ a^T(zI - \frac{1}{\sqrt{n}}W)^{-1}a &= \frac{1}{\beta}.\end{aligned}$$

Note that for large n, we have that

$$s_W(z) = \frac{1}{n}\mathrm{tr}\left(\frac{1}{\sqrt{n}}W_n - zI)^{-1}\right) \approx a^T\left(zI - \frac{1}{\sqrt{n}}W^{-1}a\right).$$

This we do not currently have high probability bounds for and would be required for a full proof. By substitution,

$$a^T(zI - \frac{1}{\sqrt{n}}W)^{-1}a = s_W(z) = \frac{1}{\beta}.$$

Given we have shown bounds, we will then have that $s_W(z) \to \frac{1}{\beta}$ with high probability and with certain concentration bounds. It is well known that $s_W(z) = \frac{z-\sqrt{z^2-4}}{2}$ satisfies

$s_W(z)^2 - z \cdot s_W(z) + 1 = 0$. Assume for now that $s_W(z) = \frac{1}{\beta}$, then

$$\begin{aligned}
\frac{1}{\beta^2} - z\frac{1}{\beta} + 1 &= 0 \\
\frac{1}{\beta} - z + \beta &= 0 \\
\frac{1}{\beta} + \beta &= z.
\end{aligned}$$

Therefore, we have our eigenvalue $z = \beta + \frac{1}{\beta}$. Note that because z lies outside of $\text{supp}(\lambda_i(\frac{1}{\sqrt{n}}W)) \subseteq [-2, 2]$, we know that z is exactly the largest eigenvalue after using Weyl's Inequality from 2, which will separate any smaller eigenvalues with high probability. At this point the proof would be completed, and we have separated the maximum eigenvalue $\lambda_{\max}$ with high probability in the Planted Spike Model.

8 Conclusion

We have shown that there exists a N such that the $\lambda_{\max}(W')$ is well concentrated around $\beta + \frac{1}{\beta}$, but there still remains the question of whether or not we can do this without an unspecified N. Further work on this problem would be to remove this unspecified N possibly with the following methods. The Trace Method provides a clean formula, but requires careful counting techniques. The Stieltjes Transform Method is a promising alternative, but introduces two other places where concentration bounds needs to be shown.

References

[1] D. Féral and S. Péché, *The Largest Eigenvalue of Rank One Deformations of Large Wigner Matrices*, Communications in Mathematical Physics, 272(1):185–228, 2007.

[2] A. S. Bandeira, A. Singer, and T. Strohmer, *A Cheeger Inequality for the Graph Connection Laplacian*, draft manuscript. Available at: https://people.math.ethz.ch/~abandeira/BandeiraSingerStrohmer-MDS-draft.pdf

[3] T. Kemp, *Math 247A Notes: Random Matrix Theory*, University of California, San Diego. Available at: https://mathweb.ucsd.edu/~tkemp/247A.Notes.pdf

[4] T. Tao, *254A, Notes 4: The Semi-Circular Law*, 2010. Available at: https://terrytao.wordpress.com/2010/02/02/254a-notes-4-the-semi-circular-law/#esn2

[5] W. J. Sherman and W. J. Morrison, *Adjustment of an Inverse Matrix Corresponding to Changes in the Elements of a Given Column or a Given Row of the Original Matrix*, Annals of Mathematical Statistics, 21(1):124–127, 1950.