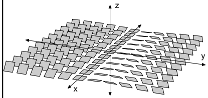
Concave symplectic embeddings and relations in mapping class groups of surfaces

Laura Starkston

University of Texas at Austin

December 6, 2014

Contact
3-manifolds
(M^3, ξ)



Seifert fibered$/S^2$

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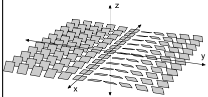

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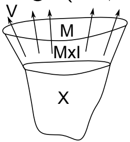

$$\mathcal{L}_V \omega = \omega$$

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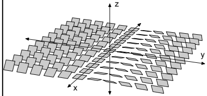

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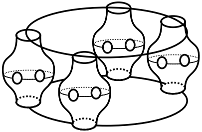
Open book decomposition $\pi : M \setminus L \to S^1$



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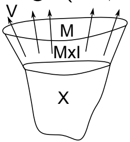

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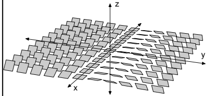

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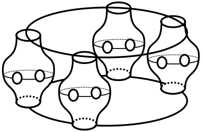

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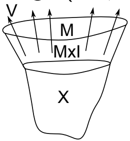

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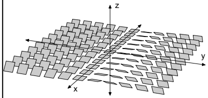

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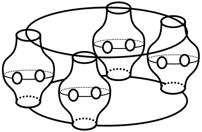

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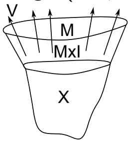

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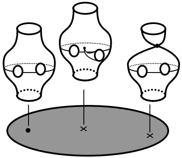
Lefschetz fibration



vanishing cycles $c_1, \cdots, c_n$

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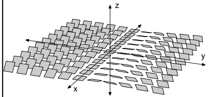

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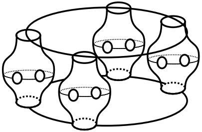

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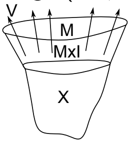

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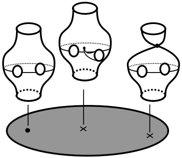

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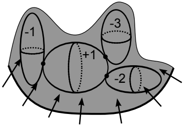
Concave boundary of $\nu(\Sigma_1 \cup \cdots \cup \Sigma_n)$



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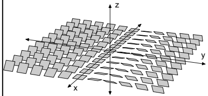

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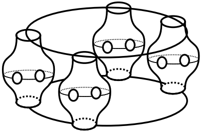

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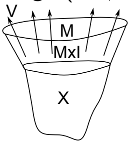

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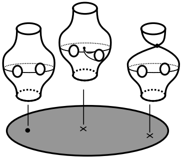

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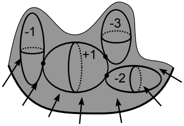

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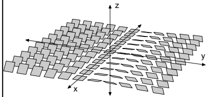

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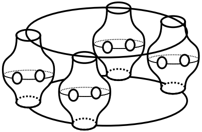

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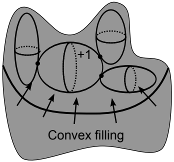
Complement of embedded surfaces in $\mathbb{C}P^2 \# N \overline{\mathbb{C}P^2}$



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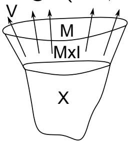

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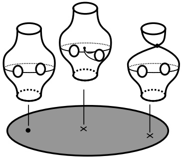

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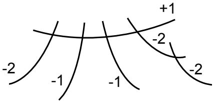
Concave Caps and Convex Fillings

$\Sigma_1, \cdots, \Sigma_n$ surfaces in a 4-manifold intersecting positively transversely $\partial(\nu(\Sigma_1 \cup \cdots \cup \Sigma_n))$ is a Seifert fibered space or more general graph manifold



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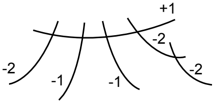

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There exists ω on $\nu(\Sigma_1 \cup \cdots \cup \Sigma_n)$ with

convex boundary	*negative definite*
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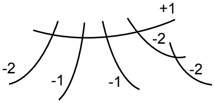

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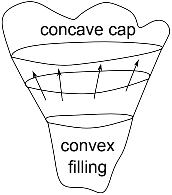
Convex fillings are more rare than *concave caps*.



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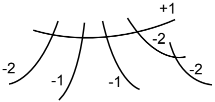

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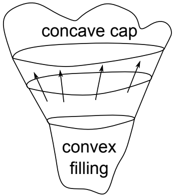


Idea: Use concave caps to find convex fillings.

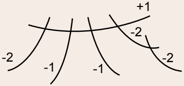
Theorem [McDuff]
A closed symplectic manifold containing a symplectic positive S^2 is symplectomorphic to $\mathbb{C}P^2 \# N \overline{\mathbb{C}P^2}$.

Concave embedding approach

Theorem (S.)

For a Seifert fibered space Y over S^2 with k singular fibers and $e_0 \leq -k-1$, with its canonical contact structure ξ_{can}:

1. *Every convex filling of (Y, ξ_{can}) is the complement of a symplectic embedding of a concave star-shaped plumbing of spheres into $\mathbb{C}P^2 \# N \overline{\mathbb{C}P^2}$.*



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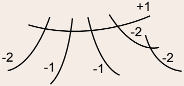

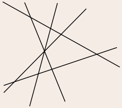

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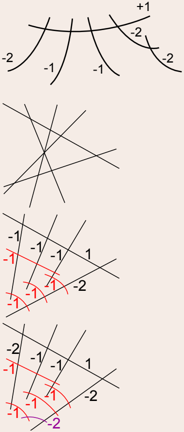
    - *blow-up at N points, including proper transforms of the $\mathbb{C}P^1$'s and exceptional spheres into the concave plumbing*



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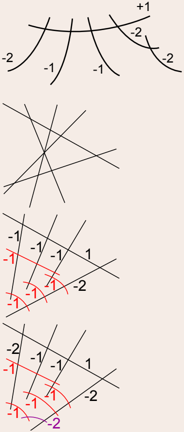
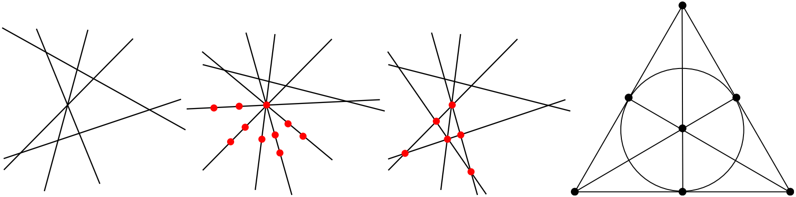
3. *For many such (Y, ξ_{can}) the isotopy class of the embedding is determined by combinatorial/homological data (sufficient conditions: $k \leq 6$ or $e_0 \leq -k-2$).*



Fix an almost complex structure J on $\mathbb{C}P^2$ tamed by ω_{std}.
A *J-line* is a J-holomorphic sphere homologous to $\mathbb{C}P^1$.

1. Through any two points, there is a unique J-line. [Gromov]
2. Any two distinct J-lines transversally, positively, at one point.



Question

Does the topology of the moduli space of J-line arrangements with given combinatorial data depend on J?

Question

Does the combinatorial intersection data determine the isotopy class of the line arrangement?

Pseudoholomorphic Line Arrangments

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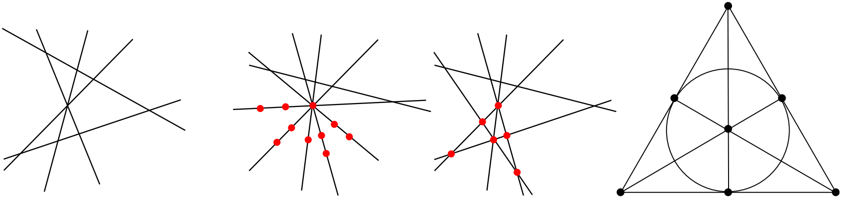


Lemma

If $\mathcal{I}$ is an intersection pattern where no line contains more than two multi-intersection points, and $\mathcal{M}_J(\mathcal{I})$ is the space of J-line arrangements with intersection data $\mathcal{I}$, then

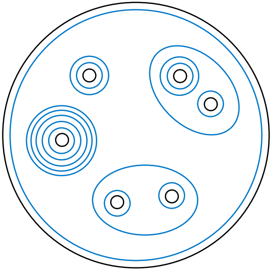
$$\bigcup_{J:\omega_{std}\text{-tame}} \mathcal{M}_J(\mathcal{I})$$

is path-connected and non-empty.

Monodromy factorization approach

[Gay-Mark] For these Seifert fibered spaces, (Y, ξ_{can}), there are open book decompositions with

- Planar pages
- Monodromy $\phi = D_{c_1} D_{c_2} \cdots D_{c_n}$, $c_1, \cdots, c_n$ disjoint

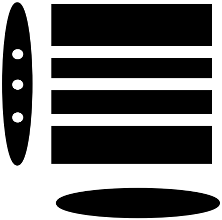


Theorem (Wendl)

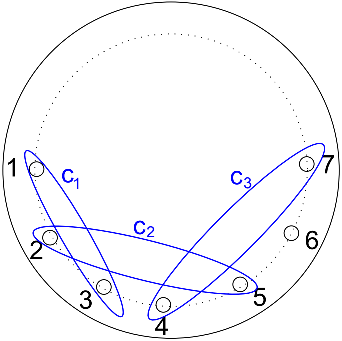
Because these contact structures are planar, each (minimal) convex filling of (Y, ξ_{can}) corresponds to a different positive factorization of ϕ.

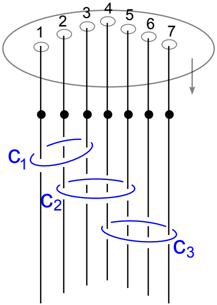
Goal: Establish correspondence between concave embeddings and positive monodromy factorizations.

What does a planar Lefchetz fibration look like?



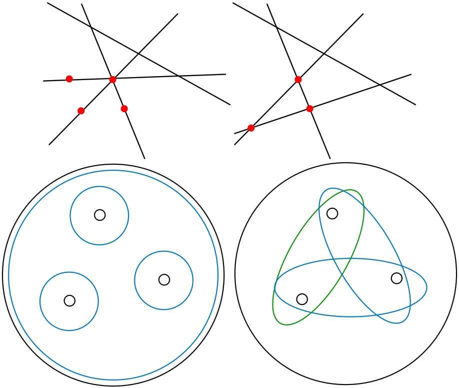
Dotted circle notation: Carve out a disk in B^4 whose boundary in S^3 is the dotted circle.





Lefschetz fibrations and mapping class relations

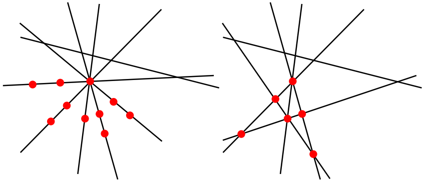
Lantern relation:

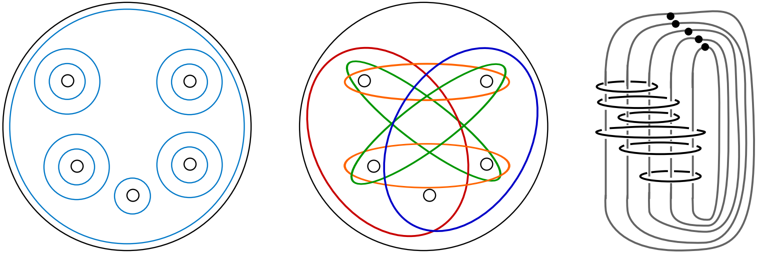


$$D_1 D_2 D_3 D_{1,2,3} = D_{1,2} D_{1,3} D_{2,3}$$

New Configurations

By blowing up more interesting configurations of lines, we get new relations:





$$D_1^2 D_2^2 D_3 D_4^2 D_5^2 D_{1,2,3,4,5} = D_{1,2,3} D_{1,4} D_{1,5} D_{2,4} D_{2,5} D_{3,4,5}$$

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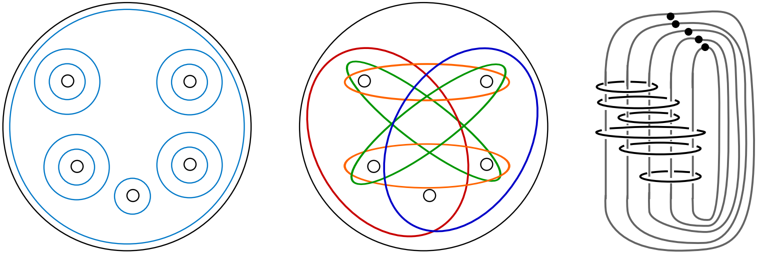
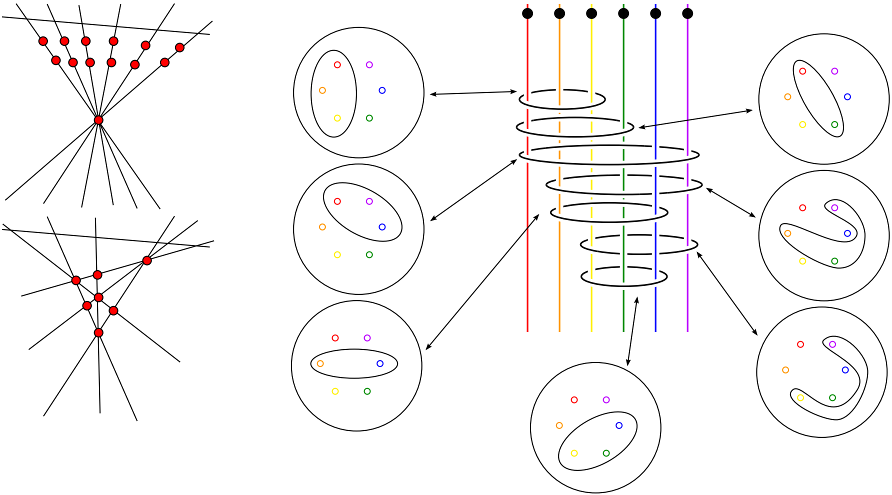


$$D_1^2 D_2^2 D_3 D_4^2 D_5^2 D_{1,2,3,4,5} = D_{1,2,3} D_{1,4} D_{1,5} D_{2,4} D_{2,5} D_{3,4,5}$$

Concave embedding strategy shows: no other fillings $\Rightarrow$ no other + factorizations.

Such *indecomposable* relations are essential relators for elements in Dehn$^+$.

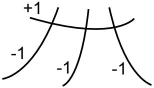
There is an infinite family of indecomposable relations generalizing this example.

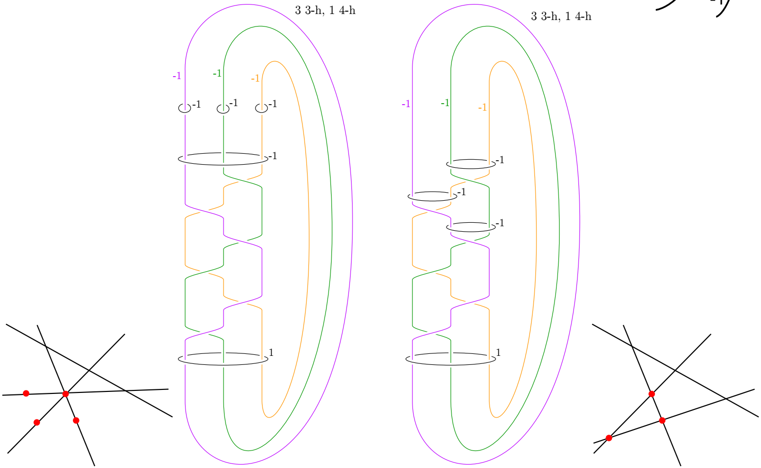


$$D_1^2 D_2^2 D_3^2 D_4^2 D_5^2 D_6^2 D_{1,2,3,4,5,6} =$$

$$D_{1,2,3} D_{1,4} D_{1,5,6} (D_{5,6}^{-1} D_{2,4,6} D_{5,6}) D_{2,5} (D_{4,5,6}^{-1} D_{3,6} D_{4,5,6}) D_{3,4,5}$$

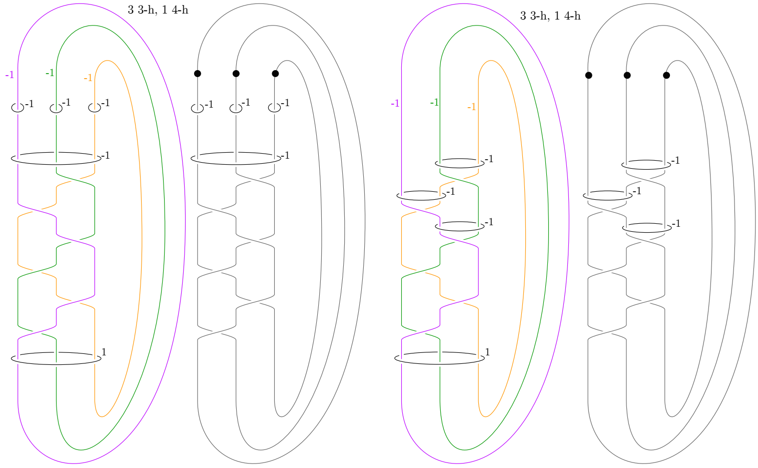
Lantern relation – two embeddings of concave plumbings





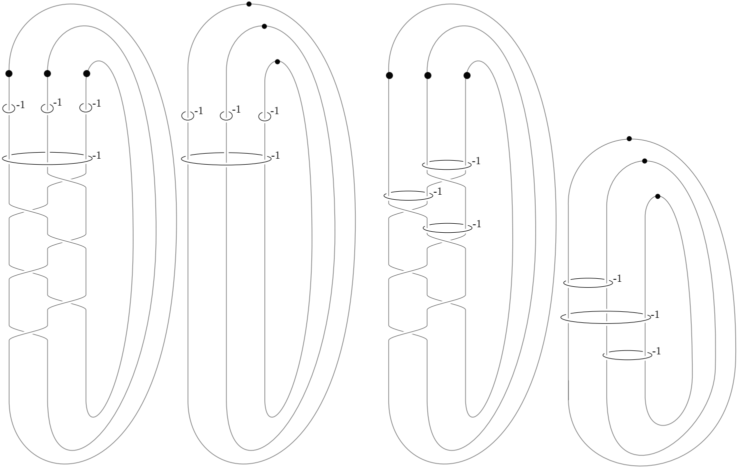
Lefschetz fibrations and mapping class relations

Lantern relation – two embeddings of concave plumbings and their complements

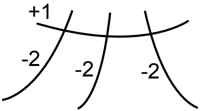


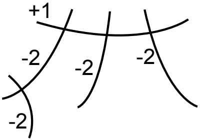
Lefschetz fibrations and mapping class relations

Lantern relation – complements of two embeddings of concave plumbings

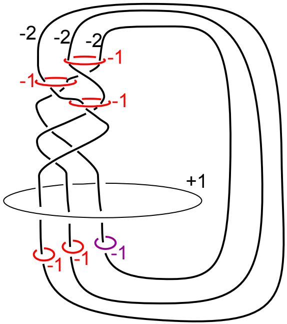


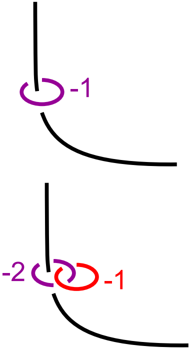
Longer Arms



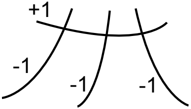


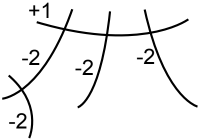
cancelling 3-handles



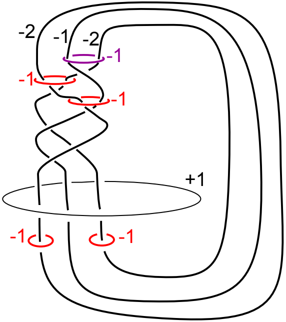


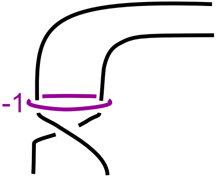
Longer Arms

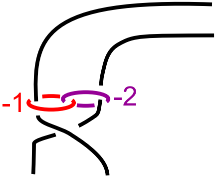


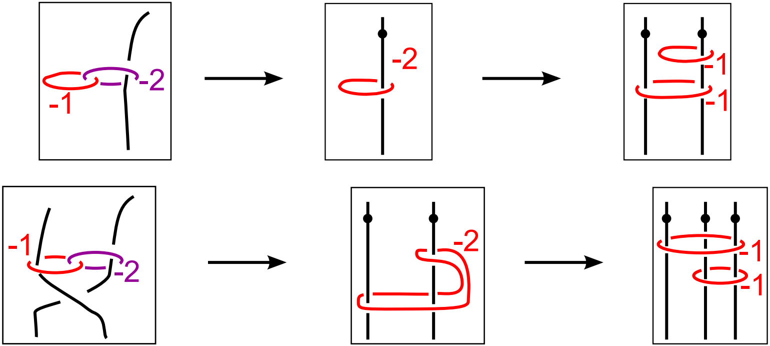


cancelling 3-handles








-1
-2
-2
-1
-1
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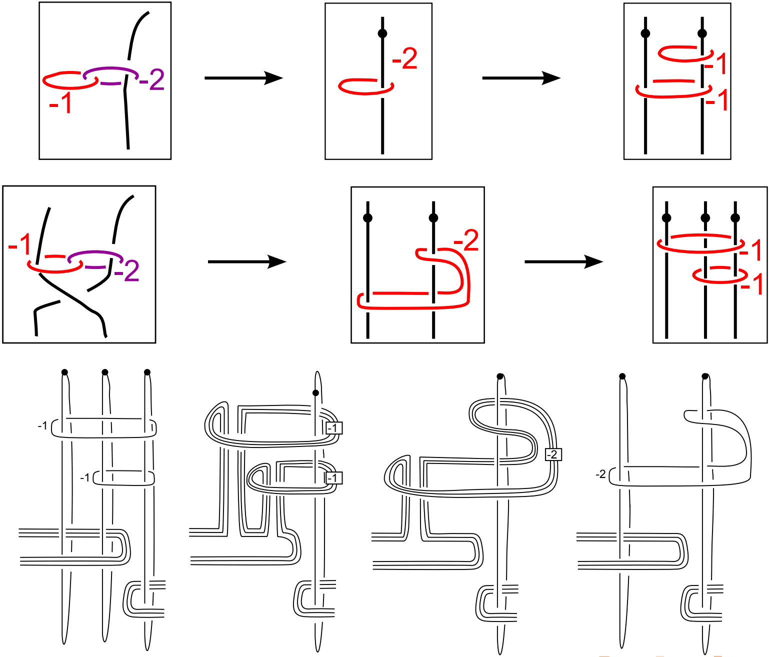
-2
-1
-1


-1
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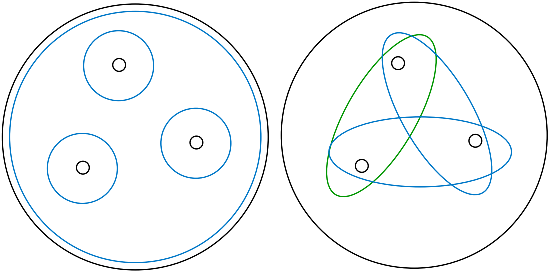
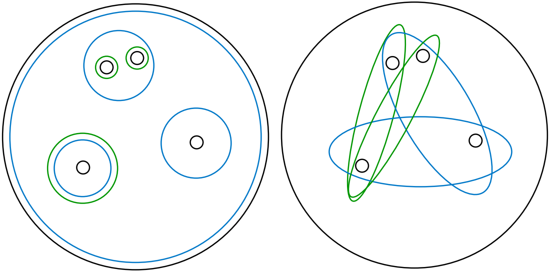
$$D_1 D_2 D_3 D_{1,2,3} = D_{1,2} D_{1,3} D_{2,3}$$



$$D_{1_a} D_{1_b} D_{1a1b} D_2^2 D_3 D_{1_a,1_b,2,3} = D_{1_a,2} D_{1_b,2} D_{1_a,1_b,3} D_{2,3}$$



Moving towards a complete dictionary: other moves

- What is a complete list of embedding moves, and how do they each translate to moves on mapping class group relations?
- How does a sequence of embedding moves translate to a sequence of mapping class group relations?