Concave symplectic embeddings and relations in mapping class groups of surfaces

Laura Starkston

University of Texas at Austin

December 6, 2014
Contact 3-manifolds \((M^3, \xi)\)

Seifert fibered/\(S^2\)
Contact 3-manifolds \((M^3, \xi)\)

Seifert fibered/\(S^2\)

Symplectic convex fillings \((X^4, \omega)\)

\[\mathcal{L}_V \omega = \omega\]
Contact 3-manifolds \((M^3, \xi)\)

Seifert fibered/\(S^2\)

Open book decomposition
\[\pi : M \setminus L \rightarrow S^1\]

Symplectic convex fillings \((X^4, \omega)\)
\[\mathcal{L}_V \omega = \omega\]
Contact 3-manifolds \((M^3, \xi)\)

Seifert fibered/S\(^2\)

Open book decomposition
\[ \pi : M \setminus L \to S^1 \]

\[ \Sigma = \pi^{-1}(0) \]

"page"
\[ \phi : \Sigma \to \Sigma \]

monodromy

Symplectic convex fillings \((X^4, \omega)\)

\[ \mathcal{L}_V \omega = \omega \]
Contact 3-manifolds $(M^3, \xi)$

Seifert fibered/$S^2$

Open book decomposition

$\pi : M \setminus L \to S^1$

$\Sigma = \pi^{-1}(0)$

“page”

$\phi : \Sigma \to \Sigma$

monodromy

Symplectic convex fillings $(X^4, \omega)$

$\mathcal{L}_V \omega = \omega$

Lefschetz fibration

vanishing cycles $c_1, \cdots, c_n$
Contact 3-manifolds \((M^3, \xi)\)

Seifert fibered/\(S^2\)

Open book decomposition
\[\pi : M \setminus L \to S^1\]

\[\Sigma = \pi^{-1}(0)\]

"page"
\[\phi : \Sigma \to \Sigma\]
monodromy

Symplectic convex fillings \((X^4, \omega)\)

\[\mathcal{L}_V \omega = \omega\]

Lefschetz fibration
vanishing cycles \(c_1, \ldots, c_n\)

factorization of monodromy
\[\phi = D_{c_1} \cdots D_{c_n}\]
Right-handed Dehn twists
Concave boundary of $\nu(\Sigma_1 \cup \cdots \cup \Sigma_n)$

Contact 3-manifolds $(M^3, \xi)$

Seifert fibered/$S^2$

Open book decomposition $\pi : M \setminus L \to S^1$

$\Sigma = \pi^{-1}(0)$

“page”

$\phi : \Sigma \to \Sigma$ monodromy

Symplectic convex fillings $(X^4, \omega)$

$L_v \omega = \omega$

Lefschetz fibration

vanishing cycles $c_1, \cdots, c_n$

factorization of monodromy $\phi = D_{c_1} \cdots D_{c_n}$

Right-handed Dehn twists
Concave boundary of $\nu(\Sigma_1 \cup \cdots \cup \Sigma_n)$

Contact 3-manifolds $(M^3, \xi)$

Seifert fibered/$S^2$

Open book decomposition

$\pi : M \setminus L \to S^1$

Seifert fibered/$S^2$

Contact 3-manifolds $(M^3, \xi)$

Open book decomposition

$\pi : M \setminus L \to S^1$

$\Sigma = \pi^{-1}(0)$

"page"

$\phi : \Sigma \to \Sigma$

monodromy

Complement of embedded surfaces in $\mathbb{CP}^2 \# N \overline{\mathbb{CP}^2}$

Symplectic convex fillings $(X^4, \omega)$

$L_V \omega = \omega$

Lefschetz fibration

vanishing cycles $c_1, \cdots, c_n$

factorization of monodromy

$\phi = D_{c_1} \cdots D_{c_n}$

Right-handed Dehn twists

Laura Starkston (University of Texas at Austin)  Concave symplectic embeddings and relations in mapping class groups of surfaces  December 6, 2014  2 / 1
\[ \Sigma_1, \cdots, \Sigma_n \] surfaces in a 4-manifold intersecting positively transversely \( \partial(\nu(\Sigma_1 \cup \cdots \cup \Sigma_n)) \) is a Seifert fibered space or more general graph manifold.
Concave Caps and Convex Fillings

$\Sigma_1, \cdots, \Sigma_n$ surfaces in a 4-manifold intersecting positively transversely $\partial(\nu(\Sigma_1 \cup \cdots \cup \Sigma_n))$ is a Seifert fibered space or more general graph manifold.

Theorem (Gay-Stipsicz, Li-Mak)

There exists $\omega$ on $\nu(\Sigma_1 \cup \cdots \cup \Sigma_n)$ with

<table>
<thead>
<tr>
<th>convex boundary</th>
<th>negative definite</th>
</tr>
</thead>
<tbody>
<tr>
<td>concave boundary</td>
<td>with enough $b_2^+$</td>
</tr>
</tbody>
</table>
Concave Caps and Convex Fillings

$\Sigma_1, \cdots, \Sigma_n$ surfaces in a 4-manifold intersecting positively transversely $\partial(\nu(\Sigma_1 \cup \cdots \cup \Sigma_n))$ is a Seifert fibered space or more general graph manifold

Theorem (Gay-Stipsicz, Li-Mak)

There exists $\omega$ on $\nu(\Sigma_1 \cup \cdots \cup \Sigma_n)$ with

<table>
<thead>
<tr>
<th>convex boundary</th>
<th>negative definite</th>
</tr>
</thead>
<tbody>
<tr>
<td>concave boundary</td>
<td>with enough $b_2^+$</td>
</tr>
</tbody>
</table>

Convex fillings are more rare than concave caps.
Σ₁, · · · , Σₙ surfaces in a 4-manifold intersecting positively transversely ∂(ν(Σ₁ ∪ · · · ∪ Σₙ)) is a Seifert fibered space or more general graph manifold

Theorem (Gay-Stipsicz, Li-Mak)

There exists ω on ν(Σ₁ ∪ · · · ∪ Σₙ) with

| convex boundary | negative definite |
| concave boundary | with enough b₂⁺ |

Convex fillings are more rare than concave caps.

Idea: Use concave caps to find convex fillings.

Theorem [McDuff]
A closed symplectic manifold containing a symplectic positive S² is symplectomorphic to CP² # N CP².
Theorem (S.)

For a Seifert fibered space $Y$ over $S^2$ with $k$ singular fibers and $e_0 \leq -k - 1$, with its canonical contact structure $\xi_{\text{can}}$:

1. Every convex filling of $(Y, \xi_{\text{can}})$ is the complement of a symplectic embedding of a concave star-shaped plumbing of spheres into $\mathbb{CP}^2 \# N \mathbb{CP}^2$. 

Laura Starkston (University of Texas at Austin)  
Concave symplectic embeddings and relations in mapping class groups of surfaces  
December 6, 2014
Theorem (S.)

For a Seifert fibered space $Y$ over $S^2$ with $k$ singular fibers and $e_0 \leq -k - 1$, with its canonical contact structure $\xi_{\text{can}}$:

1. Every convex filling of $(Y, \xi_{\text{can}})$ is the complement of a symplectic embedding of a concave star-shaped plumbing of spheres into $\mathbb{CP}^2 \# N \mathbb{CP}^2$.

2. Every such embedding can be built from
   - A collection of pseudoholomorphic $\mathbb{CP}^1$'s
Concave embedding approach

Theorem (S.)

For a Seifert fibered space $Y$ over $S^2$ with $k$ singular fibers and $e_0 \leq -k - 1$, with its canonical contact structure $\xi_{\text{can}}$:

1. Every convex filling of $(Y, \xi_{\text{can}})$ is the complement of a symplectic embedding of a concave star-shaped plumbing of spheres into $\mathbb{CP}^2 \# N \mathbb{CP}^2$.

2. Every such embedding can be built from:
   - A collection of pseudoholomorphic $\mathbb{CP}^1$’s
   - blow-up at $N$ points, including proper transforms of the $\mathbb{CP}^1$’s and exceptional spheres into the concave plumbing
Theorem (S.)

For a Seifert fibered space $Y$ over $S^2$ with $k$ singular fibers and $e_0 \leq -k - 1$, with its canonical contact structure $\xi_{\text{can}}$:

1. Every convex filling of $(Y, \xi_{\text{can}})$ is the complement of a symplectic embedding of a concave star-shaped plumbing of spheres into $\mathbb{C}P^2 \# N \mathbb{C}P^2$.

2. Every such embedding can be built from
   - A collection of pseudoholomorphic $\mathbb{C}P^1$’s
   - blow-up at $N$ points, including proper transforms of the $\mathbb{C}P^1$’s and exceptional spheres into the concave plumbing

3. For many such $(Y, \xi_{\text{can}})$ the isotopy class of the embedding is determined by combinatorial/homological data (sufficient conditions: $k \leq 6$ or $e_0 \leq -k - 2$).
Pseudoholomorphic Line Arrangements

Fix an almost complex structure $J$ on $\mathbb{CP}^2$ tamed by $\omega_{\text{std}}$. A $J$-line is a $J$-holomorphic sphere homologous to $\mathbb{CP}^1$.

1. Through any two points, there is a unique $J$-line. [Gromov]
2. Any two distinct $J$-lines transversally, positively, at one point.

Question

Does the topology of the moduli space of $J$-line arrangements with given combinatorial data depend on $J$?

Question

Does the combinatorial intersection data determine the isotopy class of the line arrangement?
Fix an almost complex structure $J$ on $\mathbb{C}P^2$ tamed by $\omega_{std}$. A $J$-line is a $J$-holomorphic sphere homologous to $\mathbb{C}P^1$.

**Lemma**

If $\mathcal{I}$ is an intersection pattern where no line contains more than two multi-intersection points, and $\mathcal{M}_J(\mathcal{I})$ is the space of $J$-line arrangements with intersection data $\mathcal{I}$, then

$$\bigcup_{J:\omega_{std}\text{-tame}} \mathcal{M}_J(\mathcal{I})$$

is path-connected and non-empty.
**Monodromy factorization approach**

[Gay-Mark] For these Seifert fibered spaces, \((Y, \xi_{can})\), there are open book decompositions with

- Planar pages
- Monodromy \(\phi = D_{c_1}D_{c_2}\cdots D_{c_n}\),
  \(c_1, \cdots, c_n\) disjoint

---

**Theorem (Wendl)**

*Because these contact structures are planar, each (minimal) convex filling of \((Y, \xi_{can})\) corresponds to a different positive factorization of \(\phi\).*

**Goal:** Establish correspondence between concave embeddings and positive monodromy factorizations.
What does a planar Lefchetz fibration look like?

**Dotted circle notation:** Carve out a disk in $B^4$ whose boundary in $S^3$ is the dotted circle.
Lefschetz fibrations and mapping class relations

Lantern relation:

\[ D_1 D_2 D_3 D_{1,2,3} = D_{1,2} D_{1,3} D_{2,3} \]
New Configurations

By blowing up more interesting configurations of lines, we get new relations:

$$D_1^2 D_2^2 D_3 D_4^2 D_5^2 D_{1,2,3,4,5} = D_{1,2,3} D_{1,4} D_{1,5} D_{2,4} D_{2,5} D_{3,4,5}$$
Concave embedding strategy shows: no other fillings $\Rightarrow$ no other $+$ factorizations.

Such *indecomposable* relations are essential relators for elements in $\text{Dehn}^+$. There is an infinite family of indecomposable relations generalizing this example.
\[ D_1^2 D_2^2 D_3^2 D_4^2 D_5^2 D_6^2 D_{1,2,3,4,5,6} = D_{1,2,3} D_{1,4} D_{1,5,6} (D_{5,6}^{-1} D_{2,4,6} D_{5,6}) D_{2,5} (D_{4,5,6}^{-1} D_{3,6} D_{4,5,6}) D_{3,4,5} \]
Lantern relation – two embeddings of concave plumbings
Lantern relation – two embeddings of concave plumbings and their complements
Lefschetz fibrations and mapping class relations

Lantern relation – complements of two embeddings of concave plumbings

Laura Starkston (University of Texas at Austin)
Longer Arms

cancelling 3-handles

Laura Starkston (University of Texas at Austin)  Concave symplectic embeddings and relations in map
Longer Arms

cancelling 3-handles
\[ D_1 D_2 D_3 D_{1,2,3} = D_{1,2} D_{1,3} D_{2,3} \]

\[ D_{1_a} D_{1_b} D_{1a1b} D_2^2 D_3 D_{1,a,1b,2,3} = D_{1_a,2} D_{1b,2} D_{1a,1b,3} D_{2,3} \]
Moving towards a complete dictionary: other moves

- What is a complete list of embedding moves, and how do they each translate to moves on mapping class group relations?
- How does a sequence of embedding moves translate to a sequence of mapping class group relations?

```
- a b a-1 b-1
- a-1 b-2
- a-2 b-2
```

Laura Starkston (University of Texas at Austin)
Concave symplectic embeddings and relations in mapping class groups of surfaces
December 6, 2014