Lie bracket as linearized adjoint action

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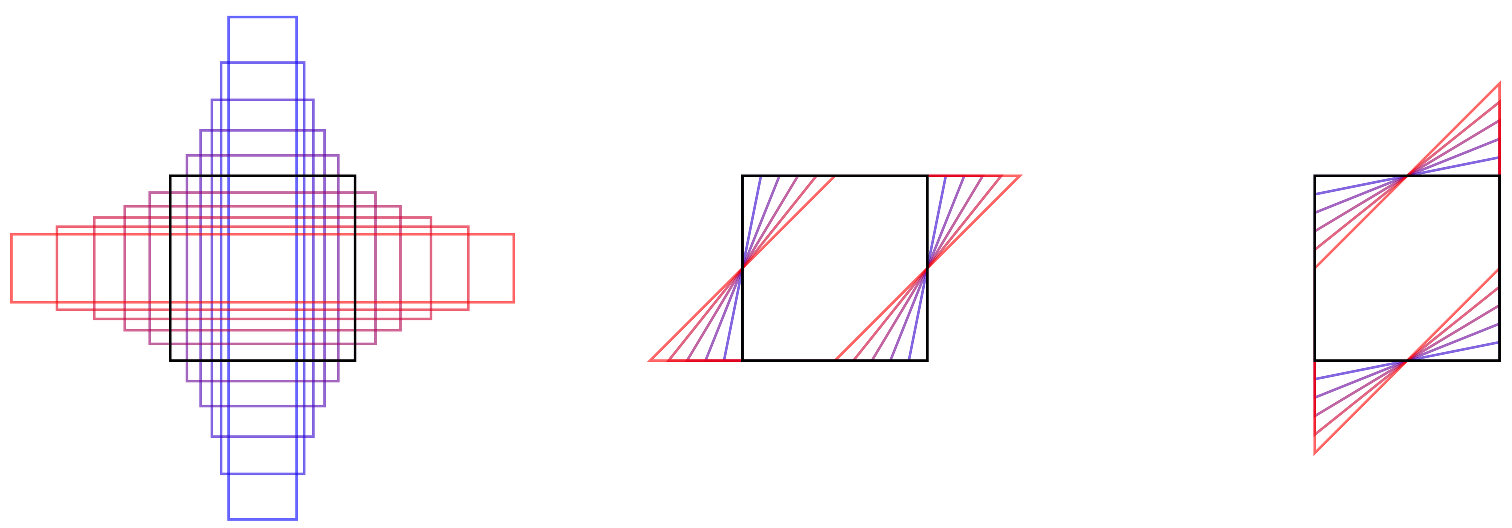
Lie Group

A Lie group is a *group* that is also a *manifold* (a group with a continuum).

Examples

- Area preserving linear transformations
  $\mathrm{SL}(2) = \{ M \in \mathrm{GL}(2;\mathbb{R}) \mid \det M = 1 \}$



$$\begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \qquad \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}$$

- Translation operators $\{ \mathscr{T}_t : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}) \mid t \in \mathbb{R} \}$ defined by $\mathscr{T}_t[f](x) = f(x+t)$.
- Rotation matrices in 3 dimensions $\mathrm{SO}(3)$

$$R_{xy}(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix},$$

$$\begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

Lie Algebra

Let point p on the manifold $\mathcal{M}$ act on function $f \in C^\infty(\mathcal{M})$ by $p(f) = f(p)$. We can define the difference between two points p_1, p_2 as

$$(p_1 - p_2) f = f(p_1) - f(p_2).$$

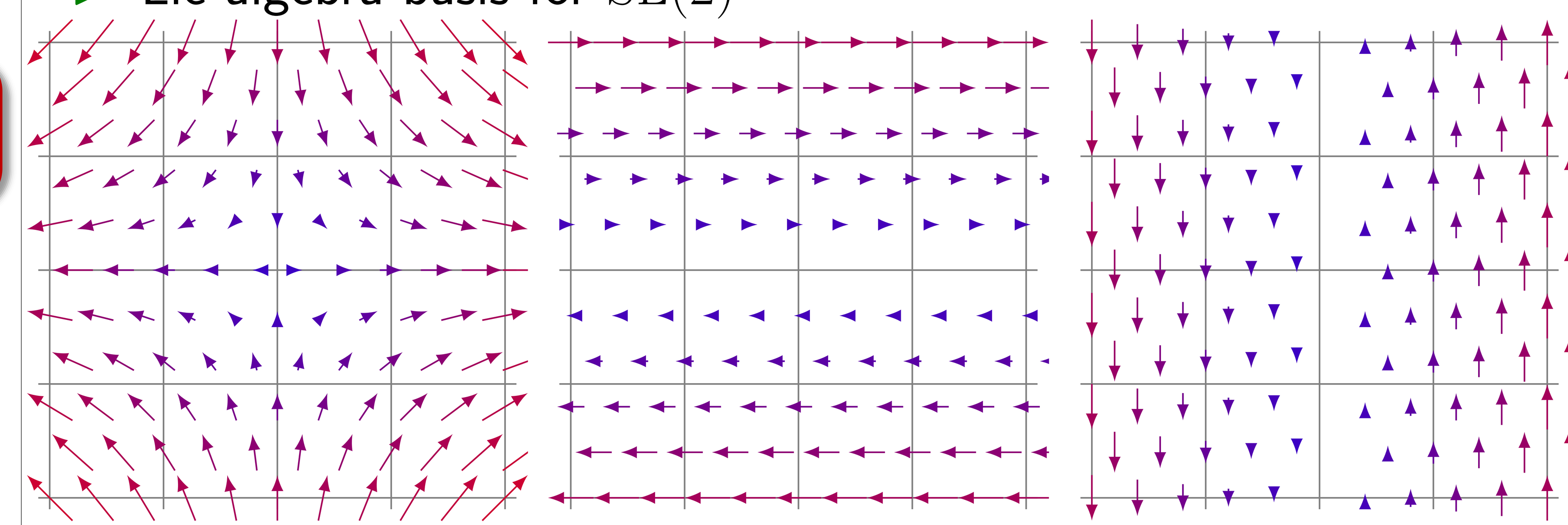
A *vector* is the "velocity" of a parameterized curve σ_t at σ_0, that is

$$\left.\frac{d\sigma_t}{dt}\right|_{t=0} = \lim_{t\to 0} \frac{\sigma_t - \sigma_0}{t}.$$

The *Lie algebra* of a Lie group is the space of all vectors at the identity element.

Examples

- Lie algebra basis for $\mathrm{SL}(2)$



$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

- The derivative operator $\frac{d}{dx}$ is a Lie algebra element of translation operator because

$$\begin{aligned} \left(\left.\frac{d}{dt}\mathscr{T}_t\right|_{t=0} \right) [f](x) &= \lim_{t\to 0} \frac{\mathscr{T}_t[f](x) - \mathscr{T}_0[f](x)}{t} \\ &= \lim_{t\to 0} \frac{f(x+t) - f(x)}{t} \\ &= \frac{df}{dx}(x). \end{aligned}$$

- Lie algebra basis for $\mathrm{SO}(3)$

$$L_{xy} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

Exponential map

For every A in Lie algebra, there exists a unique one-parameter group, σ_t, which is a curve on the Lie group, that satisfies

1. $\left.\frac{d\sigma_t}{dt}\right|_{t=0} = A$, and
2. $\sigma_t \sigma_s = \sigma_{t+s}$.

We suggestively write this unique map as

$$\sigma_t = \exp(tA).$$

We call A the *infinitesimal generator* of the one-parameter group σ_t.

Examples

- Let X be a vector field on $\mathbb{R}^2$, then $\exp(tX)$ generates a *flow* (one-parameter group of diffeomorphism) on $\mathbb{R}^2$
- $\exp\left(t\frac{d}{dx}\right) = \mathscr{T}_t$
- $\exp(\theta L_{xy}) = R_{xy}(\theta)$

Adjoint representation

A Lie group can be almost completely captured by its Lie algebra through adjoint representation!

Let G be a Lie group. Consider adjoint action $\mathrm{Ad}_x : G \to G$ for $x \in G$ defined by

$$\mathrm{Ad}_x(y) = xyx^{-1}.$$

The push forward (aka the adjoint representation) on Lie algebra element Y is

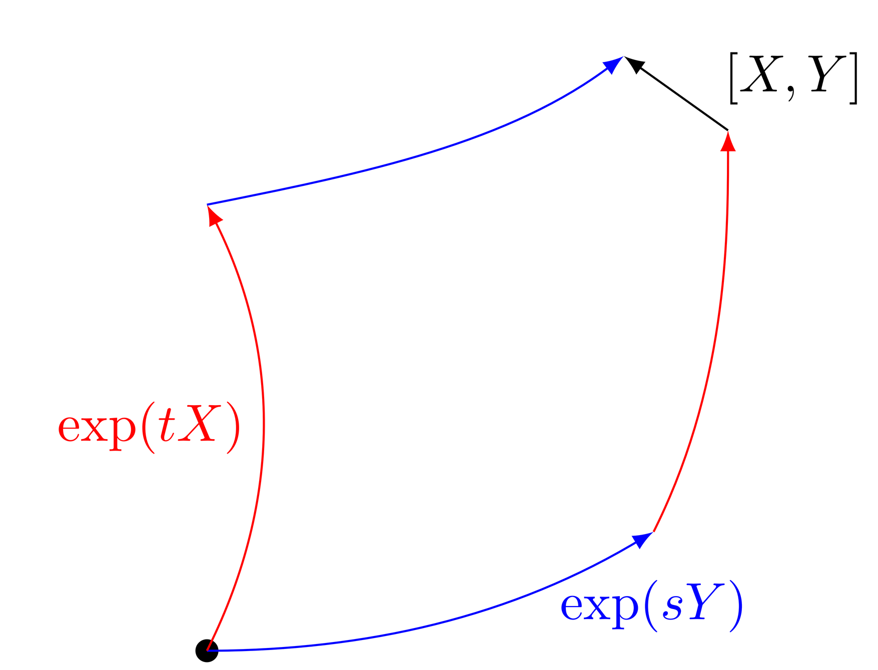
$$\begin{aligned} \mathrm{Ad}_{x*} Y &= \left.\frac{d}{dt} \mathrm{Ad}_x \exp(tY)\right|_{t=0} \\ &= \lim_{t\to 0} \frac{x \exp(tY) x^{-1} - e}{t} \end{aligned}$$

where e is the identity element.

We linearize x as $\left.\frac{d}{dt}\exp(tX)\right|_{t=0}$, and we obtain

$$\begin{aligned} &\left.\frac{d}{dt} \mathrm{Ad}_{\exp(tX)*} Y\right|_{t=0} \\ &= \left.\frac{d}{dt} \exp(tX) Y \exp(-tX)\right|_{t=0} \\ &= \left.\frac{d}{dt}\exp(tX)\right|_{t=0} Y + Y \left.\frac{d}{dt}\exp(-tX)\right|_{t=0} \\ &= XY - YX \\ &= [X, Y]. \end{aligned}$$

Note that the commutator (aka Lie bracket) $[X, Y]$ is a bilinear operation, and it empowers Lie algebra to capture some structure of the Lie group.



$$[X, Y] = \lim_{\substack{s\to 0 \\ t\to 0}} \frac{\exp(tX)\exp(sY) - \exp(sY)\exp(tX)}{st}$$