The $3n + 1$ Problem

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The 3n + 1 Problem (also called the "Collatz Conjecture") is an unsolved problem in mathematics. It was proposed by Lother Collatz in 1937 and is sometimes referred to as "one of the most elementary unsolved problem in mathematics." It's simplicity lures many mathematicians, but experts of the problem warn that the problem is a trap that wastes the time of capable mathematicians and should be avoided. Some believe that the field of mathematics is not yet developed enough to solve this problem.

The Simple Problem

Consider the following recursive function where the output is plugged back into f(n). This function is called the Collatz Function

$$f(n) = \begin{cases} \frac{n}{2} & n \equiv 0 \text{ (mod 2)} \\ 3n+1 & n \equiv 1 \text{ (mod 2)} \end{cases}$$

It is believed that for any natural number inputted into the function, 1 will be outputted after a finite amount of iterations.

Sequences produced by the Collatz Function

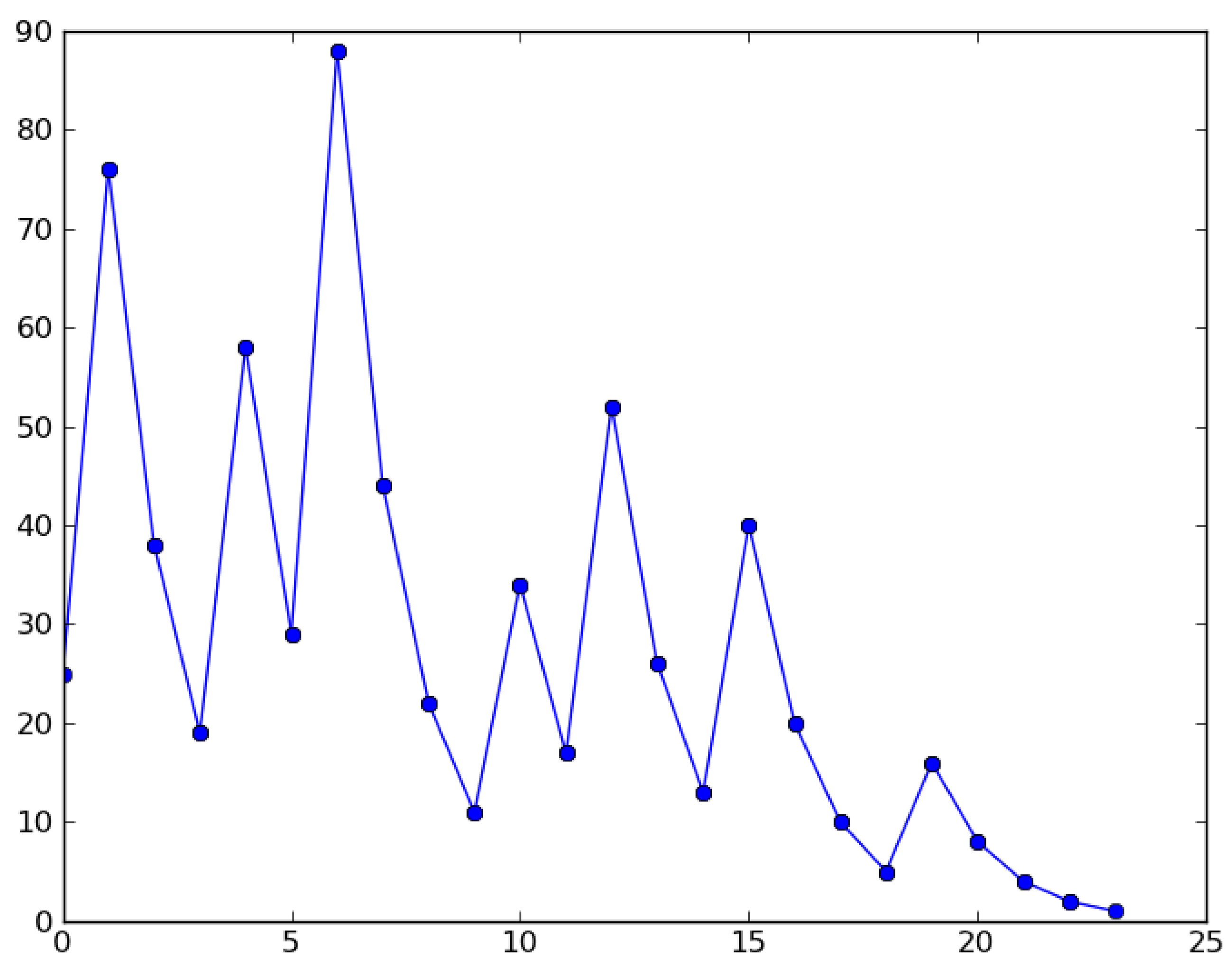
$f^0(n)$	$f^1(n)$	$f^2(n)$	$f^3(n)$	$f^4(n)$	$f^5(n)$	$f^6(n)$	$f^7(n)$	$f^8(n)$	$f^9(n)$
1	4	2	1	4	2	1	4	2	1
2	1	4	2	1	4	2	1	4	2
3	10	5	16	8	4	2	1	4	2
4	2	1	4	2	1	4	2	1	4
5	16	8	4	2	1	4	2	1	4
6	3	10	5	16	8	4	2	1	4

Collatz Terminology

1. **Collatz Number** a number that will generate 1 after a finite number of iterations through the Collatz Function.
2. **Total Stopping Distance** the minimum number of iterations required for the Collatz Function to output 1 for a given input.
$$n_{\text{tsd}} = \min\{k \in \mathbb{N} : f^k(n) = 1\}$$
3. **Stopping Distance** the number of iterations required for the the Collatz Function to output a number smaller than the original input.
$$n_{\text{sd}} = \min\{k \in \mathbb{N} : f^k(n) < n\}$$

Sequence produced by 25 iterating through the Collatz Function:

25_{tsd} = 23, 25_{sd} = 3



Interesting Result From Strong Induction

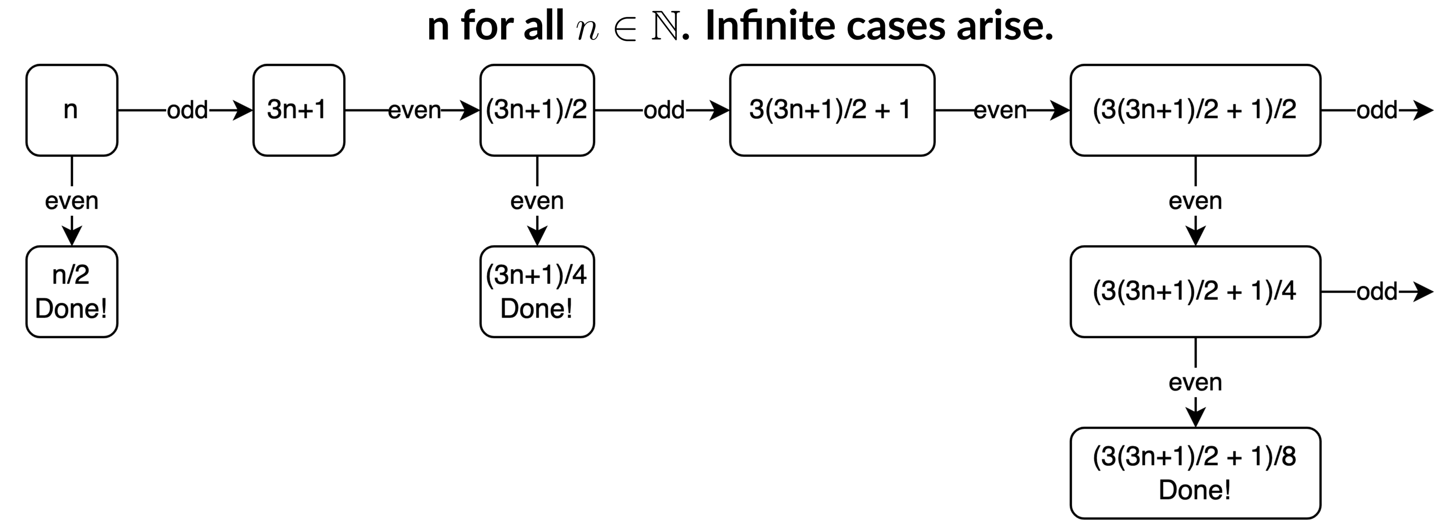
- **Testing Base Case:** $n = 2$ generates 1 after a single iteration through the Collatz Function. Therefore the base case holds.
- **Strong Induction Hypothesis:** Assume that n_{tsp} is finite for all $n \in \mathbb{N} < k$.
- **Showing that $P(k)$ Implies $P(k+1)$:** If $n = k + 1$ generates a number smaller than itself when plugged into the Collatz Function, then k+1 will generate another Collatz Number as a result of the induction hypothesis. Thus k+1 will also be a Collatz Number.

Thus the 3n + 1 Problem is now a problem of showing that n_{sp} is finite for all $n \in \mathbb{N}$

If $n \equiv 0$ (mod 2), then after one iteration, the Collatz Function will output $\frac{n}{2}$ which is smaller than n.

However, if $n \equiv 1$ (mod 2), then after one iteration, the Collatz Function will output $3n + 1$.

It's not simple to show that the Collatz Function will output a smaller number than n for all $n \in \mathbb{N}$. Infinite cases arise.



In 1994, Ivan Korec showed that at least 99.99% of all natural numbers will generate a number smaller than $n^{0.7925}$. This is so close to proving the Collatz Conjecture!

A Broader Question

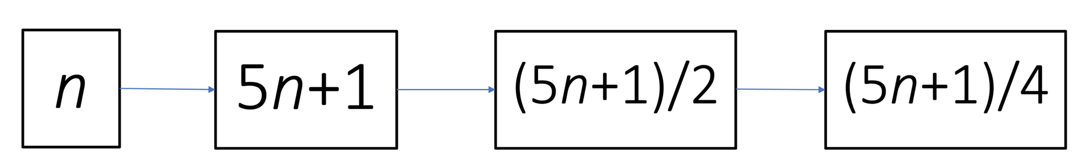
Consider the "5n + 1 Problem." This problem has been shown to be false.

The "13" loop in the 5n + 1 Problem

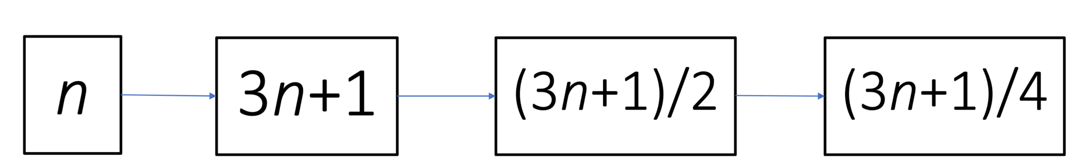
$f^0(n)$	$f^1(n)$	$f^2(n)$	$f^3(n)$	$f^4(n)$	$f^5(n)$	$f^6(n)$	$f^7(n)$	$f^8(n)$	$f^9(n)$	$f^{10}(n)$
13	66	33	166	83	416	208	104	52	26	13

Heuristical evidence helps predict the behavior of a "kn + 1 problem"

Again consider the 5n + 1 Problem. If an odd number is inputted, then Probability Theory tells us that on average, the next odd number is expected to equal approximately (5n + 1)/4, which is greater than n. This shows that some inputs may tend to infinity or get stuck in a cycle. Thus it makes sense that 5n + 1 Problem is false.



With the Collatz Function, alternatively, Probability Theory tells us that when an odd number is inputted into the Collatz Function, on average, the next odd number generated will be approximately (3n + 1)/4, which is less than n. This suggests that most inputs will follow a decreasing trend.



Equivalent Problem Statements

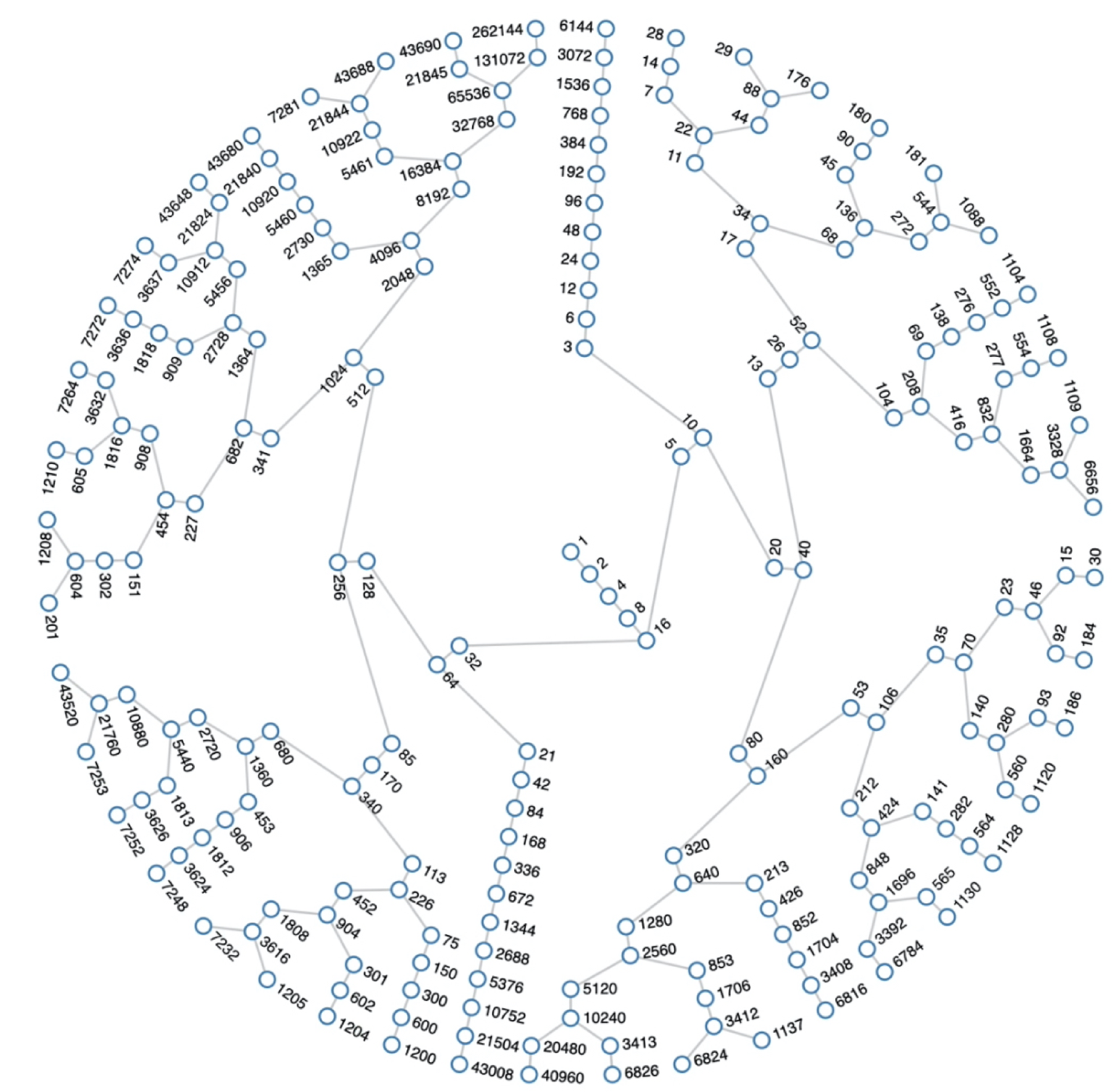
The 3n + 1 Problem can be asked in many ways. Changing the problem statement can inspire new ideas and make the problem easier.

1. Prove that for any natural number input, the Collatz Function will output a Collatz Number after a finite number of iterations.
2. Prove that no natural number input will cause the Collatz Function to either diverge to infinity or create an infinite loop that does not include 1.
3. Prove that the total stopping distance of all natural numbers is finite.
4. Prove that the stopping distance of all natural numbers is finite.

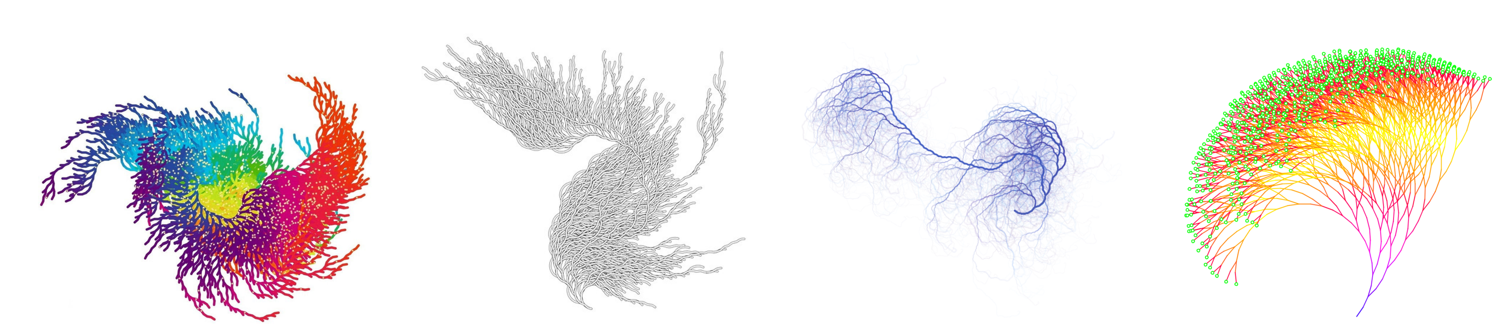
Interesting Patterns/Further Questions

1. Over half of the Collatz Numbers are part of lists of at least 2 consecutive numbers that have the same stopping distance.

Concentric circles have the same n_{tsd}. Groups of consecutive numbers are frequently in the same circle



2. Total stopping distances are not normally distributed. When testing all inputs from 1 to 100, all stopping distance were either less that 35 or greater than 85.
3. The randomness and unpredictability of the Collatz Function generates beautiful designs.



References

[1] Terence Tao. The notorious collatz conjecture. *Wordpress*, 2020.