Stokes Flow and Finite Difference Methods

Andrey Pravdić

UC Davis Directed Reading Program   Advised by Benjamin Godkin

Introduction

Fluid flows are governed by the Navier-Stokes equations. Making assumptions about the fluid flow can result easier solutions or allow study of certain fluid flows of interest.

At small scales, linear differences terms approximate differentials. This is the basis of finite difference methods. These linear differences result in a system of equations, which can be stored in a matrix and solved by a computer.

Stokes Flow

Fluid flows on small length scales, with viscous fluids, or with slow fluids are called Stokes Flows.

These three factors define the Reynolds Number, which approaches zero for any of the aforementioned conditions. Starting with the Navier-Stokes equations,

$$\text{Re}(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}) + \nabla p = \nabla^2\mathbf{u}$$

$$\nabla\cdot\mathbf{u} = 0$$

and letting Re $\to 0$, we obtain the Stokes Equations:

$$\nabla p = \nabla^2\mathbf{u},$$

$$\nabla\cdot\mathbf{u} = 0.$$

These equations are linear. Their solutions are unique for given boundary conditions.

Finite Difference Methods

The first step is discretize a domain by dividing it into a fine grid of, say, n subdivisions. The value of the solution will be calculated at each grid point and stored in a vector, $\mathbf{v}$.

Next, we appoximate derivatives using finite differences. This approximation will relate adjacent values of the grid, and eventually lead to a solvable system of equations. Below are some finite difference approximations.

If j is the entry number of v corresponding to a coordinate x_j,

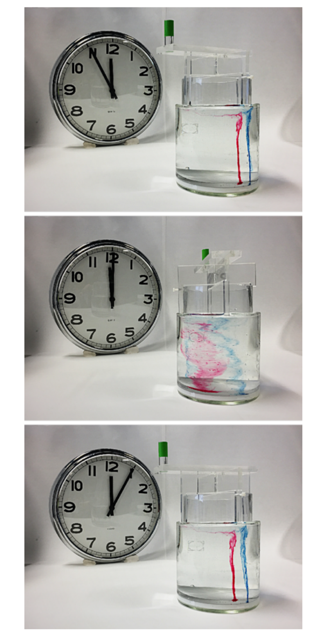
$$\frac{\partial^2 u}{\partial x^2}(x_j) \approx \frac{v_{j+1} - 2v_j + v_{j-1}}{h^2};$$

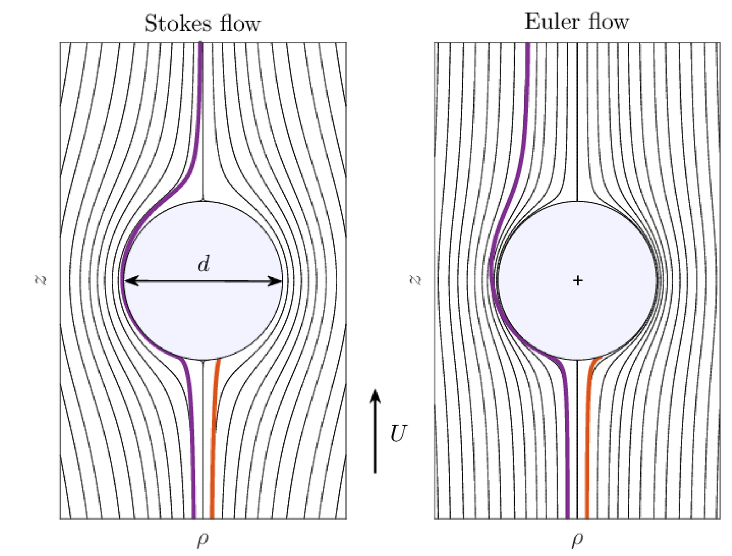
$$\frac{\partial u}{\partial x}(x_j) \approx \frac{v_{j+1} - v_j}{h};$$

where h is the length of the subintervals of the discretization.

Applying this relation to the whole vector results in n equations and n unknowns, allowing us to utilize a matrix operation.

Images of Stokes Flow





FDM: 1-Dimensional Advection Equation

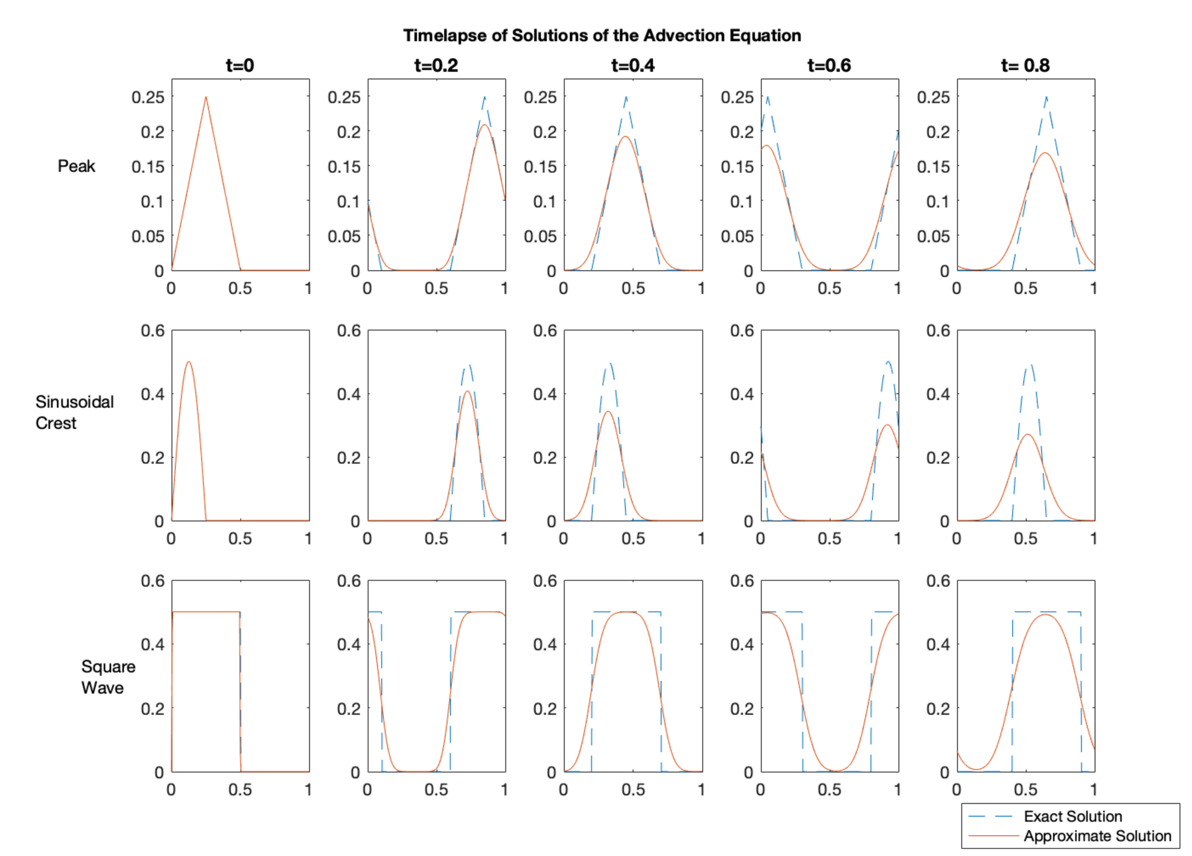
The Advection Equation describes the transport of some quantity over time, such as the shifting of dunes across the desert. We consider it with some initial condition $u_0(x)$ and assuming periodicity:

$$\begin{cases} \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0, & (x,t) \in (0,1)\times(0,T] \\ u(x,t) = u_0(x), & t = 0 \\ u(0,t) = u(1,t) & t \in [0,T) \end{cases}$$

The exact solution, given these conditions, is

$$u(x,t) = u_0(x - ct).$$

We applied three initial conditions: a triangular peak, a sinusoidal crest, and a square wave. All solutions dispersed over the course of three cycles.



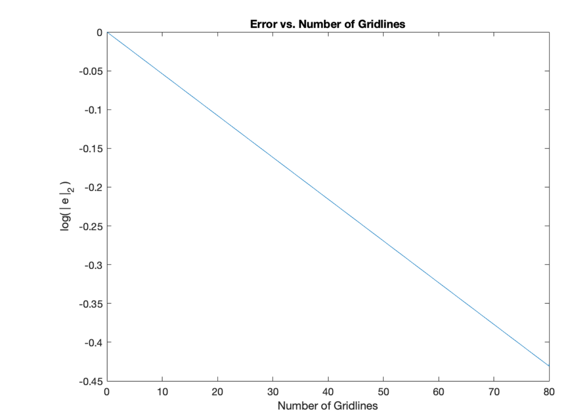
FDM: Poisson's Equation in 2 Dimensions

We also implemented an approximation for Poisson's Equation,

$$\nabla^2 u = f(x,y).$$

Solutions are determined by boundary conditions. In our case, we implemented the solution on $\Omega = [0,1]\times[0,1]$ and applied boundary conditions to $\partial\Omega$.

We found a simple analytic solution in the case of $f(x,y) = 0$ and boundary conditions $u = 1$ on the x-axis and $u = 0$ otherwise. Comparing this exact solution to our numerical solution, we found, as expected, error reduce with an increasingly fine grid.



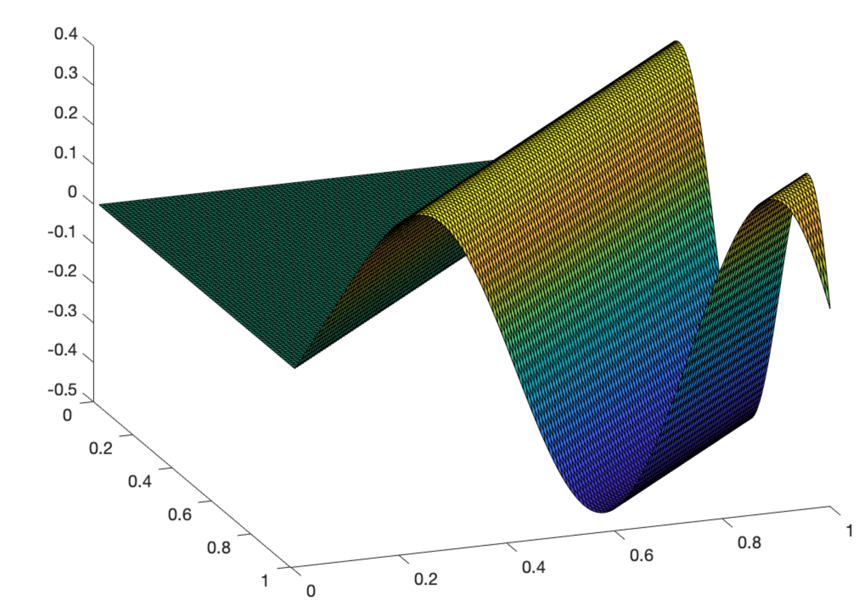


Figure 1. Solution for $f(x,y) = 1$ with zero for boundary conditions.

Future Projects

1. Implement a numerical solver for the Stokes Equation using MAC discretization.
2. Implement solutions on more interesting domains and boundaries
   1. Domains other than Ω
   2. Box with a hole
3. Implement a numerical solver for the Heat Equation

References

Chen, L. (2020). (tech.). Finite Difference Methods for Poisson Equation. Irvine, CA: UCI Math Department.

Chen, L. (2020). (tech.). Finite Difference Method for Stokes Equations: MAC Scheme.

Childress, S. (2009). An introduction to theoretical fluid mechanics. Courant Institute of Mathematical Sciences.

Heller, J. (1960). "An Unmixing Demonstration". American Journal of Physics. 28 (4): 348–353.