Schubert Calculus

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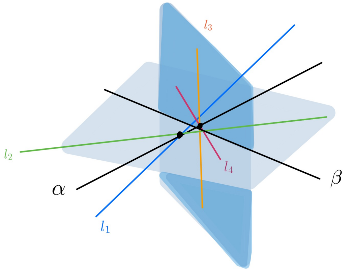
Motivating Example

How many lines intersect 4 given lines l_1, l_2, l_3, and l_4 in "general position" in 3 dimensional space?

Hermann Schubert's Intuitive Solution:



Projective Space

Definition. The n - dimensional complex projective space $\mathbb{P}^n_{\mathbb{C}}$ is the set of one dimensional linear subspaces (lines through the origin) in $\mathbb{C}^{n+1}$

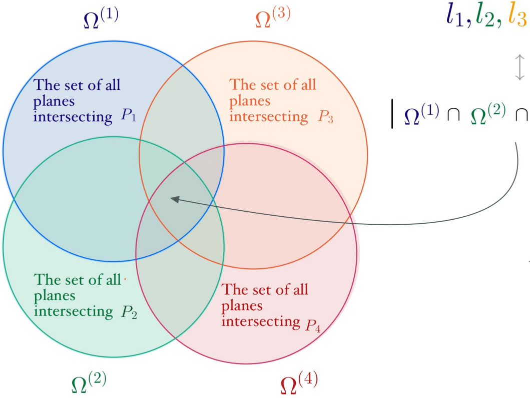
The Grassmannian

Definition. The Grassmannian $Gr(n, k)$ is the collection of k - dimensional subspaces $V \subset \mathbb{C}^n$.

$$Gr(n, k) = \{V \subset \mathbb{C}^n | dim(V) = k\}$$

Example: $Gr(4, 2)$: The set of two dimensional subspaces (planes) in $\mathbb{C}^4 \longleftrightarrow$ The set of lines in $\mathbb{P}^3_{\mathbb{C}}$
Elements in $Gr(4, 2)$: A pair of linearly independent vectors $v_1, v_2 \leftrightarrow$ Associated 2×4 matrix, with a unique representation in reduced row echelon form.

Translate Problem



The #of lines intersecting

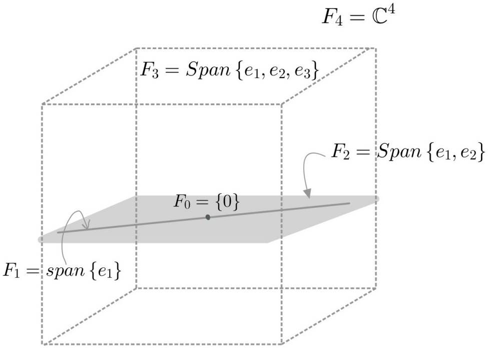
l_1, l_2, l_3, l_4

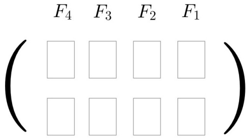
$\updownarrow$

$|\, \Omega^{(1)} \cap \Omega^{(2)} \cap \Omega^{(3)} \cap \Omega^{(4)} \,|$

Fix a flag with respect to the standard basis

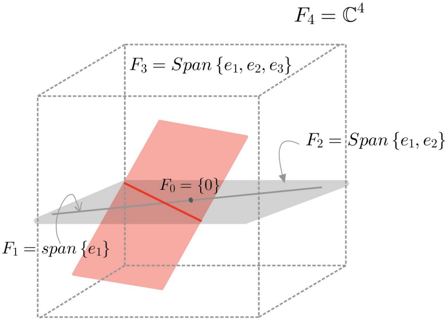
$$F_0 \subset F_1 \subset F_2 \subset F_3 \subset F_4 = \mathbb{C}^4$$





Schubert Cells and Schubert Varieties

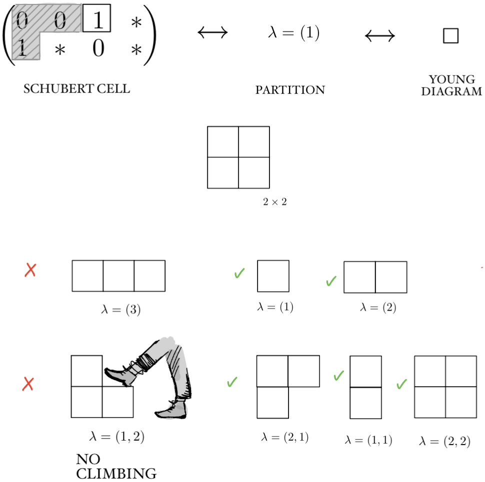
Schubert Cell of $Gr(4,2)$ The set of all planes $V \subset \mathbb{C}^4$ that satisfy certain "Schubert Conditions"



$$\begin{array}{cccc} F_4 & F_3 & F_2 & F_1 \end{array}$$
$$\begin{pmatrix} 0 & 0 & 1 & * \\ 1 & * & 0 & * \end{pmatrix}$$

The **Schubert Variety** corresponding to a list of Schubert Conditions is the set of all Schubert cells that satisfy *at least* those conditions

Partitions and Young Diagrams



The Littlewood Richardson Rule

$$\{\text{Schubert Varieties}\} \longleftrightarrow \{\text{ Schur Functions}\}$$

$$\text{Intersection of Schubert Varieties} \longleftrightarrow \text{Products of Schur Functions}$$

The Zero Dimensional Littlewood Richardson Rule: Let B be the $k \times (n-k)$ ambient rectangle and let $\lambda^{(1)} \dots \lambda^{(m)}$ be partitions fittings inside B such that $|B| = \Sigma_i |\lambda^{(i)}|$. Then

$$c^B_{\lambda^{(1)} \dots \lambda^{(m)}} := |\Omega_{\lambda^{(1)}} \cap \dots \cap \Omega_{\lambda^{(m)}}|$$

is equal to the number of chains of Littlewood-Richardson tableaux of contents $\lambda^{(1)} \dots \lambda^{(m)}$ with total shape B.

$$\sigma_{\square} \cdot \sigma_{\square} \cdot \sigma_{\square} \cdot \sigma_{\square} = \boxed{2}\,\sigma_{\boxplus}$$